Notre Dame Journal of Formal Logic
Volume XVII, Number 3, July 1976
NDJFAM

MONADS FOR REGULAR AND NORMAL SPACES

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Given an enlargement $*(X, \mathfrak{J})$ of a topological space $(X, \mathfrak{J})$, the monad of a point $x \in X$ is defined to be $\mu(x) = \bigcap\{*F: x \in F \in \mathfrak{J}\}$. It is known that for any space $(X, \mathfrak{J})$, the family of monads $\{\mu(x): x \in X\}$ contains all the information about $\mathfrak{J}$ in the sense that for each $x \in X$, $\{F \subseteq X: \mu(x) \subseteq *F\}$ is exactly the neighborhood filter at x. However, it is possible to say something about $\mathfrak{J}$ without resorting to this method. For example, a space X is Hausdorff iff for any two points x and y in X, $\mu(x) \cap \mu(y) = \emptyset$. In this paper some further relationships between the topology on X and $\{\mu(x): x \in X\}$ will be shown, and particularly nice characterizations of regular and normal spaces will be given. These characterizations will be in terms of a natural topology on $*X$, the Q-topology. Let us briefly consider the Q-topology.

It is possible to write a formal sentence expressing the fact that $\mathfrak{J}$ is a topology on X, so in any enlargement $*(X, \mathfrak{J})$, $*\mathfrak{J}$ is closed under *finite intersections (and hence under finite intersections) and under internal unions. $*\mathfrak{J}$ also contains $\emptyset$ and $*X$, so is the base for a topology on $*X$, the Q-topology. Sets in $*\mathfrak{J}$ are said to be *open, subsets of $*X$ whose complements are in $*\mathfrak{J}$ are said to be *closed, and so on. Robinson has shown that an internal set is *open iff it is Q-open and *closed iff it is Q-closed. Also, a standard set A is open iff $*A$ is *open. We now introduce a new type of refinement relation which is particularly suited for studying Q-topologies.

Definition 1 We shall say that the covering $\mathfrak{U}_1$ of X fills the covering $\mathfrak{U}_2$ of X if for each $V \in \mathfrak{U}_2$, $V = \bigcup\{U \in \mathfrak{U}_1: U \subseteq V\}$.

Let $\mathfrak{G}$ be the collection of all finite open coverings of a given space X and let FR be the filling relation restricted to $\mathfrak{G} \times \mathfrak{G}$. The left domain of FR is $\mathfrak{G}$ since every covering fills itself and for each finite collection $\mathfrak{U}_1, \ldots, \mathfrak{U}_n$ of coverings in $\mathfrak{G}$, $\{U_1 \cap \ldots \cap U_n: U_1 \in \mathfrak{U}_1, \ldots, U_n \in \mathfrak{U}_n\}$ is a covering in $\mathfrak{G}$ filling each of $\mathfrak{U}_1, \ldots, \mathfrak{U}_n$, so the relation FR is concurrent. Hence, there is a covering of $*X$ in $*\mathfrak{G}$, say φ_F, such that if $\mathfrak{U}$ is a finite open covering of X, φ_F fills $*\mathfrak{U}$. In general φ_F is not unique and we shall speak of an arbitrary but fixed φ_F. For each $x \in *X$, $\{P \in \varphi_F: x \in P\}$ is an

Received December 13, 1974

internal subset of the *finite set φ_F and is also *finite, so $\bigcap\{P \in \varphi_F : x \in P\}$ is *open. Set $\varphi_F(x) = \bigcap\{P \in \varphi_F : x \in P\}$ for each $x \in {}^*X$ so that $\{\varphi_F(x) : x \in {}^*X\}$ is a *finite *open covering of $^*X$ filling φ_F and in turn filling $^*\mathfrak{U}$ for each finite open covering $\mathfrak{U}$ of X. Moreover, $\{\varphi_F(x) : x \in {}^*X\}$ has the additional useful property that for each $z \in {}^*X$, $\varphi_F(z)$ is the smallest set in $\{\varphi_F(x) : x \in {}^*X\}$ containing z.

