Notre Dame Journal of Formal Logic
Volume XVII, Number 4, October 1976
NDJFAM

THE AXIOMS FOR LATTICOIDS AND THEIR ASSOCIATIVE EXTENSIONS

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By definition, *cf.*, e.g., [1], p. 23, a latticoid is an algebraic system satisfying the following formulas:[1]

A1 $[ab]: a, b \epsilon A .\supset. a \cap b = b \cap a$
A2 $[ab]: a, b \epsilon A .\supset. a \cup b = b \cup a$
A3 $[ab]: a, b \epsilon A .\supset. a = a \cap (a \cup b)$
A4 $[ab]: a, b \epsilon A .\supset. a = a \cup (a \cap b)$

The addition of each (but, obviously, not both) of the following two formulas:

N1 $[abc]: a, b, c \epsilon A .\supset. a \cap (b \cap c) = (a \cap b) \cap c$
N2 $[abc]: a, b, c \epsilon A .\supset. a \cup (b \cup c) = (a \cup b) \cup c$

as a new axiom to $\{A1; A2; A3; A4\}$ generates two different systems which can be called latticoid with meet-associative law and latticoid with join-associative law, respectively.

In this note it will be shown that, although these three systems are rather weak, their respective axiom-systems can be shortened considerably. Namely, I shall prove that:

Any algebraic system

$$\mathfrak{A} = \langle A, \cup, \cap \rangle$$

where $\cup$ *and* $\cap$ *are two binary operations defined on the carrier set A, is either a latticoid or a latticoid with meet-associative law or a latticoid with join-associative law, if it satisfies respectively one of the groups of postulates* (A), (B), *and* (C) *which are given below:*

(A) *For latticoids:*

B1 $[abcdf]: a, b, c, d, f \epsilon A .\supset. c \cap ((a \cup b) \cap d) = ((b \cup a) \cap ((f \cap d) \cup d)) \cap c$
B2 $[ab]: a, b \epsilon A .\supset. a = (a \cup b) \cap a$

1. Throughout this paper A indicates an arbitrary but fixed carrier set. The so-called closure axioms are assumed tacitly.

Received January 7, 1971

(B) *For latticoids with meet-associative law:*

E1 $[abcdef]:a,b,c,d,e,f\epsilon A .\supset. e \cup (((a \cap b) \cap c) \cup d) = (((b \cap c) \cap a) \cup ((f \cup d) \cap d)) \cup e$

E2 $[abc]:a,b,c\epsilon A .\supset. a = ((a \cap b) \cap c) \cup a$

(C) *For latticoids with join-associative law:*

F1 $[abcdef]:a,b,c,d,e,f\epsilon A .\supset. e \cap (((a \cup b) \cup c) \cap d) = (((b \cup c) \cup a) \cap ((f \cap d) \cup d)) \cap e$

F2 $[abc]:a,b,c\epsilon A .\supset. a = ((a \cup b) \cup c) \cap a$

Remark I: It should be noted that the forms of postulates given in (A), (B), and (C) are suggested by Kalman's postulate system for lattices, *cf.* [2]. But, obviously, the deductions presented below differ in several points from Kalman's.

1 *Proof of* (A): Since it is self-evident that axioms *A1, A2, A3,* and *A4* imply *B1* and *B2,* it remains only to prove that the former formulas are the consequences of the latter. Hence, let us assume *B1* and *B2.* Then:

B3 $[ac]:a,c\epsilon A .\supset. c \cap a = (a \cup a) \cap c$

PR $[ac]:\mathrm{Hp}(1) .\supset.$

$c \cap a = c \cap ((a \cup (a \cup a)) \cap a) = (((a \cup a) \cup a) \cap (((a \cup a) \cap a) \cup a)) \cap c$ $[1, B2, b/a \cup a; B1, b/a \cup a, d/a, f/a \cup a]$

$= (((a \cup a) \cup a) \cap (a \cup a)) \cap c = (a \cup a) \cap c$ $[B2, b/a; B2, a/a \cup a, b/a]$

