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A FIRST-ORDER LOGIC OF KNOWLEDGE AND BELIEF WITH IDENTITY. I

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In establishing the semantic completeness of a first-order system, we customarily show how to generate, from a given hypothetically unprovable formula, a set of formulae which provide, in a rather direct way, a countermodel for the given formula. The "model sets" obtained by the completeness procedure make possible a syntactic treatment of semantics.

By generalizing the notion of model set to that of "model system," Hintikka [1] has been able to provide insight into the logic of knowledge and belief. However, his informal approach tends to obscure the underlying semantical assumptions. In this paper, Hintikka's informal, partly syntactic, partly semantic notion of model system is analyzed into the syntactic and semantic components of a formal first-order Gentzen-type system. With the semantics plainly open to view, it appears that some of the difficulties [2] encountered in Hintikka evaporate or are at least to be located elsewhere. In Part II the system is shown to be semantically complete.

1 *Language of* $\mathcal{I}\langle \mathsf{K}, \mathsf{B}\rangle$ The primitive basis of $\mathcal{I}\langle \mathsf{K}, \mathsf{B}\rangle$ consists of the seven improper symbols

$$\mathsf{N}\ \mathsf{C}\ \mathsf{E}\ \mathsf{K}\ \mathsf{B} \rightarrow ,$$

and the following proper symbols:

(1) the 1-ary functional constant P;
(2) the 2-ary functional constant I;
(3) an infinite set F of free individual variables;
(4) an infinite set B of bound individual variables;
(5) an infinite set of propositional variables; and
(6) for each n, an infinite set of n-ary functional variables.

We shall not specify the contents of these sets. However, as usual, we shall assume that they are pairwise disjoint, that no improper symbol or functional constant of $\mathcal{I}\langle \mathsf{K}, \mathsf{B}\rangle$ belongs to any of them, that each is well-ordered (alphabetically), and that membership in each is effectively

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decidable. We shall use a, a', a_1, a_1', a_2, a_2', . . . as metalinguistic variables ranging over F, x as a metalinguistic variable ranging over B, and p as a metalinguistic variable ranging over propositional variables. In giving the formation rules of $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ we shall use symbols of $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ as names for themselves and juxtaposition for juxtaposition. In defining the *quasi-formulae* of $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ we simultaneously define some functions on quasi-formulae:

qf1. A propositional variable standing alone is a quasi-formula. $(p)_0 = p$.
qf2. If f is an n-ary functional constant or variable and for each i such that $1 \leqslant i \leqslant n$, α_i is a free or bound individual variable, then $f\alpha_1 \ldots \alpha_n$ is an atomic quasi-formula of index n. $(f\alpha_1 \ldots \alpha_n)_0 = f$; if $1 \leqslant i \leqslant n$, then $(f\alpha_1 \ldots \alpha_n)_i = \alpha_i$.
qf3. If A is a quasi-formula, $\mathsf{N}A$ is a quasi-formula. $(\mathsf{N}A)_0 = \mathsf{N}$; $(\mathsf{N}A)_1 = A$.
qf4. If A_1 and A_2 are quasi-formulae, $\mathsf{C}A_1A_2$ is a quasi-formula. $(\mathsf{C}A_1A_2)_0 = \mathsf{C}$; $(\mathsf{C}A_1A_2)_1 = A_1$; $(\mathsf{C}A_1A_2)_2 = A_2$.
qf5. If A is a quasi-formula, $\mathsf{E}xA$ is a quasi-formula. $(\mathsf{E}xA)_0 = \mathsf{E}$; $(\mathsf{E}xA)_1 = x$; $(\mathsf{E}xA)_2 = A$.
qf6. If A is a quasi-formula and α is a free or bound individual variable, $\mathsf{K}\alpha A$ and $\mathsf{B}\alpha A$ are quasi-formulae. $(\mathsf{K}\alpha A)_0 = \mathsf{K}$; $(\mathsf{B}\alpha A)_0 = \mathsf{B}$; $(\mathsf{K}\alpha A)_1 = (\mathsf{B}\alpha A)_1 = \alpha$; $(\mathsf{K}\alpha A)_2 = (\mathsf{B}\alpha A)_2 = A$.

An object shall be a quasi-formula of $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ iff it can be shown to be by a finite number of applications of (qf1)-(qf6). A similar understanding shall govern all other recursive definitions. Subsequently we shall use A, A', A_1, A_1', A_2, A_2', . . . as metalinguistic variables ranging over quasi-formulae.

Suppose X is a set such that no member of X is a finite sequence of members of X, and let $\sigma = \langle x_1, \ldots, x_n\rangle$ be a finite sequence of members of X. If $x \in X$, then $\sigma * x = \langle x_1, \ldots, x_n, x\rangle$; if $\tau = \langle y_1, \ldots, y_m\rangle$ is a finite sequence of members of X, then $\sigma * \tau = \langle x_1, \ldots, x_n, y_1, \ldots, y_m\rangle$; if s and t are members of X or finite sequences of members of X, then $\sigma * s * t = (\sigma * s) * t$. Now let X be the non-negative integers, and let ξ_0 be the null sequence of members of X. We define $(A)_{\xi_0} = A$, and if ξ is a finite sequence of elements of X, $x \in X$, and $((A)_\xi)_x$ is defined, then $(A)_{\xi * x} = ((A)_\xi)_x$.