Theorem 1 *A subset F of a topological space X is open iff for each $x \in {}^*F$, $\varphi_F(x) \subseteq {}^*F$.*

Proof: If $\varphi_F(x) \subseteq {}^*F$ for each $x \in {}^*F$, then it is true that $^*F$ is a *neighborhood of each of its points, so $^*F$ is *open and F is open.

Suppose that F is open. Then $\{X, F\}$ is a finite open covering of X and $^*\{X, F\} = \{{}^*X, {}^*F\}$ is filled by φ_F. By the above, $^*F = \bigcup\{P \in \varphi_F : P \subseteq {}^*F\}$. Thus for each point $x \in {}^*F$ there is a set $P \in \varphi_F$ such that $x \in P \subseteq {}^*F$ so $\varphi_F(x)$, which is a subset of each set in φ_F containing x, is a subset of $^*F$.

The previous theorem contrasts with the theorem that $F \subseteq X$ is open iff for each $x \in F$, $\mu(x) \subseteq {}^*F$ in two ways: $\varphi_F(x)$ is internal and x need not be standard.

Lemma 1 *For each collection $\mathfrak{U}$ of open sets in any space X, $G = \bigcap\{{}^*U : U \in \mathfrak{U}\}$ is Q-open.*

Proof: If $G = \emptyset$ we are done, so suppose that $G \neq \emptyset$. Then for each point $z \in G$ and set $U \in \mathfrak{U}$, $\varphi_F(z) \subseteq {}^*U$, so $G = \bigcup\{\varphi_F(z) : z \in G\}$ is the union of a family of *open sets.

Corollary 1 *For any space X and $x \in X$, $\mu(x)$ is Q-open.*

For any topological space $(X, \mathfrak{T})$ and subset A of X, the monad of A is defined to be $\bigcap\{{}^*U : A \subseteq U \in \mathfrak{T}\}$ and is denoted by $\mu(A)$.

Corollary 2 *For any space X and $A \subseteq X$, $\mu(A)$ is Q-open.*

Corollary 3 *For any space X and family $\mathfrak{F}$ of closed subsets of X, $\bigcup\{{}^*F : F \in \mathfrak{F}\}$ is Q-closed. In particular, if X is a T_1-space, then for each $A \subseteq X$, $A = \bigcup\{\{x\} : x \in A\}$ is Q-closed in $^*X$ and if X is infinite, then it is not dense in $^*X$ when $^*X$ is given the Q-topology.*

We hope to discuss Q-topologies in greater detail in a later paper. The following theorem is central to this paper.

Theorem 2 *Let P be a set property which can be expressed formally and which is closed under finite intersections. Then for any topological space $(X, \mathfrak{T})$ and $x \in X$, the following conditions are equivalent:*

(i) *There is an internal *neighborhood $V \subseteq \mu(x)$ of x with property $^*P$.*
(ii) $\mu(x) = \bigcap\{{}^*U : U$ *is a neighborhood of x with property P*$\}$.
(iii) *The neighborhoods of x with property P form a base for the neighborhood filter at x.*

*Moreover, the *neighborhood of x in condition* (i) *can be taken to be *open iff there is a base for the neighborhood filter at x composed of open neighborhoods of x with property P.*

Proof: (i) $\Longrightarrow$ (ii). Suppose that there is an internal *neighborhood of x which is a subset of $\mu(x)$ with property $*P$ and let G be any open neighborhood of x. Then $*G$ contains a *neighborhood of x with property $*P$, so G contains a neighborhood of x with property P. For each open neighborhood G of x, let $V_G \subseteq G$ be a neighborhood of x with property P and let $W_G \subseteq V_G$ be an open neighborhood of x. Then,

$$\mu(x) \subseteq \bigcap\{*W_G : X \in G \in \mathfrak{J}\} \subseteq \bigcap\{*V_G : x \in G \in \mathfrak{J}\} \subseteq \bigcap\{*G: x \in G \in \mathfrak{J}\} = \mu(x).$$

Notice that the assumption that P is closed under finite intersections was not used in this portion of the proof.