B4 $[a]:a\epsilon A .\supset. a = a \cap a$ $[B2, b/a; B3, c/a]$

B5 $[a]:a\epsilon A .\supset. a = a \cup a$ $[B2, b/a; B3, c/a \cup a; B4, a/a \cup a]$

A1 $[ab]:a,b\epsilon A .\supset. a \cap b = b \cap a$

PR $[ab]:\mathrm{Hp}(1) .\supset.$

$a \cap b = a \cap (b \cap b) = a \cap ((b \cup b) \cap b)$ $[1; B4, a/b; B5, a/b]$

$= ((b \cup b) \cap (((b \cup b) \cap b) \cup b)) \cap a$ $[B1, a/b, c/a, d/b, f/b \cup b]$

$= (b \cap (b \cup b)) \cap a = (b \cap b) \cap a = b \cap a$ $[B5, a/b; B2, a/b; B5, a/b; B4, a/b]$

B6 $[abdf]:a,b,d,f\epsilon A .\supset. (a \cup b) \cap d = (b \cup a) \cap ((f \cap d) \cup d)$

PR $[abdf]:\mathrm{Hp}(1) .\supset.$

$(a \cup b) \cap d = ((a \cup b) \cap d) \cap ((a \cup b) \cap d)$ $[1; B4, a/(a \cup b) \cap d]$

$= ((b \cup a) \cap ((f \cap d) \cup d)) \cap ((a \cup b) \cap d)$ $[B1, c/(a \cup b) \cap d]$

$= ((b \cup a) \cap ((f \cap d) \cup d)) \cap ((b \cup a) \cap ((f \cap d) \cup d))$ $[B1, c/(b \cup a) \cap ((f \cap d) \cup d)]$

$= (b \cup a) \cap ((f \cap d) \cup d)$ $[B4, a/(b \cup a) \cap ((f \cap d) \cup d)]$

B7 $[abc]:a,b,c\epsilon A .\supset. c \cap a = c \cap ((a \cap b) \cup a)$

PR $[abc]:\mathrm{Hp}(1) .\supset.$

$c \cap a = (c \cup c) \cap a = (c \cup c) \cap ((b \cap a) \cup a) = c \cap ((a \cap b) \cup a)$ $[1; B5, a/c; B6, a/c, b/c, d/a, f/b; B5, a/c; A1]$

B8 $[abd]:a,b,d\epsilon A .\supset. (a \cup b) \cap d = (b \cup a) \cap d$

PR $[abd]:\mathrm{Hp}(1) .\supset.$

$(a \cup b) \cap d = (b \cup a) \cap (((d \cup b) \cap d) \cup d)$ $[1; B6, f/d \cup b]$

$= (b \cup a) \cap (d \cup d) = (b \cup a) \cap d$ $[B2, a/d; B5, a/d]$

A2 $[ab]: a, b \in A \,.\supset.\, a \cup b = b \cup a$

PR $[ab]: \mathrm{Hp}(1) \,.\supset.$

$a \cup b = (a \cup b) \cap (a \cup b) = (b \cup a) \cap (a \cup b) = (a \cup b) \cap (b \cup a)$ [1; *B4*, $a/a \cup b$; *B8*, $d/a \cup b$; *A1*, $a/b \cup a$, $b/a \cup b$]

$= (b \cup a) \cap (b \cup a) = b \cup a$ [*B8*, $d/b \cup a$; *B4*, $a/b \cup a$]

A3 $[ab]: a, b \in A \,.\supset.\, a = a \cap (a \cup b)$ [*B2*; *A1*, $a/a \cup b$, b/a]

A4 $[ab]: a, b \in A \,.\supset.\, a = a \cup (a \cap b)$

PR $[ab]: \mathrm{Hp}(1) \,.\supset.$

$a = a \cap a = a \cap ((a \cap b) \cup a) = ((a \cap b) \cup a) \cap a$ [1; *B4*; *B7*, c/a; *A1*, $b/(a \cap b) \cup a$]

$= ((a \cap b) \cup a) \cap ((a \cap b) \cup a) = (a \cap b) \cup a = a \cup (a \cap b)$ [*B7*, $c/(a \cap b) \cup a$; *B4*, $a/(a \cap b) \cup a$; *A2*, $a/a \cap b$, b/a]

Since it is shown above that *A1*, *A2*, *A3*, and *A4* are the consequences of *B1* and *B2*, the proof is complete.