The logical symbols $\exists$ ("there exists"), $\forall$("for all"), $\supset$ ("(materially) implies"), $\equiv$ ("if and only if" or "iff"), & ("and"), $\vee$ ("or"), and - ("not") will be used informally to clarify metalinguistic explanations. We define $Q(\xi_1, \xi_2, A) \equiv \{((A)_{\xi_2 * 0} = \mathsf{E})\ \&\ ((A)_{\xi_1} = (A)_{\xi_2 * 1})\ \&\ \exists \xi[(\xi_2 * 2 * \xi = \xi_1)\ \&\ -\exists \xi_3\{\exists \xi_4(\xi_3 * \xi_4 = \xi)\ \&\ ((A)_{\xi_2 * 2 * \xi_3 * 0} = \mathsf{E})\ \&\ ((A)_{\xi_2 * 2 * \xi_3 * 1} = (A)_{\xi_1})\}]\}$. The predicate is obviously functional, and for future use we define $\mathbf{q}(\xi_1, A) = \xi_2$ iff $Q(\xi_1, \xi_2, A)$. A quasi-formula A is a *formula* of $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ iff $\forall \xi_1[((A)_{\xi_1} \in B) \supset \exists \xi_2 Q(\xi_1, \xi_2, A)]$. $A(a/x)$ is defined by $\forall \xi\{[Q(2*\xi, \xi_0, \mathsf{E}xA) \supset ((A(a/x))_\xi = a)]\ \&\ [-Q(2*\xi, \xi_0, \mathsf{E}xA) \supset ((A(a/x))_\xi = (A)_\xi)]\}$. Evidently, if $\mathsf{E}xA$ is a formula, so is $A(a/x)$.

By an *expression* of $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ we shall understand a finite linear array of symbols of $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$; for convenience of exposition we admit the 'array' consisting of no symbols of $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ as the *empty* expression. Empty and non-empty *formula-sequences* of $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ are certain expressions of $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ defined as follows:

fs1. The empty expression is an empty formula-sequence.
fs2. A formula standing alone is a non-empty formula-sequence.
fs3. If Γ is a non-empty formula-sequence and A is a formula, then Γ, A is a non-empty formula-sequence.

We shall use Δ, Γ, Γ_1, Γ_2, . . . as metalinguistic variables ranging over formula-sequences. The value of $,\Gamma$ shall be $,A_1, \ldots, A_n$ if the value of Γ is $A_1, \ldots, A_n$ and the empty expression if the value of Γ is the empty formula-sequence; $\Gamma,$ is defined similarly. If (the value of) Γ is $A_1, \ldots, A_n$, then $|\Gamma| = \{A_1, \ldots, A_n\}$ and $|\mathsf{N}\Gamma| = \{\mathsf{N}A_1, \ldots, \mathsf{N}A_n\}$; if (the value of) Γ is the empty formula-sequence, then $|\Gamma| = |\mathsf{N}\Gamma| = \emptyset$. *Sequents* of $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ are expressions of the form $\Gamma \to \Delta$. We shall use S, S_1, S_2, . . . as metalinguistic variables ranging over sequents. We write $|\Gamma \to \Delta| = |\Gamma| \cup |\mathsf{N}\Delta|$.

2 *Defensibility* Before describing the axioms and rules of inference of $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ we define and relate the notions of defensibility and satisfiability.

If μ is a set of formulae, then $\mathsf{v}(\mu)$ will be the set of free individual variables occurring in the members of μ. A set μ of formulae is a *model set* provided:

m1. If $\mathsf{N}A \in \mu$, then $A \notin \mu$.
m2. If $\mathsf{NN}A \in \mu$, then $A \in \mu$.
m3. If $\mathsf{C}A_1A_2 \in \mu$, then $\mathsf{N}A_1 \in \mu$ or $A_2 \in \mu$.
m4. If $\mathsf{NC}A_1A_2 \in \mu$, then $A_1 \in \mu$ and $\mathsf{N}A_2 \in \mu$.
m5. If $\mathsf{E}xA \in \mu$, then $A(a/x) \in \mu$ and $\mathsf{P}a \in \mu$ for some a.
m6. If $\mathsf{NE}xA \in \mu$ and $\mathsf{P}a \in \mu$, then $\mathsf{N}A(a/x) \in \mu$.
m7. If $a \in \mathsf{v}(\mu)$, then $\mathsf{I}aa \in \mu$.
m8. If f is an n-ary functional constant or variable, $fa_1 \ldots a_n \in \mu$, $\mathsf{I}a_ia \in \mu$, and $1 \leqslant i \leqslant n$, then $fa_1 \ldots a_{i-1}aa_{i+1} \ldots a_n \in \mu$.