(ii) $\Longrightarrow$ (i). Assume that $\mu(x) = \bigcap\{*U\colon U$ is a neighborhood of x with property $P\}$. If $U_1, \ldots, U_n$ is any finite collection of neighborhoods of x with property P, then $U_1 \cap \ldots \cap U_n$ is a neighborhood of x with property P and a subset of each of $U_1, \ldots, U_n$. Hence, there is a *neighborhood V of x with property $*P$ which is a subset of $*U$ for every neighborhood U of x with property P, so $V \subseteq \mu(x) = \bigcap\{*U\colon U$ is a neighborhood of x with property $P\}$.

(i) $\Longrightarrow$ (iii). Suppose that there is a *neighborhood of x which is a subset of $\mu(x)$ with property $*P$. Then for every neighborhood U of x, $*U$ contains a *neighborhood of x with property $*P$ and U contains a neighborhood of x with property P, so the neighborhoods of x with property P form a base for the neighborhood filter.

(iii) $\Longrightarrow$ (i). Suppose that the neighborhoods of x with property P form a base. Then for each finite collection $U_1, \ldots, U_n$ of neighborhoods of x there is a neighborhood of x with property P which is a subset of $U_1 \cap \ldots \cap U_n$, and so is a subset of each of $U_1, \ldots, U_n$. By concurrence, there is a *neighborhood V of x with property $*P$ which is a subset of $*U$ for each neighborhood U of x, so $V \subseteq \mu(x) = \bigcap\{*U\colon U$ is a neighborhood of $x\}$.

Notice that again the closure of P under finite intersections was not used in this portion of the proof.

The further result can be shown by considering the set property P' defined by: a set A has property P' iff it is open and has property P. If P is closed under finite intersections, then so is P'.

One obvious corollary to this theorem is Robinson's theorem that for each topological space X and point $x \in X$ there is an internal *open *neighborhood of x in $\mu(x)$.

We shall say that a regular Hausdorff space is T_3 and that a normal Hausdorff space X is T_4.

Corollary 4 *The following conditions are equivalent for every space X:*

(i) *For each point* $x \in X$ *there is a *open *neighborhood* V *of* x *such that* $\overline{V} \subseteq \mu(x)$.
(ii) *For each point* $x \in X$ *and every *open *neighborhood* $V \subseteq \mu(x)$ *of* x, $\overline{V} \subseteq \mu(x)$.
(iii) *For each point* $x \in X$, $\mu(x)$ *is* Q-*closed.*
(iv) *For each point* $x \in X$, $\mu(x)$ *is a* Q-*zero-set.*
(v) *For each point* $x \in X$, $\mu(x)$ *is* Q-*clopen.*
(vi) X *is regular.*
(vii) *For each point* $x \in X$ *the closed neighborhoods of* x *form a base for the neighborhood filter at* x.

Proof: Here P is the property of being closed. Condition (vii) is a well-known and obvious equivalent to condition (vi). By the previous theorem, (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (vii).

(ii) $\Rightarrow$ (i). We know that for any space X and point $x \in X$, there is a *open *neighborhood $V \subseteq \mu(x)$ of x.

(iii) $\Rightarrow$ (ii). If $\mu(x)$ is Q-closed, then the Q-closure of any *open $V \subseteq \mu(x)$ is also a subset of $\mu(x)$. The Q-closure of an internal set is its *closure, so $\overline{V} \subseteq \mu(x)$ for each *open $V \subseteq \mu(x)$.

(iii) $\Leftrightarrow$ (v). Trivial in the light of the result that $\mu(x)$ is Q-open for any point x of any space X.

(v) $\Rightarrow$ (iv). Let $\mu(x)$ be Q-clopen and define f: $^*X \to \mathcal{R}$ by: $f(z) = 0$ for $z \in \mu(x)$, $f(z) = 1$ otherwise. Then f is continuous and $\mu(x) = f^{-1}(0)$.

(iv) $\Rightarrow$ (iii). Trivial; zero-sets are closed.