Remark II: We have to note that axioms *B1* and *B2* are inferentially equivalent to the following two formulas:

C1 $[abcdf]: a, b, c, d, f \in A \,.\supset.\, c \cup ((a \cap b) \cup d) = ((b \cap a) \cup ((f \cup d) \cap d)) \cup c$

C2 $[ab]: a, b \in A \,.\supset.\, a = (a \cap b) \cup a$

We omit here a proof of this fact, since it is completely banal.

2 *Proof of* (B): Since it is obvious that formulas *E1* and *E2* are the consequences of axioms *A1*, *A2*, *A3*, *A4*, and *N1*, we have only to prove that the former formulas imply the latter. Therefore, let us assume *E1* and *E2*. Then:

E3 $[ae]: a, e \in A \,.\supset.\, e \cup a = (a \cap a) \cup e$

PR $[ae]: \mathrm{Hp}(1) \,.\supset.$

$e \cup a = e \cup ((a \cap (a \cap a)) \cap a) \cup a)$ [1; *E2*, $b/a \cap a$, c/a]

$= ((((a \cap a) \cap a) \cap a) \cup ((((a \cap a) \cap a) \cup a) \cap a)) \cup e$ [*E1*, $b/a \cap a$, c/a, d/a, $f/(a \cap a) \cap a$]

$= ((((a \cap a) \cap a) \cap a) \cup (a \cap a)) \cup e$ [*E2*, b/a, c/a]

$= (a \cap a) \cup e$ [*E2*, $a/a \cap a$, b/a, c/a]

A4 $[ab]: a, b \in A \,.\supset.\, a = a \cup (a \cap b)$

PR $[ab]: \mathrm{Hp}(1) \,.\supset.$

$a = ((a \cap b) \cap (a \cap b)) \cup a = a \cup (a \cap b)$ [1; *E2*, $c/a \cap b$; *E3*, $a/a \cap b$, e/a]

E4 $[a]: a \in A \,.\supset.\, a = a \cap a$

PR $[a]: \mathrm{Hp}(1) \,.\supset.$

$a = ((a \cap a) \cap (a \cap a)) \cup a = (a \cap a) \cup ((a \cap a) \cap (a \cap a)) = a \cap a$ [1; *E2*, b/a, $c/a \cap a$; *E3*, $e/(a \cap a) \cap (a \cap a)$; *A4*, $a/a \cap a$, $b/a \cap a$]

A2 $[ab]: a, b \in A \,.\supset.\, a \cup b = b \cup a$ [*E3*, a/b, e/a; *E4*, a/b]

E5 $[a]: a \in A \,.\supset.\, a = a \cup a$ [*A4*, b/a; *E4*]

E6 $[abcdf]: a, b, c, d, f \in A \,.\supset.\, ((a \cap b) \cap c) \cup d = ((b \cap c) \cap a) \cap ((f \cup d) \cap d)$

PR $[abcdf]: \mathrm{Hp}(1) \,.\supset.$

$((a \cap b) \cap c) \cup d = (((a \cap b) \cap c) \cup d) \cup (((a \cap b) \cap c) \cup d)$ [1; *E5*, $a/((a \cap b) \cap c) \cup d$]

$= (((b \cap c) \cap a) \cup ((f \cup d) \cap d)) \cup (((a \cap b) \cap c) \cup d)$
[$E1$, $e/((a \cap b) \cap c) \cup d$]
$= (((b \cap c) \cap a) \cup ((f \cup d) \cap d)) \cup (((b \cap c) \cap a) \cup ((f \cup d) \cap d))$
[$E1$, $e/((b \cap c) \cap a) \cup ((f \cup d) \cap d)$]
$= ((b \cap c) \cap a) \cup ((f \cup d) \cap d)$
[$E5$, $a/((b \cap c) \cap a) \cup ((f \cup d) \cap d)$]