Let X be a set. X is countable iff there exists a one-to-one function f defined on X with values in ω, the set of non-negative integers. The union of X is denoted by ΣX, the power set of X by $\mathcal{P}(X)$, and the cardinality of X by $\overline{\overline{X}}$. If Y is a set, the set of functions from X into Y is denoted by Y^X and the cartesian product of X with Y by $X \times Y$; we write $X^1 = X$ and $X^{n+1} = (X^n) \times X$. If f is a function defined on X, the image of X under f is denoted by $f(X)$, and if $Y \in \mathcal{P}(X)$, the restriction of f to Y is denoted by $f|Y$.

If X is a countable set of sets of formulae, $x \in X$, and $R \in (\mathcal{P}(X^2))^{\mathsf{v}(\Sigma X)}$, then $A(x, X, R) \in (\mathcal{P}(X))^{(\mathsf{v}(\Sigma X)^2)}$ is defined by (i) if $\mathsf{I}a_1a_2 \in x$ and $\langle x, y\rangle \in R(a_2)$, then $y \in A(x, X, R)(a_1, a_2)$, and (ii) if $y \in A(x, X, R)(a_1, a)$, $\langle y, z\rangle \in R(a_2)$, and $\mathsf{I}aa_2 \in y$, then $z \in A(x, X, R)(a_1, a_2)$. We let $A(x, X, R)(a) = A(x, X, R)(a, \mathsf{v}(\Sigma X))$.

Let Ω be a countable set of model sets and let R_k and R_b be elements of $(\mathcal{P}(\Omega^2))^{\mathsf{v}(\Sigma\Omega)}$. $\langle \Omega, R_k\rangle$ is a *k-model system* provided, for each $a \in \mathsf{v}(\Sigma\Omega)$:

k1. $R_k(a)$ is reflexive.
k2. If $\mathsf{K}aA \in \mu \in \Omega$ and $\nu \in A(\mu, \Omega, R_k)(a)$, then $A \in \nu$.
k3. If $\mathsf{NK}aA \in \mu \in \Omega$, then for each a_1 such that $\mathsf{I}aa_1 \in \mu$ there exists some $\nu \in \Omega$ such that $\langle \mu, \nu\rangle \in R_k(a_1)$ and $\mathsf{N}A \in \nu$.

$\langle \Omega, R_b \rangle$ is a *b-model system* provided, for each $a \in \mathbf{v}(\Sigma\Omega)$:

b1. If $\mu \in \Omega$, then there exists some $\nu \in \Omega$ such that $\langle \mu, \nu \rangle \in R_b(a)$.
b2. If $\mathsf{B}aA \in \mu \in \Omega$ and $\nu \in A(\mu, \Omega, R_b)(a)$, then $A \in \nu$.
b3. If $\mathsf{NB}aA \in \mu \in \Omega$, then for each a_1 such that $\mathsf{I}aa_1 \in \mu$ there exists some $\nu \in \Omega$ such that $\langle \mu, \nu \rangle \in R_b(a_1)$ and $\mathsf{N}A \in \nu$.

$\langle \Omega, R_k, R_b \rangle$ is a *kb-model system* provided:

kb1. $\langle \Omega, R_k \rangle$ is a k-model system.
kb2. $\langle \Omega, R_b \rangle$ is a b-model system.
kb3. If $a \in \mathbf{v}(\Sigma\Omega)$, then $R_b(a) \subset R_k(a)$.

We shall say that Ω is a k-model system provided there exists a function R_k such that $\langle \Omega, R_k \rangle$ is a k-model system; we shall say that Ω is a b-model system provided there exists a function R_b such that $\langle \Omega, R_b \rangle$ is a b-model system; we shall say that Ω is a kb-model system provided there exist functions R_k and R_b such that $\langle \Omega, R_k, R_b \rangle$ is a kb-model system. A set λ of formulae is *k/b/kb-defensible* iff there exists some k/k/kb-model system Ω such that $\lambda \subset \mu$ for some $\mu \in \Omega$.

3 *Satisfiability* If λ is a set of formulae, $F_n(\lambda)$ is the set of n-ary functional constants and variables occurring in the numbers of λ, and $A(\lambda)$ is the set of propositional variables occurring in the members of λ.

An *interpretation* $\mathfrak{I}(\lambda)$ *of a set* λ *of formulae* is a 10-tuple $\langle X, x_0, Y, \psi, Z, R_k, R_b, \chi, \theta, \phi \rangle$ such that:

i1. X is a non-empty countable set with $x_0 \in X$.
i2. Y is a non-empty countable set with $\psi \in (\mathcal{P}(Y))^X$.
i3. Z is a non-empty countable subset of Y^X.
i4. R_k and R_b are elements of $(\mathcal{P}(X^2))^Z$ such that if $f \in Z$, then

a. $R_k(f)$ is reflexive.
b. $R_b(f) \subset R_k(f)$.
c. If $x \in X$, then there is some $y \in X$ such that $\langle x, y \rangle \in R_b(f)$.

i5. $\chi \in Z^{\mathbf{v}(\lambda)}$.
i6. If $f \in F_n(\lambda)$, then $\theta(f) \in 2^{(Y^n)}$ is such that

a. $\theta(\mathsf{I})(\alpha, \beta) = 1$ iff $\alpha = \beta$.
b. If $g \in Z$, then $\theta(\mathsf{P})(g(x)) = 1$ iff $g(x) \in \psi(x)$.

i7. $\phi \in 2^{A(\lambda) \times X}$.