If in the previous theorem and its proof we substitute a set A for the point x, then we have another theorem and the following:

Corollary 5 *The following conditions are equivalent for every space* X:

(i) *For each closed subset* A *of* X *there is a *open *neighborhood* V *of* $^*A$ *such that* $\overline{V} \subseteq \mu(A)$.
(ii) *For each closed subset* A *of* X *and every *open *neighborhood* $V \subseteq \mu(A)$ *of* $^*A$, $\overline{V} \subseteq \mu(A)$.
(iii) *For each closed subset* A *of* X, $\mu(A)$ *is* Q-*closed.*
(iv) *For each closed subset* A *of* X, $\mu(A)$ *is a* Q-*zero-set.*
(v) *For each closed subset* A *of* X, $\mu(A)$ *is* Q-*clopen.*
(vi) X *is normal.*
(vii) *For each closed subset* A *of* X *the closed neighborhoods of* A *form a base for the filter of neighborhoods of* A.

We omit the proof, which is essentially identical to the preceding one.

In each of the following corollaries, condition (iii) is known to be equivalent to condition (iv).

Corollary 6 *The following conditions are equivalent for every space* X:

(i) *There is a *zero-set *neighborhood* $V \subseteq \mu(x)$ *of* x *for each point* $x \in X$.
(ii) $\mu(x) = \bigcap\{{}^*U$: *U is a zero-set neighborhood of* $x\}$ *for each point* $x \in X$.
(iii) *X is completely regular.*
(iv) *For each point* $x \in X$ *the zero-set neighborhoods of x form a base for the filter of neighborhoods of x.*

Corollary 7 *The following conditions are equivalent for every space X:*

(i) *There is a *clopen *neighborhood* $V \subseteq \mu(x)$ *of x for each point* $x \in X$.
(ii) $\mu(x) = \bigcap\{{}^*U$: *U is a clopen neighborhood of* $x\}$ *for each point* $x \in X$.
(iii) *X is zero-dimensional.*
(iv) *Each point in X has a neighborhood base of clopen sets.*

Corollary 8 *The following conditions are equivalent for every space X:*

(i) *There is a *precompact (*open) *neighborhood* $V \subseteq \mu(x)$ *for each point* $x \in X$.
(ii) $\mu(x) = \bigcap\{{}^*U$: *U is a precompact (open) neighborhood of* $x\}$ *for each point* $x \in X$.
(iii) *X is locally compact.*
(iv) *Each point in X has a neighborhood base of precompact (open) sets.*

Corollary 9 *The following conditions are equivalent for every space X:*

(i) *There is a *connected *neighborhood* $V \subseteq \mu(x)$ *for each point* $x \in X$.
(ii) $\mu(x) = \bigcap\{{}^*U$: *U is a connected neighborhood of* $x\}$ *for each point* $x \in X$.
(iii) *X is locally connected.*
(iv) *Each point in X has a neighborhood base of connected sets.*

Corollary 10 *The following conditions are equivalent for every space X:*

(i) *There is a *regular open *neighborhood* $V \subseteq \mu(x)$ *for each point* $x \in X$.
(ii) $\mu(x) = \bigcap\{{}^*U$: *U is a regular open neighborhood of* $x\}$ *for each point* $x \in X$.
(iii) *X is semi-regular.*
(iv) *Each point in X has a neighborhood base of regular open sets.*

These corollaries have been particularly helpful in our study of Q-topologies. Let us consider a few examples of how they can be used to prove standard results.

Theorem 3 *Any product of topological spaces is regular iff each factor space is regular.*

Proof: Recall that for any point $x \in \prod_{a \in \mathfrak{A}} X_a$ and each point $z \in {}^*\left(\prod_{a \in \mathfrak{A}} X_a\right)$, $z \in \mu(x)$ iff $z_a \in \mu(x_a)$ for each standard index a. Recall also that every cartesian product of sets is closed iff each set is closed. Now it is trivial that for each point $x \in \prod_{a \in \mathfrak{A}} X_a$, $\mu(x)$ is Q-closed iff $\mu(x_a)$ is Q-closed for each standard index a.