E7 $[abc]: a, b, c \epsilon A .\supset. c \cup a = c \cup ((b \cup a) \cap a)$

PR $[abc]: \mathrm{Hp}(1) .\supset.$
$c \cup a = (c \cap c) \cup a = ((c \cap c) \cap c) \cup a = ((c \cap c) \cap c) \cup ((b \cup a) \cap a)$
[1; $E4$, a/c; $E4$, a/c; $E6$, a/c, b/c, d/a, f/b]
$= (c \cap c) \cup ((b \cup a) \cap a) = c \cup ((b \cup a) \cap a)$ [$E4$, a/c; $E4$, a/c]

E8 $[ab]: a, b \epsilon A .\supset. a = (b \cup a) \cap a$

PR $[ab]: \mathrm{Hp}(1) .\supset.$
$a = a \cup a = a \cup ((b \cup a) \cap a) = ((b \cup a) \cap a) \cup a$
[1; $E5$; $E7$, c/a; $A2$, $b/(b \cup a) \cap a$]
$= ((b \cup a) \cap a) \cup ((b \cup a) \cap a) = (b \cup a) \cap a$
[$E7$, $c/(b \cup a) \cap a$; $E5$, $a/(b \cup a) \cap a$]

E9 $[abcd]: a, b, c, d \epsilon A .\supset. ((a \cap b) \cap c) \cup d = ((b \cap c) \cap a) \cup d$

PR $[abcd]: \mathrm{Hp}(1) .\supset.$
$((a \cap b) \cap c) \cup d = ((b \cap c) \cap a) \cup ((d \cup d) \cap d)$ [1; $E6$, f/d]
$= ((b \cap c) \cap a) \cup d$ [$E8$, a/d, b/d]

E10 $[abc]: a, b, c \epsilon A .\supset. (a \cap b) \cap c = (b \cap c) \cap a$

PR $[abc]: \mathrm{Hp}(1) .\supset.$
$(a \cap b) \cap c = ((a \cap b) \cap c) \cup ((a \cap b) \cap c)$ [1; $E5$, $a/(a \cap b) \cap c$]
$= ((b \cap c) \cap a) \cup ((a \cap b) \cap c)$ [$E9$, $d/(a \cap b) \cap c$]
$= ((a \cap b) \cap c) \cup ((b \cap c) \cap a)$
[$A2$, $a/(b \cap c) \cap a$, $b/(a \cap b) \cap c$]
$= ((b \cap c) \cap a) \cup ((b \cap c) \cap a)$ [$E9$, $d/(b \cap c) \cap a$]
$= (b \cap c) \cap a$ [$E5$, $a/(b \cap c) \cap a$]

A1 $[ab]: a, b \epsilon A .\supset. a \cap b = b \cap a$

PR $[ab]: \mathrm{Hp}(1) .\supset.$
$a \cap b = (a \cap a) \cap b = (b \cap a) \cap a = ((b \cap b) \cap a) \cap a$
[1; $E4$; $E10$, a/b, b/a, c/a; $E4$, a/b]
$= ((b \cap a) \cap b) \cap a = (b \cap a) \cap (b \cap a) = b \cap a$
[$E10$, a/b, c/a; $E10$, $a/b \cap a$, c/a; $E4$, $a/b \cap a$]

A3 $[ab]: a, b \epsilon A .\supset. a = a \cap (a \cup b)$

PR $[ab]: \mathrm{Hp}(1) .\supset.$
$a = (b \cup a) \cap a = a \cap (b \cup a) = a \cap (a \cup b)$ [1; $E8$; $A1$, $a/b \cup a$, b/a; $E5$]

N1 $[abc]: a, b, c \epsilon A .\supset. a \cap (b \cap c) = (a \cap b) \cap c$

PR $[abc]: \mathrm{Hp}(1) .\supset.$
$a \cap (b \cap c) = (b \cap c) \cap a = (a \cap b) \cap c$ [1; $A1$, $b/b \cap c$; $E10$]

Since it is shown above that *E1* and *E2* imply *A1*, *A2*, *A3*, *A4*, and *N1*, the proof is complete.

