Notre Dame Journal of Formal Logic
Volume XVIII, Number 1, January 1977
NDJFAM

A NOTE ON THE COMPLETENESS PROOF FOR NATURAL DEDUCTION

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Brief as it is, the argument of my earlier paper[1] can be further simplified to put it easily within reach of beginning students of logic. As in that paper, let a system of natural deduction be based on *negation*, *conjunction*, and *universal quantification*, with the standard rules of *indirect proof*, *simplification* and *conjunction*, and *instantiation* and *generalization* governing these three operations, respectively. For easier exposition we also include now another rule, clearly redundant, for simplifying double negations.

Let a deduction D be given, each of whose assumptions is undischarged in D and remains undischarged in any extension of D. Then the following rules for appending steps to D will define a certain extension D′ of D whose assumptions are likewise undischarged and undischargeable. If the first step in D is of a form treated by one of the rules 1-5 below introduce a new formula or formulas as the rule instructs, go on to the second step, and so on until all the formulas of D have been harvested and D′ has been reached. Repetitions may be omitted.

(1) From a step in D of the form $P\&Q$ introduce inferences in D′ of the forms P and Q, by *simplification.*
(2) From a step in D of the form $\forall xFx$ introduce inferences in D′ of the form Fa, by *instantiation,* using each free variable in D and the first free variable not in D. (Free and bound variables are distinguished typographically.)
(3) From a step in D of the form $--P$ introduce an inference in D′ of the form P, by *double negation.*
(4) From a step in D of the form $-(P\&Q)$ introduce an assumption in D′ of form $-P$, or, in case this would be dischargeable, introduce an assumption

1. "An elementary completeness proof for a system of natural deduction," *Notre Dame Journal of Formal Logic*, vol. XIV (1973), pp. 430-432.

Received January 31, 1974

of the form $-Q$ instead. (If an assumption of the form $-P$ would be dischargeable then it is clear from the rule of *conjunction* that an assumption of the form $-Q$ would be undischargeable.)
(5) From a step in D of the form $-\forall x Fx$ introduce an assumption in D′ of the form $-Fa$, where the free variable is the first one not already in use. (It is clear from the rule of *generalization* that the new assumption would be undischargeable.)

The completeness of the system is now easily shown. Let α be some arbitrary formula which cannot be proved. Then the assumption $-\alpha$ is undischargeable in the one-step deduction D consisting of this assumption alone. Call a formula *attainable* (relative to $-\alpha$) if it appears as a step in some one of the deductions D, D′, D″, ... Finally, let an interpretation of the system be given as follows: let quantifiers range over the universe of natural numbers; let free variables (in their given order) designate natural numbers; let an atomic formula be true if and only if it is attainable; let a non-atomic formula be true or false according to the customary interpretation of the logical operators.

Under this interpretation every attainable formula is true. For suppose not, and let β be a shortest attainable formula which is false, clearly, β cannot be an atomic formula. Nor can it be the negation of an atomic formula (for then the atomic formula and its negation would both be attainable, permitting the discharge of some assumption). So β must have one of the forms treated in the rules 1-5. Then, by the appropriate rule, a shorter false formula is easily attained, contrary to supposition.

It follows at once that every formula α which cannot be proved cannot be universally valid (since an interpretation exists in which $-\alpha$ is true). Q.E.D.

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