We are to think of X as a set of possible worlds and of x_0 as the real world. Y is a set of possible individuals; $\psi(x)$ is the subset of these actually existing in x. Z is a set of trans-world personalities; each free individual variable is assigned such a personality by χ. θ assigns to each function symbol (the characteristic function of) an extension in Y; I is assigned identity and P is made to correspond to real individuals. ϕ assigns truth or falsity to primitive statements made in each possible world.

If $x \in X$ and $R \in (\mathcal{P}(X^2))^Z$, then $A(x, X, R) \in (\mathcal{P}(X))^{(Z^2)}$ is defined by (i) if $f_1(x) = f_2(x)$ and $\langle x, y\rangle \in R(f_2)$, then $y \in A(x, X, R)(f_1, f_2)$, and (ii) if $y \in A(x, X, R)(f_1, f)$, $\langle y, z\rangle \in R(f_2)$, and $f(y) = f_2(y)$, then $z \in A(x, X, R)(f_1, f_2)$. We write $A(x, X, R)(f) = A(x, X, R)(f, Z)$. If ξ is not in the domain of a function g, then a function f is said to be an *extension* of g to ξ provided the domain of f consists of the domain of g and the element ξ and the restriction of f to the domain of g is g. If we consider a function $f \in B^A$ as a triple $\langle A, B, \{\langle a, f(a)\rangle | a \in A\}\rangle$, then there is nothing odd about functions defined on $\emptyset$; accordingly we let $\eta_0 = \langle \emptyset, Z, \emptyset\rangle$. We define $W_{\mathcal{J}(\lambda)}$ as follows:

w1. If $A \in \lambda$, then $\langle A, \xi_0, \eta_0, x_0\rangle \in W_{\mathcal{J}(\lambda)}$.
w2. If $\langle A, \xi, \eta, x\rangle \in W_{\mathcal{J}(\lambda)}$, then

a. If $(A)_{\xi*0} = \mathbf{N}$, then $\langle A, \xi*1, \eta, x\rangle \in W_{\mathcal{J}(\lambda)}$.
b. If $(A)_{\xi*0} = \mathbf{C}$, then $\langle A, \xi*1, \eta, x\rangle \in W_{\mathcal{J}(\lambda)}$ and $\langle A, \xi*2, \eta, x\rangle \in W_{\mathcal{J}(\lambda)}$.
c. If $(A)_{\xi*0} = \mathbf{E}$, then $\langle A, \xi*2, \nu, x\rangle \in W_{\mathcal{J}(\lambda)}$ for each extension ν of η to ξ such that $\nu(\xi) \in Z$ and $\nu(\xi)(x) \in \psi(x)$.
d. If $(A)_{\xi*0} = \mathbf{K/B}$, then $\langle A, \xi*2, \eta, y\rangle \in W_{\mathcal{J}(\lambda)}$ for each y such that $y \in A(x, X, R_{k/b})(f)$ for some $f \in Z$ such that $f(x) = \chi((A)_{\xi*1})(x)$ if $(A)_{\xi*1} \in F$ and $f(x) = \eta(\mathbf{q}(\xi*1, A))(x)$ if $(A)_{\xi*1} \in B$.

$V_{\mathcal{J}(\lambda)}$ is defined on a portion of $W_{\mathcal{J}(\lambda)}$ as follows:

v1. If $(A)_{\xi*0}$ is a propositional variable, then $V_{\mathcal{J}(\lambda)}(A, \xi, \eta, x) = \phi((A)_{\xi*0}, x)$.
v2. If $(A)_{\xi*0}$ is an n-ary functional constant or variable, then $V_{\mathcal{J}(\lambda)}(A, \xi, \eta, x) = \theta((A)_{\xi*0})(\alpha_1, \ldots, \alpha_n)$, where $\alpha_i = \chi((A)_{\xi*i})(x)$ if $(A)_{\xi*i} \in F$, and $\alpha_i = \eta(\mathbf{q}(\xi*i, A))(x)$ if $(A)_{\xi*i} \in B$.
v3. If $(A)_{\xi*0} = \mathbf{N}$, then $V_{\mathcal{J}(\lambda)}(A, \xi, \eta, x) = 0/1$ if $V_{\mathcal{J}(\lambda)}(A, \xi*1, \eta, x) = 1/0$.
v4. If $(A)_{\xi*0} = \mathbf{C}$, then $V_{\mathcal{J}(\lambda)}(A, \xi, \eta, x) = 0$ if $V_{\mathcal{J}(\lambda)}(A, \xi*1, \eta, x) = 1$ and $V_{\mathcal{J}(\lambda)}(A, \xi*2, \eta, x) = 0$, and $V_{\mathcal{J}(\lambda)}(A, \xi, \eta, x) = 1$ if $V_{\mathcal{J}(\lambda)}(A, \xi*1, \eta, x) = 0$ or $V_{\mathcal{J}(\lambda)}(A, \xi*2, \eta, x) = 1$.
v5. If $(A)_{\xi*0} = \mathbf{E}$, then $V_{\mathcal{J}(\lambda)}(A, \xi, \eta, x) = 0$ if $V_{\mathcal{J}(\lambda)}(A, \xi*2, \nu, x) = 0$ for each ν such that $\langle A, \xi*2, \nu, x\rangle \in W_{\mathcal{J}(\lambda)}$ and $V_{\mathcal{J}(\lambda)}(A, \xi, \eta, x) = 1$ if $V_{\mathcal{J}(\lambda)}(A, \xi*2, \nu, x) = 1$ for some ν such that $\langle A, \xi*2, \nu, x\rangle \in W_{\mathcal{J}(\lambda)}$.
v6. If $(A)_{\xi*0} = \mathbf{K/B}$, then $V_{\mathcal{J}(\lambda)}(A, \xi, \eta, x) = 0$ if for each $f \in Z$ such that $f(x) = \chi((A)_{\xi*1})(x)$ if $(A)_{\xi*1} \in F$ and $f(x) = \eta(\mathbf{q}(\xi*1, A))(x)$ if $(A)_{\xi*1} \in B$ there exists some y such that $\langle x, y\rangle \in R_{k/b}(f)$ and $V_{\mathcal{J}(\lambda)}(A, \xi*2, \eta, y) = 0$, and $V_{\mathcal{J}(\lambda)}(A, \xi, \eta, x) = 1$ if $V_{\mathcal{J}(\lambda)}(A, \xi*2, \eta, y) = 1$ for each y such that $\langle A, \xi*2, \eta, y\rangle \in W_{\mathcal{J}(\lambda)}$.

We say that $\mathcal{J}(\lambda)$ gives $A \in \lambda$ the *value* $0/1$ as $V_{\mathcal{J}(\lambda)}(A, \xi_0, \eta_0, x_0) = 0/1$. An *interpretation of a formula A* is an interpretation of $\{A\}$. An interpretation $\mathcal{J}(\lambda)$ of a set λ of formulae (*simultaneously*) *satisfies* λ provided $\mathcal{J}(\lambda)$ gives each $A \in \lambda$ the value 1. A set λ of formulae is (*simultaneously*) *satisfiable* if there is an interpretation of λ which (simultaneously) satisfies λ.

4 *Equivalence of Defensibility and Satisfiability* We consider now the relationship between defensibility and satisfiability.

Theorem 1 *If λ is a kb-defensible set of formulae, then λ is satisfiable.*

Proof: We suppose that $\lambda \subset \mu \in \Omega$, where (Ω, R_k, R_b) is a kb-model system. If $x \in \Omega$ and $a \in \mathbf{v}(\Sigma\Omega)$, we correlate with $\langle x, a\rangle$ an object $\zeta(x, a)$ subject to the condition that $\zeta(x, a_1) = \zeta(y, a_2)$ iff $x = y$ and $\mathsf{I}a_1a_2 \in x$. Let $Y = \zeta(\Omega, \mathbf{v}(\Sigma\Omega))$ and define $\psi \in (\mathscr{P}(Y))^{\Omega}$ by $\psi(x) = \{\zeta(x, a) \mid \mathsf{P}a \in x\}$. Z shall be the set of functions $f \in Y^{\Omega}$ such that for some $a \in \mathbf{v}(\Sigma\Omega)$, $f(x) = \zeta(x, a)$ for all $x \in \Omega$. If $f \in Z$ and a is the least a_1 such that $f(x) = \zeta(x, a_1)$ for each $x \in \Omega$, then $S_{k/b}(f) = R_{k/b}(a)$. If $a \in \mathbf{v}(\lambda)$, $\chi(a)(x) = \zeta(x, a)$ for each $x \in \Omega$. Define $\phi \in 2^{A(\lambda)\times\Omega}$ by $\phi(p, x) = 1$ iff $p \in x$. If $f \in F_n(\lambda)$, define $\theta(f) \in 2^{(Y^n)}$ by $\theta(f)(\alpha_1, \ldots, \alpha_n) = 1$ iff there is some $x \in \Omega$ and for each i such that $1 \leqslant i \leqslant n$ some $a_i \in \mathbf{v}(\Sigma\Omega)$ such that $\zeta(x, a_i) = \alpha_i$ and $fa_1 \ldots a_n \in x$.

It is easily verified that $\mathfrak{I}(\lambda) = \langle \Omega, \mu, Y, \psi, Z, S_k, S_b, \chi, \theta, \phi\rangle$ is an interpretation of λ. To simplify the notation we drop the subscript '$\mathfrak{I}(\lambda)$', replacing $W_{\mathfrak{I}(\lambda)}/V_{\mathfrak{I}(\lambda)}$ by W/V. If $\langle A, \xi, \eta, x\rangle \in W$, let $S(A_1, A, \xi, \eta, x)$ iff for each ξ_1, $(A_1)_{\xi_1}$ is a free individual variable such that $\zeta((A_1)_{\xi_1}, x) = \eta(\mathbf{q}(\xi * \xi_1, A))(x)$ if $(A)_{\xi * \xi_1}$ is a bound individual variable such that for all ξ_2, $\mathbf{q}(\xi * \xi_1, A) \neq \xi * \xi_2$, and $(A_1)_{\xi_1}$ is $(A)_{\xi * \xi_1}$ otherwise. Note that if $A \in \lambda$, then $S(A, A, \xi_0, \eta_0, \mu)$. To show that $\mathfrak{I}(\lambda)$ satisfies λ it therefore suffices to prove the following:

Lemma *If* $\langle A, \xi, \eta, x\rangle \in W$, $S(A_1, A, \xi, \eta, x)$, *and* $A_1/\mathsf{N}A_1 \in x$, *then* $V(A, \xi, \eta, x) = 1/0$.