Remark III: It should be noted that the proof of *A1* given above, i.e., that *E4* and *E10* hold *A1*, is due to Padmanabhan, *cf.* [4], but the deductions presented here differ from his.

3 *Proof of* (C): Since it is obvious that axioms *A1, A2, A3, A4*, and *N2* imply *F1* and *F2*, it remains only to prove that the latter formulas hold the former. But, since *F1* and *F2* are duals of *E1* and *E2* respectively, it is self-evident that the deductions, which are exactly analogous and dual to the proofs presented in section **2**, will show at once that *A1, A2, A3, A4*, and *N2* are the consequences of *F1* and *F2*. Thus, we have

$$\{A1;\ A2;\ A3;\ A4;\ N2\} \rightleftarrows \{F1;\ F2\}$$

4 The mutual independence of axioms contained in each of the sets $\{B1;\ B2\}$, $\{C1;\ C2\}$, $\{E1;\ E2\}$, and $\{F1;\ F2\}$ is established by using the following algebraic table:[2]

𝔐1

∪	α	β
α	α	α
β	α	α

∩	α	β
α	α	β
β	α	β

𝔐2

∪	α	β
α	α	β
β	α	β

∩	α	β
α	α	α
β	α	α

𝔐3

∪	α	β
α	α	α
β	α	α

∩	α	β
α	α	α
β	α	β

𝔐4

∪	α	β	γ	δ	η
α	α	β	γ	δ	η
β	β	β	η	δ	η
γ	γ	η	γ	δ	η
δ	δ	δ	δ	δ	η
η	η	η	η	η	η

∩	α	β	γ	δ	η
α	α	α	α	α	α
β	α	β	α	β	β
γ	α	α	γ	γ	γ
δ	α	β	γ	δ	δ
η	α	β	γ	δ	η

𝔐5

∪	α	β	γ	δ	η
α	α	α	α	α	α
β	α	β	α	β	β
γ	α	α	γ	γ	γ
δ	α	β	γ	δ	δ
η	α	β	γ	δ	η

∩	α	β	γ	δ	η
α	α	β	γ	δ	η
β	β	β	η	δ	η
γ	γ	η	γ	δ	η
δ	δ	δ	δ	δ	η
η	η	η	η	η	η

Namely:

(a) *B1* and *B2* are mutually independent, since:

2. Concerning 𝔐1 and 𝔐3, *cf.* [3], pp. 385-386. It is self-evident that the tables 𝔐5 and 𝔐4 are isomorphic with the diagram given in [1], p. 22, figure 5, and its dual respectively.

(α) $\mathfrak{M}1$ verifies *B2*, but falsifies *B1* for a/α, b/α, c/α, d/β, and f/β: (i) $\alpha \cap ((\alpha \cup \alpha) \cap \beta) = \alpha \cap (\alpha \cap \beta) = \alpha \cap \beta = \beta$, (ii) $((\alpha \cup \alpha) \cap ((\beta \cap \beta) \cup \beta)) \cap \alpha = (\alpha \cap (\beta \cup \beta)) \cap \alpha = (\alpha \cap \alpha) \cap \alpha = \alpha \cap \alpha = \alpha$.
(β) $\mathfrak{M}3$ verifies *B1*, but falsifies *B2* for a/β and b/β: (i) $\beta = \beta$, (ii) $(\beta \cup \beta) \cap \beta = \alpha \cap \beta = \alpha$.

