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AN ARITHMETICAL RECONSTRUCTION OF THE LIAR'S ANTINOMY USING ADDITION AND MULTIPLICATION

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The present note gives an improvement on [8]. There it was shown that the liar's antinomy can be reconstructed in any recursively enumerable arithmetical theory in which all elementary functions are definable, if the theory is assumed to be complete.[1] Here the same construction is done by requiring only definability of addition and multiplication. This constitutes a natural and therefore straight-forward proof of a strong version of the incompleteness theorem for arithmetical theories. The improvement on [8] consists in the fact that addition and multiplication are obviously fewer and less complex than all elementary functions: the former belong respectively to the first and to the second class of Grzegorczyk's hierarchy [5], whereas the latter constitute its third class. The present result can be considered optimal in so far as it is impossible to obtain the same result for the less complex function used because definability of addition alone allows completeness, see [1].

1 *Nomenclature* A traditional first-order arithmetical language is usually constructed from the following items: individual variables $x_0, x_1, \ldots$; an individual constant 0 to represent the number zero; a unary function symbol s to represent the successor function; a finite (possibly empty) set of binary operation symbols $\{\circ_i \mid 1 \leqslant i \leqslant a\}$, where a is any natural number, to represent e.g., addition, multiplication, etc.; the equality symbol $=$; connectives, say $\neg$ and $\rightarrow$; quantifiers, say $\wedge$ and $\vee$.

We consider any such language $\mathcal{L}$. Let $\mathbf{N}$ be the set of natural numbers and $n, n_1, n_2, \ldots$ be any elements of $\mathbf{N}$. Let $\theta, \theta_1, \theta_2$ be any terms of $\mathcal{L}$ and let Φ, Φ_1, Φ_2 be any formulas of $\mathcal{L}$. We have to define a computable injection g: $\mathcal{L} \rightarrow \mathbf{N}$. We set

$$b = a + 8$$
$$\langle n_1, n_2 \rangle = (n_1 + n_2)^2 + n_1$$

1. Note in [8] the following misprints: p. 377, l. 23: ε^2 *instead of* ε^3; p. 378, l. 10: quantifier *instead of* quantifier $\vee$; l. 17: $i \in n$ *instead of* $i \leqslant n$; l. 30: negative *instead of* nonnegative; p. 379, ll. 15, 17, 19, 22, 23, 24, 26, 27: $\forall$ *instead of* $\vee$; p. 380, l. 2: sur *instead of* zur.

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and define the injection g by recursion on the structure of $\mathcal{L}$ as follows:

$$g(0) = 0$$
$$g(s\theta) = \begin{cases} g(\theta) + b & \text{if } b \mid g(\theta) \\ b.g(\theta) + 1 & \text{else} \end{cases}$$
$$g(x_i) = b.i + 2$$
$$g(\theta_1 \circ_i \theta_2) = b.\langle g(\theta_1), g(\theta_2)\rangle + 2 + i \quad \textit{for } 1 \leqslant i \leqslant a$$
$$g(\theta_1 = \theta_2) = b.\langle g(\theta_1), g(\theta_2)\rangle + a + 3$$
$$g(\neg\Phi) = b.g(\Phi) + a + 4$$
$$g(\Phi_1 \rightarrow \Phi_2) = b.\langle g(\Phi_1), g(\Phi_2)\rangle + a + 5$$
$$g(\wedge x_i\Phi) = b.\langle g(x_i), g(\Phi)\rangle + a + 6$$
$$g(\vee x_i\Phi) = b.\langle g(x_i), g(\Phi)\rangle + a + 7$$

We will also make use of the function d: $\mathbf{N} \rightarrow \mathbf{N}$ defined as

$$d(u) = b.\langle 2,\ b.\langle b.\langle 2,\ b.u\rangle + a + 3,\ u\rangle + a + 5\rangle + a + 6$$

which obviously can be represented by an eighth-degree polynominal in u with natural number coefficients. We call it the diagonal function because

$$d(g(\Phi)) = g(\wedge x_0(x_0 = s^{g(\Phi)}0 \rightarrow \Phi)),$$

compare [4].

A formula of $\mathcal{L}$ is called a sentence iff no variable occurs free in it. A set of sentences is called a theory iff it contains any sentence which can be deduced from it by first-order logic with equality. A set of sentences is called arithmetical iff it is satisfied by a realization whose universe is $\mathbf{N}$, which maps the constant 0 on the number zero and which maps the constant s on the successor function. Such a realization maps any digit s^n0 on the number n.

We say that a set $X \subseteq \mathbf{N}$ is definable in a set of sentences Y iff there is a formula Φ such that

$$n \in X \text{ iff } \Phi(s^n0) \in Y.$$

We say that truth is definable in a theory $\mathbf{T}$ iff the set $g(\mathbf{T})$ of g values of the sentences of $\mathbf{T}$ is definable in $\mathbf{T}$. This implies that there is a formula Θ such that

$$\Phi \in \mathbf{T} \text{ iff } \Theta(s^{g(\Phi)}0) \in \mathbf{T}.$$

Note that the above notion of definability is weaker (and therefore more general) than that of Gödel [2] and Tarski [4], provided Y is consistent. In particular the above notion of definability of truth proves, for consistent $\mathbf{T}$, to be weaker than that of Tarski [4]. On the other hand the above notion of definability of truth is weaker than that of Tarski [3], see [8].

We say that a function f: $\mathbf{N}^k \rightarrow \mathbf{N}$ is definable in a set of sentences Y iff there is a term θ such that

$$\theta(s^{n_1}0, \ldots, s^{n_k}0) = s^{f(n_1,\ldots,n_k)}0 \in Y \text{ for any } n_1, \ldots, n_k.$$

2 *The construction of the antinomy*

Lemma 1 *Let* $\mathbf{T}$ *be any theory. If addition and multiplication are definable in* $\mathbf{T}$, *then every polynominal function with natural number coefficients is definable in* $\mathbf{T}$.

