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ON NACHBIN'S CHARACTERIZATION OF A BOOLEAN LATTICE

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A classical theorem of L. Nachbin [6] characterizes Boolean lattices as those bounded distributive lattices in which each prime ideal is maximal. This result has been generalized and applied to non-bounded distributive lattices by G. Grätzer and E. T. Schmidt, see [3], especially p. 276. Recently, D. Adams ([1], Theorem 1) has given a version of Nachbin's theorem for bounded non-distributive lattices. The object of this note is to give a transparent alternative proof of Grätzer and Schmidt's generalization and also to establish a theorem akin to that of Adams.

The notation and terminology follows that of [2] and Stone's Theorem ([2], Theorem 15, p. 74) will be used freely. Incidentally, a proof of Nachbin's Theorem is given in [2], Theorem 22, p. 76; it is a simplication (possibly due to boundedness) of the proof in [3]. For elements x and y of a lattice $\mathfrak{L}$, let $\langle x,y\rangle = \{z \in L\colon\ x \wedge z \leqslant y\}$. When L is distributive, $\langle x,y\rangle$ is an ideal. For a detailed account of such ideals, see Mandelker [5].

The following lemma is an extension of [4], Lemma 12.

Lemma 1 *A distributive lattice $\mathfrak{L}$ is relatively complemented if and only if for each $x, y \in L$, $(x] \vee \langle x,y\rangle = L$.*

Proof: Suppose $\mathfrak{L}$ is relatively complemented and x, y, z are in L. Let w be the complement of x in $[x \wedge y \wedge z, x \vee y \vee z]$. Then, $z = z \wedge (x \vee y \vee z) = z \wedge (x \vee w) = (z \wedge x) \vee (z \wedge w)$. Since $z \wedge x \in (x]$ and $z \wedge w \in \langle x, y\rangle$, it follows that $(x] \vee \langle x, y\rangle = L$.

Conversely, suppose the ideal-theoretic condition holds. Let $c \in [a, b]$. Then, $b \in (c] \vee \langle c, a\rangle$ and so $b = c_1 \vee d$ for some $c_1 \leqslant c$ and $d \in L$ such that $c \wedge d \leqslant a$. Then $b = c \vee d$ and $(d \vee a) \wedge b$ is the relative complement of c.

Lemma 2 *The set of prime ideals of a distributive lattice $\mathfrak{L}$ is unordered by set-inclusion if and only if, for each $x, y \in L$, $(x] \vee \langle x, y\rangle = L$.*

Proof: Suppose the set of prime ideals is unordered. If $(x] \vee \langle x, y\rangle \neq L$ then there is a prime ideal P such that $(x] \vee \langle x, y\rangle \subseteq P$. Since the set of prime filters is unordered, $L\backslash P$ is a maximal filter. But $x \notin L\backslash P$. Hence,

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$y \in L = [x) \vee (L \backslash P)$, and so $x \wedge a \leqslant y$ for some $a \in L \backslash P$. Then, $a \in \langle x, y \rangle \subseteq P$ yields a contradiction. Hence, $(x] \vee \langle x, y \rangle = L$.

Suppose $(x] \vee \langle x, y \rangle = L$ for any $x, y \in L$. Let P and Q be prime ideals such that $P \subseteq Q$. If $P \neq Q$ then choose $a \in Q \backslash P$ and $b \in P$. Since $(a] \cap \langle a, b \rangle = (a \wedge b]$, it follows that $\langle a, b \rangle \subseteq P$, whence $L = (a] \vee \langle a, b \rangle \subseteq Q$. This is a contradiction and so $P = Q$.

Theorem 1 (Grätzer and Schmidt [3]) *A distributive lattice is relatively complemented if and only if its set of prime ideals is unordered by set-inclusion.*

The proof of the following lemma is the same as that of [2], Lemma 5, p. 71; see also [7], Lemma 1.

Lemma 3 *Let I and J be ideals of a modular lattice. If $I \cap J$ and $I \vee J$ are principal then so are I and J.*

Theorem 2 *A lattice $\mathfrak{L}$ with 0 is a generalized Boolean lattice if and only if each of the following conditions is satisfied.*

(i) *$\mathfrak{L}$ is modular.*
(ii) *Each ideal $J \neq L$ is contained in a prime ideal.*
(iii) *The set of prime ideals of L is unordered by set-inclusion.*
(iv) *Each filter $F \neq L$ is contained in a prime filter.*

Proof: It is sufficient to prove that (i) - (iv) imply that each initial segment of $\mathfrak{L}$ is a Boolean lattice. Condition (iv) is clearly equivalent to each of the following conditions:

(v) $(0]$ is an intersection of prime ideals.
(vi) For each $x \in L$, $(x]^* = \langle x, 0 \rangle$ is an ideal.

Thus, (ii), (iii) and (iv) imply that $(x] \vee (x]^* = L$ for each $x \in L$, *cf.* the proof of Lemma 1 or Theorem 1 of Adams [1].

Now let $a \in [0, b]$. As $\mathfrak{L}$ is modular, $(b] = (a] \vee ((a]^* \cap (b])$ while $(0] = (a] \cap ((a]^* \cap (b])$. By Lemma 3, there exists $c \in L$ such that $(a]^* \cap (b] = (c]$. It follows that $[0, b]$ is pseudocomplemented and c is the pseudocomplement a^+ of a in $[0, b]$. Also $b = a \vee a^+ = a^{++} \vee a^+$, $a \wedge a^+ = a^{++} \wedge a^+ = 0$, and $a \leqslant a^{++}$. As $\mathfrak{L}$ is modular, $a = a^{++}$. Hence, by Glivenko's Theorem ([2], Theorem 4, p. 58), $[0, b]$ is a Boolean lattice.

As is shown by the five element non-modular lattice, conditions (ii), (iii) and (iv) are independent of (i), while (i), (ii) and (iii) are satisfied by the lattice obtained by adjoining a new largest element to the five element modular non-distributive lattice.

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