(b) *C1* and *C2* are mutually independent, since:

(α) $\mathfrak{M}2$ verifies *C2*, but falsifies *C1* for a/α, b/α, c/α, d/β, and f/β: (i) $\alpha \cup ((\alpha \cap \alpha) \cup \beta) = \alpha \cup (\alpha \cup \beta) = \alpha \cup \beta = \beta$, (ii) $((\alpha \cap \alpha) \cup ((\beta \cup \beta) \cap \beta)) \cup \alpha = (\alpha \cup (\beta \cap \beta)) \cup \alpha = (\alpha \cup \alpha) \cup \alpha = \alpha \cup \alpha = \alpha$.
(β) $\mathfrak{M}3$ verifies *C1*, but falsifies *C2* for a/β and b/β: (i) $\beta = \beta$, (ii) $(\beta \cap \beta) \cup \beta = \beta \cup \beta = \alpha$.

(c) *E1* and *E2* are mutually independent, since:

(α) $\mathfrak{M}5$ verifies *E2*, but falsifies *E1* for a/β, b/γ, c/δ, d/η, e/η, and f/η: (i) $\eta \cup (((\beta \cap \gamma) \cap \delta) \cup \eta) = \eta \cup ((\eta \cap \delta) \cup \eta) = \eta \cup (\eta \cup \eta) = \eta \cup \eta = \eta$, (ii) $(((\gamma \cap \delta) \cap \beta) \cup ((\eta \cup \eta) \cap \eta)) \cup \eta = ((\delta \cap \beta) \cup (\eta \cap \eta)) \cup \eta = (\delta \cup \eta) \cup \eta = \delta \cup \eta = \delta$.
(β) $\mathfrak{M}3$ verifies *E1*, but falsifies *E2* for a/β and b/β: (i) $\beta = \beta$, (ii) $((\beta \cap \beta) \cap \beta) \cup \beta = (\beta \cap \beta) \cup \beta = \beta \cup \beta = \alpha$.

(d) *F1* and *F2* are mutually independent, since:

(α) $\mathfrak{M}4$ verifies *F2*, but falsifies *F1* for a/β, b/γ, c/δ, d/η, e/η, and f/η: (i) $\eta \cap (((\beta \cup \gamma) \cup \delta) \cap \eta) = \eta \cap ((\eta \cup \delta) \cap \eta) = \eta \cap (\eta \cap \eta) = \eta \cap \eta = \eta$, (ii) $(((\gamma \cup \delta) \cup \beta) \cap ((\eta \cap \eta) \cup \eta)) \cap \eta = ((\delta \cup \beta) \cap (\eta \cup \eta)) \cap \eta = (\delta \cap \eta) \cap \eta = \delta \cap \eta = \delta$.
(β) $\mathfrak{M}3$ verifies *F1*, but falsifies *F2* for a/β and b/β: (i) $\beta = \beta$, (ii) $((\beta \cup \beta) \cup \beta) \cap \beta = (\alpha \cup \beta) \cap \beta = \alpha \cap \beta = \alpha$.

5 It is well known[3] that a latticoid with meet-associative law and a latticoid with join-associative law are two different systems. We can prove it easily using tables $\mathfrak{M}4$ and $\mathfrak{M}5$. Namely:

(1) $\mathfrak{M}4$ verifies *A1, A2, A3, A4,* and *N1*, but falsifies *N2* for a/β, b/γ, and c/δ: (i) $\beta \cup (\gamma \cup \delta) = \beta \cup \delta = \delta$, (ii) $(\beta \cup \gamma) \cup \delta = \eta \cup \delta = \eta$.

(2) $\mathfrak{M}5$ verifies *A1, A2, A3, A4,* and *N2*, but falsifies *N1* for a/β, b/γ, and c/δ: (i) $\beta \cap (\gamma \cap \delta) = \beta \cap \delta = \delta$, (ii) $(\beta \cap \gamma) \cap \delta = \eta \cap \delta = \eta$.

Thus, the systems $\{A1; A2; A3; A4; N1\}$ and $\{A1; A2; A3; A4; N2\}$, although they are duals, are different.

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3. *Cf.*, e.g., [1], p. 22, Example 3.

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