Theorem 4 *The continuous, open and closed image of a regular space is regular.*

Proof: Let X be regular and let $f: X \to Y$ be a continuous, open and closed surjection. By continuity, $f(\mu(x)) \subseteq \mu(f(x))$ for each $x \in X$ so if $V \subseteq \mu(x)$ is a *open *neighborhood of x, then $f(V) \subseteq \mu(f(x))$ is a *open *neighborhood of $f(x)$. Now, $\overline{V} \subseteq \mu(x)$ so $f(\overline{V}) = \overline{f(V)} \subseteq \mu(f(x))$ is a *closed *neighborhood of $f(x)$.

Theorem 5 *Every subspace of a regular space is regular.*

Proof: Given $U \subseteq X$, $\mu_U(x) = \mu_X(x) \cap {}^*U$ for each point $x \in U$ and $\mu_X(x)$ is Q-closed, so $\mu_U(x)$ is Q-closed in the relative topology.

Theorem 6 *Every compact Hausdorff space is normal.*

Proof: Let X be compact Hausdorff and let $A \subseteq X$ be closed. Every neighborhood of A is a neighborhood of each point $x \in A$, so $\mu(A) \supseteq \bigcup\{\mu(x): x \in A\}$. Since A is closed, $\bigcup\{\mu(x): x \in A\} \supseteq {}^*A$. It is known that X is compact Hausdorff iff $\{\mu(x): x \in X\}$ partitions ${}^*X$ and each $\mu(x)$ is Q-open, so $\bigcup\{\mu(x): x \in A\} = {}^*X/\bigcup\{\mu(x): x \notin A\}$ is Q-closed. Now, $\bigcup\{\varphi_F(z): z \in {}^*A\} \subseteq \bigcup\{\varphi_F(z): z \in \bigcup\{\mu(x): x \in A\}\} = \bigcup\{\mu(x): x \in A\}$, so $\overline{\bigcup\{\varphi_F(z): z \in {}^*A\}} \subseteq \mu(A)$ is a *closed *neighborhood of ${}^*A$.

Theorem 7 *The continuous closed image of a normal space is normal.*

Proof: Let X be normal and let $f: X \to Y$ be closed and continuous. For any closed $A \subseteq Y$, $B = f^{-1}(A)$ is closed so there is a *closed *neighborhood $U \subseteq \mu(B)$ of ${}^*B$. By continuity, $f(\mu(B)) \subseteq \mu(f(B))$ so $f(U) \subseteq \mu(A)$ and $f(U)$ is *closed. Note that $V = {}^*Y/(f({}^*X/{}^\circ U)) \subseteq f(U)$ is a *open *neighborhood of ${}^*A$ and $\overline{V} \subset f(U)$, so Y is normal.

Theorem 8 *Suppose that any space X has the hereditary property P_1 iff for each point $x \in X$, $\mu(x)$ contains a ${}^*P_2$ *neighborhood of x. Suppose further that for any spaces X and Y and $A \times B \subseteq X \times Y$, $A \times B$ has property P_2 whenever A and B have property P_2, and that every space X has property P_2. Then an arbitrary product space has property P_1 iff each factor space has property P_1.*

In our proof it might be considered an abuse of notation to write ${}^*\left(\prod_{a \in \mathfrak{A}} X_a\right) = \prod_{a \in {}^*\mathfrak{A}} {}^*X_a$. No confusion should result, however, and the proof will be simplified. We will denote the projection of $V \subseteq \prod_{a \in \mathfrak{A}} X_a$ onto X_b by $\rho_b(V)$.

Proof: For each $a \in \mathfrak{A}$, X_a is homeomorphic to a subspace of $\prod_{a \in \mathfrak{A}} X_a$, so if $\prod_{a \in \mathfrak{A}} X_a$ has property P_1, then each factor space has property P_1.