Proof: The proof is by induction on the complexity of $(A)_\xi$. The *complexity* $\mathbf{c}(A)$ of a quasi-formula A is defined as follows: (i) if A is defined by (qf1) or (qf2), $\mathbf{c}(A) = 1$; (ii) $\mathbf{c}(\mathsf{N}A) = \mathbf{c}(A) + 1$, $\mathbf{c}(\mathsf{C}A_1A_2) = \mathbf{c}(A_1) + \mathbf{c}(A_2) + 1$, and if α is a free or bound individual variable, $\mathbf{c}(\mathsf{E}xA) = \mathbf{c}(\mathsf{K}\alpha A) = \mathbf{c}(\mathsf{B}\alpha A) = \mathbf{c}(A) + 1$. We suppose that $\langle A, \xi, \eta, x\rangle \in W$ and $S(A_1, A, \xi, \eta, x)$.

I. $\mathbf{c}((A)_\xi) = 1$.

A. $(A)_\xi = p$. Then $A_1 = p$. If $p \in x$, then $\phi(p, x) = 1$, so $V(A, \xi, \eta, x) = 1$. If $\mathsf{N}p \in x$, then by (m1), $p \notin x$, so $\phi(p, x) = 0$, and therefore $V(A, \xi, \eta, x) = 0$.

B. $(A)_\xi = f\alpha_1 \ldots \alpha_n$, where f is an n-ary functional constant or variable and each α_i is a free or bound individual variable. Then $A_1 = fa_1 \ldots a_n$, where $a_i = \alpha_i$ if $\alpha_i \in F$ and $\zeta(x, a_i) = \eta(\mathbf{q}(\xi * i, A))(x)$ if $\alpha_i \in B$. $V(A, \xi, \eta, x) = \theta(f)(\beta_1, \ldots, \beta_n)$, where $\beta_i = \chi(\alpha_i)(x)$ if $\alpha_i \in F$ and $\beta_i = \eta(\mathbf{q}(\xi * i, A))(x)$ if $\alpha_i \in B$. Therefore $V(A, \xi, \eta, x) = \theta(f)(\zeta(x, a_1), \ldots, \zeta(x, a_n))$. If $A_1 \in x$, then $\theta(f)(\zeta(x, a_1), \ldots, \zeta(x, a_n)) = 1$. Now suppose $\mathsf{N}A_1 \in x$ and $V(A, \xi, \eta, x) = 1$. Then for some $a_1', \ldots, a_n'$ and some $y \in \Omega$ we have $fa_1' \ldots a_n' \in y$ and for each i, $\zeta(y, a_i') = \zeta(x, a_i)$. But then $y = x$ and $\mathsf{I}a_i'a_i \in x$ for each i, so by (m8), $fa_1 \ldots a_n \in x$, contradicting (m1).

II. $\mathbf{c}((A)_\xi) > 1$. We assume that if $\mathbf{c}((A')_{\xi'}) < \mathbf{c}((A)_\xi)$, $\langle A', \xi', \eta', x'\rangle \in W$, $S(A_1', A', \xi', \eta', x')$, and $A_1'/\mathsf{N}A_1' \in x'$, then $V(A', \xi', \eta', x') = 1/0$.

A. $(A)_{\xi * 0} = \mathsf{C}$. Then $\langle A, \xi * 1, \eta, x\rangle \in W$ and $\langle A, \xi * 2, \eta, x\rangle \in W$. We verify easily that $(A_1)_0 = \mathsf{C}$, $S((A_1)_1, A, \xi * 1, \eta, x)$, and $S((A_1)_2, A, \xi * 2, \eta, x)$.

1. Suppose $A_1 \in x$. Then by (m3), $\mathsf{N}(A_1)_1 \in x$ or $(A_1)_2 \in x$. If $\mathsf{N}(A_1)_1 \in x$, then $V(A, \xi * 1, \eta, x) = 0$ by the induction hypothesis, so $V(A, \xi, \eta, x) = 1$. If

$(A_1)_2 \in x$, then $V(A, \xi*2, \eta, x) = 1$ by the induction hypothesis, so $V(A, \xi, \eta, x) = 1$.
2. Suppose $\mathsf{N}A_1 \in x$. Then by (m4), $(A_1)_1 \in x$ and $\mathsf{N}(A_1)_2 \in x$. Thus by the induction hypothesis, $V(A, \xi*1, \eta, x) = 1$ and $V(A, \xi*2, \eta, x) = 0$, so $V(A, \xi, \eta, x) = 0$.