Proof: From multiplication being definable in $\mathbf{T}$ it follows that monomial functions with natural number coefficients are definable in $\mathbf{T}$. From this and from addition being definable in $\mathbf{T}$ the lemma follows immediately.

Corollary 1 *Let* $\mathbf{T}$ *be any theory. If addition and multiplication are definable in* $\mathbf{T}$, *then the diagonal function is definable in* $\mathbf{T}$.

Lemma 2 *Let* $\mathbf{T}$ *be any recursively enumerable arithmetical theory. If every polynomial function with natural number coefficients is definable in* $\mathbf{T}$, *then truth is definable in* $\mathbf{T}$.

Proof: Consider any recursively enumerable set X. From [7] we know that X is diophantine. Therefore there are two polynomials $\mathrm{p}_1(u, u_1, \ldots, u_h)$ and $\mathrm{p}_2(u, u_{h+1}, \ldots, u_k)$ in $u, u_1, \ldots, u_h$ and $u, u_{h+1}, \ldots, u_k$ respectively, both with natural number coefficients, such that

$$n \in X \text{ iff } \mathrm{p}_1(n, n_1, \ldots, n_h) = \mathrm{p}_2(n, n_{h+1}, \ldots, n_k)$$

for some $n_1, \ldots, n_k$ (see e.g., [6], pp. 200-201). By the lemma above we know that there are terms θ_1 and θ_2 such that

$$\theta_1(s^n0, s^{n_1}0, \ldots, s^{n_h}0) = s^{\mathrm{p}_1(n,n_1,\ldots,n_h)}0 \in \mathbf{T}$$
$$\theta_2(s^n0, s^{n_{h+1}}0, \ldots, s^{n_k}0) = s^{\mathrm{p}_2(n,n_{h+1},\ldots,n_k)}0 \in \mathbf{T}$$

for any $n, n_1, \ldots, n_k$. From $\mathbf{T}$ being arithmetical it follows that if

$$\vee x_1 \ldots \vee x_k\, \theta_1(s^n0, x_1, \ldots, x_h) = \theta_2(s^n0, x_{h+1}, \ldots, x_k) \in \mathbf{T}$$

then

$$n \in X$$

and from $\mathbf{T}$ being a theory it follows that if $n \in X$ then

$$\vee x_1 \ldots \vee x_k\, \theta_1(s^n0, x_1, \ldots, x_h) = \theta_2(s^n0, x_{h+1}, \ldots, x_k) \in \mathbf{T}.$$

Any recursively enumerable set is therefore definable in $\mathbf{T}$ and, as a particular case, we obtain that truth is definable in $\mathbf{T}$.

Theorem on definability of truth *Let* $\mathbf{T}$ *be any recursively enumerable arithmetical theory. If addition and multiplication are definable in* $\mathbf{T}$, *then truth is definable in* $\mathbf{T}$.

Proof: From Lemma 1 and Lemma 2.

Lemma 3 *Let* $\mathbf{T}$ *be a theory. If truth and the diagonal function are both definable in* $\mathbf{T}$, *then* $\mathbf{T}$ *is incomplete or contradictory.*

Proof: Suppose, there is a formula Θ such that

$$\Phi \in \mathbf{T} \text{ iff } \Theta(s^{g(\Phi)}0) \in \mathbf{T}$$

and there is a term δ such that

$$\delta(s^n 0) = s^{d(n)} 0 \in \mathbf{T}.$$

Consider the number $m = g(\neg\Theta(\delta(x_0)))$. It results that

$$\begin{aligned}&\Theta(s^{d(m)}0) \in \mathbf{T}\\ \text{iff } &\wedge x_0(x_0 = s^m 0 \rightarrow \neg\Theta(\delta(x_0))) \in \mathbf{T}\\ \text{iff } &\neg\Theta(\delta(s^m 0)) \in \mathbf{T}\\ \text{iff } &\neg\Theta(s^{d(m)} 0) \in \mathbf{T}.\end{aligned}$$

Therefore either one of the sentences

$$\Theta(s^{d(m)} 0) \text{ and } \neg\Theta(s^{d(m)} 0)$$

does not belong to **T** or **T** is contradictory.

Incompleteness theorem *Every recursively enumerable arithmetical theory in which addition and multiplication are definable is incomplete.*

Proof: It suffices to combine the theorem on definability of truth and Corollary 1 with Lemma 3 and to remark that an arithmetical set of sentences cannot be contradictory.

As concerns applications of the above theorem we can note the following. From [4] we know that the theory **R** with the axioms

$s^{n_1}0 + s^{n_2}0 = s^{n_1+n_2}0$
$s^{n_1}0.s^{n_2}0 = s^{n_1 \cdot n_2}0$
$s^{n_1}0 \neq s^{n_2}0$, *for* $n_1 \neq n_2$
$\vee x_0(x_0 + x_1) = s^n 0 \rightarrow x_1 = s^0 0 \vee \ldots \vee x_1 = s^n 0$
$\vee x_0(x_0 + x_1) = s^n 0 \vee \vee x_0(x_0 + s^n 0) = x_1$

and every consistent theory obtained from **R** by adding new axioms with the same constants of **R** is incomplete. From the theory above we may conclude that the subtheory of **R** with the only axioms

$$s^{n_1}0 + s^{n_2}0 = s^{n_1+n_2}0$$
$$s^{n_1}0.s^{n_2}0 = s^{n_1 \cdot n_2}0$$

and every arithmetical theory obtained from it by adding new axioms (also with new constants) is incomplete.

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