Suppose now that for each index $a \in \mathfrak{A}$ and point $x_a \in X_a$, $\mu(x_a)$ contains a ${}^*P_2$ *neighborhood U_a of x_a. For any point $x \in \prod_{a \in \mathfrak{A}} X_a$ and neighborhood V of

x, $\rho_b(V)$ differs from X_b on a finite collection of indices which we shall call $\mathfrak{B}$. Define $U \subseteq {}^*\left(\prod_{a \in \mathfrak{A}} X_a\right)$ as the product with factor U_a for $a \in \mathfrak{B}$ and ${}^*X_a$ for $a \in {}^*\mathfrak{A}/\mathfrak{B}$. Then $U = \prod_{a \in \mathfrak{B}} U_a \times \prod_{a \in {}^*\mathfrak{A}/\mathfrak{B}} {}^*X_a$ is a ${}^*P_2$ *neighborhood of x and a subset of ${}^*V$, so there must be a neighborhood of x which is a subset of V with property P_2. By Theorem 2 we are done.

We mention regularity in the following corollary to show the scope of this theorem.

Corollary 11 *An arbitrary topological product is regular (respectively completely regular, zero-dimensional) iff each factor space is regular (respectively completely regular, zero-dimensional).*

Proof: Regularity (respectively complete regularity, zero-dimensionality) is hereditary. A finite product of sets is closed (respectively a zero-set, clopen) if each set is closed (respectively a zero-set, clopen). Every topological space is closed (respectively a zero-set, clopen).

We hope that these examples will make the corollaries to Theorem 2 easier to use. It was shown that any topological space X is regular iff for each point $x \in X$, $\mu(x)$ is Q-closed. This theorem can be improved for Hausdorff spaces by observing that for any Hausdorff space X and $x \in X$, if $z \in \mu(y)$ for some point $y \neq x$, then $\mu(y)$ is a Q-neighborhood of z disjoint from $\mu(x)$. Hence, $\mu(x)$ is Q-closed iff for each point $z \in {}^*X/\left(\bigcup\{\mu(y)\colon y \in X\}\right)$ there is a Q-neighborhood of z disjoint from $\mu(x)$. Notice the order of quantification: $\forall x \forall z \exists U$.

Theorem 9 *A Hausdorff space X is regular iff for each point $z \in {}^*X/\left(\bigcup\{\mu(y)\colon y \in X\}\right)$ there is a *open *neighborhood U of z such that $U \cap \mu(x) = \emptyset$ for each point $x \in X$*, i.e., *iff* $\mathrm{ns}({}^*X) = \bigcup\{\mu(n)\colon n \in X\}$ *is Q-closed.*

Proof: $\Leftarrow$ In the light of the previous comment this is trivial.

$\Rightarrow$ Suppose that X is regular and let $z \in {}^*X/\left(\bigcup\{\mu(y)\colon y \in X\}\right)$ and consider $\varphi_F(z)$, which is a fixed *open *neighborhood of z. For any point $x \in X$, $z \notin \mu(x)$ so there must be a standard neighborhood F of x such that $z \notin {}^*F$. Let $G \subseteq F$ be a closed neighborhood of x. Then $z \notin {}^*G$, $z \in {}^*(X/G)$, $\varphi_F(z) \subseteq {}^*(X/G)$ and ${}^*(X/G) \cap \mu(x) = \emptyset$, so $\varphi_F(z) \cap \mu(x) = \emptyset$.

At this point the order of quantification is $\forall z \exists U \forall x$. Moreover, U can be taken to be a *open *neighborhood of z, not just Q-open. We also point out that since $\mu(x)$ is Q-open, $\overline{\varphi_F(z)} \cap \mu(x) = \emptyset$ for each point $x \in X$. The same argument, slightly modified, proves the following:

Theorem 10 *A Hausdorff space X is normal iff for each family $\mathfrak{F}$ of closed subsets of X and each point $z \in {}^*X/\left(\bigcup\{\mu(F)\colon F \in \mathfrak{F}\}\right)$ there is a *open *neighborhood U of z such that $U \cap \mu(F) = \emptyset$ for each set $F \in \mathfrak{F}$*, i.e., *iff* $\bigcup\{\mu(F)\colon F \in \mathfrak{F}\}$ *is Q-closed.*

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