B. $(A)_{\xi*0} = \mathsf{N}$. Then $\langle A, \xi*1, \eta, x\rangle \in W$. We verify easily that $(A_1)_0 = \mathsf{N}$ and $S((A_1)_1, A, \xi*1, \eta, x)$. If $A_1 \in x$, then by the induction hypothesis, $V(A, \xi*1, \eta, x) = 0$, so $V(A, \xi, \eta, x) = 1$. If $\mathsf{N}A_1 \in x$, then by (m2), $(A_1)_1 \in x$; thus by the induction hypothesis, $V(A, \xi*1, \eta, x) = 1$, so $V(A, \xi, \eta, x) = 0$.

C. $(A)_{\xi*0} = \mathsf{E}$. Then $(A_1)_0 = \mathsf{E}$. If $\mathsf{P}a \in x, f(y) = \zeta(y, a)$ for all $y \in \Omega$, and ν is the extension of η to ξ defined by $\nu(\xi) = f$, then $\langle A, \xi*2, \nu, x\rangle \in W$ and $S((A_1)_2(a/x), A, \xi*2, \nu, x)$.

1. Suppose $A_1 \in x$. Then by (m5), $(A_1)_2(a/x) \in x$ and $\mathsf{P}a \in x$ for some a. Thus if ν is as above, $V(A, \xi*2, \nu, x) = 1$ by the induction hypothesis, so $V(A, \xi, \eta, x) = 1$.
2. Suppose $\mathsf{N}A_1 \in x$. If $\psi(x) = \emptyset$, then $V(A, \xi, \eta, x) = 0$. Otherwise, by (m6), $\mathsf{N}(A_1)_2(a/x) \in x$ for each a such that $\mathsf{P}a \in x$. Thus if ν is as above, $V(A, \xi*2, \nu, x) = 0$ by the induction hypothesis, so $V(A, \xi, \eta, x) = 0$.

D. $(A)_{\xi*0} = \mathsf{K/B}$. Then $(A_1)_0 = \mathsf{K/B}$. Let $f \in Z$ be such that $f(x) = \chi((A)_{\xi*1})(x)$ if $(A)_{\xi*1} \in F$ and $f(x) = \eta(\mathsf{q}(\xi*1, A))(x)$ if $(A)_{\xi*1} \in B$, and suppose $S_{k/b}(f) = R_{k/b}(a)$. Then $f(z) = \zeta(z, a)$ for all $z \in \Omega$. If $(A)_{\xi*1} \in F$, then $(A_1)_1 = (A)_{\xi*1}$, so $\zeta(x, (A_1)_1) = \chi((A_1)_1)(x) = \chi((A)_{\xi*1})(x) = f(x) = \zeta(x, a)$; if $(A)_{\xi*1} \in B$, then $\zeta(x, (A_1)_1) = \eta(\mathsf{q}(\xi*1, A))(x) = f(x) = \zeta(x, a)$. Therefore $\mathsf{I}(A_1)_1 a \in x$.

1. Suppose $A_1 \in x$. If $\langle A, \xi*2, \eta, y\rangle \in W$, then $S((A_1)_2, A, \xi*2, \eta, y)$ and $y \in A(x, \Omega, S_{k/b})(f)$ for some f as above. Then by (m8), $y \in A(x, \Omega, R_{k/b})((A_1)_1)$, so by (k2)/(b2), $(A_1)_2 \in y$. Thus by the induction hypothesis, $V(A, \xi*2, \eta, y) = 1$, so $V(A, \xi, \eta, x) = 1$.
2. Suppose $\mathsf{N}A_1 \in x$. By (k3)/(b3), there exists some $y \in \Omega$ such that $\langle x, y\rangle \in R_{k/b}(a)$ and $\mathsf{N}(A_1)_2 \in y$ for each a such that $\mathsf{I}(A_1)_1 a \in x$. If f is as above, $y \in A(x, \Omega, S_{k/b})(f)$, so $\langle A, \xi*2, \eta, y\rangle \in W$. But $S((A_1)_2, A, \xi*2, \eta, y)$, so $V(A, \xi*2, \eta, y) = 0$ by the induction hypothesis. Thus $V(A, \xi, \eta, x) = 0$.Q.E.D.

<u>Theorem 2</u> *<u>If $\mathcal{I}(\lambda)$</u> = $\langle X, x_0, Y, \psi, Z, R_k, R_b, \chi, \theta, \phi\rangle$ satisfies λ and $\overline{Z - \chi(\mathsf{v}(\lambda))} \leqslant \overline{F - \mathsf{v}(\lambda)}$, then λ is kb-defensible.*

Proof: If $f \in \chi(\mathsf{v}(\lambda))$, let $\mathsf{w}(f) = \{a \in \mathsf{v}(\lambda) \mid f = \chi(a)\}$. If $f \in Z - \chi(\mathsf{v}(\lambda))$, define $\mathsf{w}(f) = \{a\}$, where a is the first member of $F - \mathsf{v}(\lambda)$ not already assigned. Note that if $a \in \mathsf{w}(f)$ and $a \in \mathsf{w}(g)$, then $f = g$.

