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ONE DIMENSION IN PS AND PSI

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It is with regret that the Editor learned of the death of Professor Ivo Thomas on February 2, 1976. This paper was under composition at that time.

PS is $\{CCCpqpp, CCpqCCqrCpr\}$ with substitution and detachment; **PSI** adds Cpp. We find characteristic matrices for their one-dimensional expressions, formed only from C and p. The matrix for **PS** is:

$$\mathfrak{M}1 \qquad \begin{array}{c|cccc} C & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 1 & 1 & 1 \\ {}^{*}1 & 0 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{array}$$

that for **PSI**, $\mathfrak{M}2$, is $\mathfrak{M}1$ without the value 3. We deal explicitly only with **PS**, leaving the reader to adapt the argument to **PSI**.

Equivalence $\mathfrak{M}1$ is such that if α, β are equivalent they have the same value. Since **PS** contains both syllogisms, if α, β are equivalent, $C\varphi\alpha\varphi\beta$ holds. The latter point pervades proof of Theorem I, the former that of Theorem II.

It is obvious that $\mathfrak{M}1$ assigns to p the quadruple (0123) and to each compound α a unique quadruple of 1's and 0's. Such quadruples will be called "the value" of the expression; for compound α "val'α". Six expressions are now listed with their values and a numeral which names the value. If the numeral is prefixed to p, the result names the corresponding expression.

p	(0123)	1
Cpp	(1110)	2
$CCppp$	(0111)	3
$CpCpp$	(1100)	4
$CCCpppp$	(1111)	5
$CCppCpCpp$	(1101)	6

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The following Table records that $Cipjp$ is equivalent to kp $(1 \leqslant i, j \leqslant 6)$, $(2 \leqslant k \leqslant 6)$. Justification of its entries requires the proving of seventy-two implications, of which each turns out to be a substitution in one of twenty-four. Suffice it to say that the shortest is CPP (using capitalized variables for arbitrary implications), the most often used is $CPCQP$, the longest and most specific is $CCCCppCpCppCpCppCpp$ (which follows from $CCCpqpp$, $CCpCpqCpq$, and $CCCprrCCqrCCpqr$).

TABLE

$$
\begin{array}{cc|cccccc}
 & & & & j & & & \\
 & C & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
 & 1 & 2 & 4 & 4 & 4 & 4 & 4 \\
 & 2 & 3 & 5 & 3 & 6 & 5 & 6 \\
i & 3 & 5 & 2 & 5 & 4 & 5 & 6 \\
 & 4 & 3 & 5 & 3 & 5 & 5 & 5 \\
 & 5 & 3 & 2 & 3 & 4 & 5 & 6 \\
 & 6 & 3 & 2 & 3 & 2 & 5 & 5
\end{array}
\quad k
$$

Four theorems follow:

Theorem I *Every compound α is equivalent to some kp* $(2 \leqslant k \leqslant 6)$.

Proof is by induction on the number of C's in α. (i) α contains one C: then α is $2p$, equivalent to itself since CPP is provable. (ii) α is $C\beta\gamma$ with $n + 1$ C's and the theorem holds for all compound expressions with $\leqslant n$ C's. Three subcases arise.

a) β is elementary, γ compound: then β is $1p$, and by the induction hypothesis γ is equivalent to some jp $(2 \leqslant j \leqslant 6)$ so $C\beta\gamma$ is equivalent to $C1pjp$ which is equivalent to $4p$ (Table).
b) β is compound, γ elementary: $C\beta\gamma$ is equivalent to $3p$ or $5p$. Proof is similar to that of a).
c) β, γ are compound: by the induction hypothesis β is equivalent to some ip, γ to some jp $(2 \leqslant i, j \leqslant 6)$ so $C\beta\gamma$ is equivalent to some kp $(2 \leqslant k \leqslant 6)$ (Table). (i), (ii) prove the theorem.

Theorem II *If* val'α *is* 5, *α is equivalent to* $5p$.

Proof: By Theorem I α is equivalent to some kp $(2 \leqslant k \leqslant 6)$ and therefore has the value k. But only for $k = 5$ does kp have the value 5.

Theorem III *If* val'α *is* 5, *α is provable.*

Proof: If val'α is 5, by Theorem II, $C5p\alpha$ is provable, but $5p$ is provable being a substitution in an axiom, so α is provable.

Theorem IV *If α is provable,* val'α *is* 5.

Proof: $\mathfrak{M}1$ is a PS matrix. Theorems III and IV show that $\mathfrak{M}1$ is characteristic for the class of expressions investigated.

Historical Note In [1] Łukasiewicz gave

C	1	2	3	4
*1	1	2	4	4
2	1	1	4	4
3	2	2	x	2
4	1	2	1	1

with $x = 2$ as a **PS** matrix, with $x = 1$ as a **PSI** matrix. The former is not characteristic for one-dimensional expressions since it satisfies **6** p; however the latter can be used to do the work of $\mathfrak{M}1$. It has come to mind that C. A. Meredith in a letter of 8/23/58 gave as **PS** and **PSI** matrices $\mathfrak{M}1$ without the value 2, and $\mathfrak{M}2$. The former suffers from the same disability as Łukasiewicz's, but Meredith's mapping into two values may have sparked the idea underlying $\mathfrak{M}1$.

We remark finally that no finite matrix can be characteristic for either general system.

REFERENCE

[1] Łukasiewicz, J., "W Sprawie Aksjomatyki Implikacyjnego Rachunku Zdań," (Concerning an axiom system of the implicational propositional calculus). *VI Zjazd Matematyków Polskich*, Warszawa, September 20-23, 1948. Supplement to *Annales de la Société Polonaise de Mathématique*, Kraków, vol. 22 (1950), pp. 87-92.

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