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CONSISTENT, INDEPENDENT, AND DISTINCT PROPOSITIONS. II

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This is a continuation of my paper: "Consistent, independent, and distinct propositions," *Notre Dame Journal of Formal Logic*, vol. XIII (1972), pp. 399-406. All page-references to follow are to it. Our motivation there was to construct a system with denumerably many consistent, independent, and distinct propositions. This we did by exhibiting a matrix and defining a system by the matrix. We now present such a system, to be called S11, axiomatically.

S11 is obtained by adding to a Lewis-type formulation of S6 the axiom:

C16 $\Diamond\sim\Diamond\Diamond(p\wedge\sim p)$,

and the two rules:

R5 If $\vdash\Diamond P$ and $\vdash\Diamond\sim\Diamond P$, then $\vdash\Diamond\sim\Diamond\sim\Diamond P$;
R6 If $\vdash\Diamond P$, $\vdash\Diamond Q$, and $\vdash\sim\Diamond(P\wedge Q)$, then $\vdash\Box(\Diamond P\vee\Diamond Q)$.

We first show that S11 is consistent. To do so it suffices to produce a σ-regular S11-matrix with the following additional property: if the premises of **R5** and **R6** are verified by the matrix, so are their respective conclusions. Consider the matrix described in pp. 402-403. Our matrix $\mathfrak{M}$ is obtained from it by the following slight modification: $\mathsf{P}\{2\} = \{2, 3, \ldots\}$, i.e., $\mathsf{P}\{2\} = K$. By Theorem 2, p. 402, $\mathfrak{M}$ is a σ-regular S6-matrix. Next note that *C16* is verified by $\mathfrak{M}$. Now let $\mathsf{P}x \in D$ and $\mathsf{P} - \mathsf{P}x \in D$. Then $x \neq \Lambda$ and $x \neq \{2\}$. Observe that if x contains two or more elements, then $\mathsf{P}x = K$, i.e., $-\mathsf{P}x = \Lambda$, whence $\mathsf{P} - \mathsf{P}x \notin D$. So x is a set containing one element but not $\{2\}$, in which case $\mathsf{P} - \mathsf{P} - \mathsf{P}x \in D$. Next let $\mathsf{P}x \in D$, $\mathsf{P}y \in D$, and $-\mathsf{P}(x \cap y) \in D$. So $x \neq y$. For, if $x = y$, $\mathsf{P}x \in D$, $-\mathsf{P}(x \cap x) = -\mathsf{P}x \in D$. That this cannot happen may be seen by arguing as in p. 404. Also $x \neq \Lambda$, $y \neq \Lambda$. So $\mathsf{P}x \cup \mathsf{P}y = K$. Hence $-\mathsf{P} - (\mathsf{P}x \cup \mathsf{P}y) \in D$.

Note now that for the same reason given in p. 403, **SE** is an E-connective of S11 but not **ME**. We consider next the denumerable list of formulas: P_0, P_1, P_2, . . ., where P_1, P_2, . . . are the same formulas as in p. 403, and P_0 is $\sim\Diamond(p\wedge\sim p)$. Clearly each of these formulas is a

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proposition of S11. We shall show that they are consistent, independent, and distinct in S11, but we may no longer resort to our matrix. We have to deduce the relevant formulas in our axiomatic system. First observe that P_0, P_1, and P_2 are theorems of S11. So by repeated applications of **R5**, P_3, P_4, . . . are also theorems. It follows, as in pp. 402-403, that $\{P_0, P_1, \ldots\}$ is a consistent set of propositions. We show next:

(A) $\vdash \sim P_s \prec P_t$, *for* $s \neq t$; $s, t = 0, 1, \ldots$

Evidently it suffices to consider the case $s < t$. We proceed by induction with respect to s. We have, by S2, $\vdash \sim P_0 \prec P_t$. By the hypothesis of induction, $\vdash \sim P_{s-1} \prec P_{t-1}$. Also, $\vdash \Diamond \sim P_{s-1}$ and $\vdash \Diamond \sim P_{t-1}$. Hence by **R6**, $\vdash \sim \Diamond \sim P_{s-1} \prec \Diamond \sim P_{t-1}$, i.e., $\vdash \sim P_s \prec P_t$. Now, if $\{Q_1, Q_2, \ldots, Q_n\}$ is any finite subset of $\{P_0, P_1, \ldots\}$, then:

Z1	$\sim Q_i \prec Q_j$, *for* $i \neq j$; $i, j = 1, 2, \ldots, n$	[By (A)]
Z2	$\sim Q_i \prec (Q_1 \wedge \ldots \wedge Q_{i-1} \wedge \sim Q_i \wedge Q_{i+1} \wedge \ldots \wedge Q_n)$	[*Z1*; S1°]
Z3	$\Diamond \sim Q_i \prec \Diamond (Q_1 \wedge \ldots \wedge Q_{i-1} \wedge \sim Q_i \wedge Q_{i+1} \wedge \ldots \wedge Q_n)$	[*Z2*; S2°]
Z4	$\Diamond \sim Q_i$	[From above]
Z5	$\Diamond (Q_1 \wedge \ldots \wedge Q_{i-1} \wedge \sim Q_i \wedge Q_{i+1} \wedge \ldots \wedge Q_n)$	[*Z3*; *Z4*; S1°]

So that $\{P_0, P_1, \ldots\}$ is an independent set of propositions. By proceeding as in p. 404, $\{P_0, P_1, \ldots\}$ is a distinct set of propositions. Further, the remarks that follow are also applicable to S11.

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