As before, we omit the subscript '$\mathcal{I}(\lambda)$'. If $\langle A, \xi, \eta, x\rangle \in W$, let $F_{\mathsf{w}}(A_1, A, \xi, \eta, x)$ if for each ξ_1, (1) if $(A)_\xi$ is atomic (and $\xi_1 \neq 0$), then $(A_1)_{\xi_1} \in \mathsf{w}(f)$ for some $f \in Z$ such that $f(x) = \chi((A)_{\xi*\xi_1})(x)$ if $(A)_{\xi*\xi_1} \in F$ and $f(x) = \eta(\mathsf{q}(\xi*\xi_1, A))(x)$ if $(A)_{\xi*\xi_1} \in B$, and (2) otherwise, (a) if $(A)_{\xi*\xi_1}$ is a bound individual variable such that for all ξ_2, $\mathsf{q}(\xi*\xi_1, A) \neq \xi*\xi_2$, then (i) $(A_1)_{\xi_1} \in \mathsf{w}(\eta(\mathsf{q}(\xi*\xi_1, A)))$ and (ii) for each ξ_2, if $(A)_{\xi*\xi_2}$ is a bound individual variable such that $\mathsf{q}(\xi*\xi_1, A) = \mathsf{q}(\xi*\xi_2, A)$, then $(A_1)_{\xi_1} = (A_1)_{\xi_2}$, and (b) otherwise, $(A_1)_{\xi_1} = (A)_{\xi*\xi_1}$.

If $x \in X$, define $\mu(x) = \{A_1 | \exists A \exists \xi \exists \eta [F_w(A_1, A, \xi, \eta, x) \,\&\, (V(A, \xi, \eta, x) = 1)]\} \cup \{\mathbf{N}A_1 | \exists A \exists \xi \exists \eta [F_w(A_1, A, \xi, \eta, x) \,\&\, (V(A, \xi, \eta, x) = 0)]\} \cup \{\mathsf{P}a | \exists f [(f \in Z) \,\&\, (a \in \mathbf{w}(f)) \,\&\, (f(x) \in \psi(x))]\} \cup \{\mathsf{I}a_1a_2 | \exists f_1 \exists f_2 [(f_1 \in Z) \,\&\, (f_2 \in Z) \,\&\, (a_1 \in \mathbf{w}(f_1)) \,\&\, (a_2 \in \mathbf{w}(f_2)) \,\&\, (f_1(x) = f_2(x))]\}$. Let $\Omega = \mu(X)$; if $a \in \mathbf{v}(\Sigma\Omega)$, define $S_k(a)$ and $S_b(a)$ on Ω by $\langle \mu(x), \mu(y) \rangle \in S_{k/b}(a)$ iff $\langle x, y \rangle \in R_{k/b}(f)$, where $a \in \mathbf{w}(f)$. We claim that $\langle \Omega, S_k, S_b \rangle$ is a kb-model system.

(k1), (b1), and (kb3) follow easily from the corresponding properties of R_k and R_b.

(k2)/(b2). Suppose $\mathbf{K}aA_1/\mathbf{B}aA_1 \in \mu(x)$ and $\mu(y) \in A(\mu(x), \Omega, S_{k/b})(a)$. There exist A, ξ, and η such that $\langle A, \xi, \eta, x \rangle \in W$, $V(A, \xi, \eta, x) = 1$, and $F_w(\mathbf{K}aA_1, A, \xi, \eta, x)/F_w(\mathbf{B}aA_1, A, \xi, \eta, x)$. If $a \in \mathbf{w}(f)$, then $y \in A(x, X, R_{k/b})(f)$. Since $(A)_{\xi*0} = \mathbf{K}/\mathbf{B}$, we therefore have $\langle A, \xi*2, \eta, y \rangle \in W$, $V(A, \xi*2, \eta, y) = 1$, and $F_w(A_1, A, \xi*2, \eta, y)$; so $A_1 \in \mu(y)$.

Remark: If $\mathbf{N}A_1 \in \mu(x)$, then there exist A, ξ, and η such that $\langle A, \xi, \eta, x \rangle \in W$, $V(A, \xi, \eta, x) = 0$, and $F_w(A_1, A, \xi, \eta, x)$. For suppose $\langle A, \xi, \eta, x \rangle \in W$, $V(A, \xi, \eta, x) = 1$, and $F_w(\mathbf{N}A_1, A, \xi, \eta, x)$. Then $(A)_{\xi*0} = \mathbf{N}$, so $(A, \xi*1, \eta, x) \in W$, $V(A, \xi*1, \eta, x) = 0$, and $F_w(A_1, A, \xi*1, \eta, x)$.

(k3)/(b3). Suppose $\mathbf{NK}a_1A_1/\mathbf{NB}a_1A_1 \in \mu(x)$ and $\mathsf{I}a_1a_2 \in \mu(x)$. Then either (i) $a_1 \in \mathbf{w}(f_1)$ and $a_2 \in \mathbf{w}(f_2)$ for some $f_1 \in Z$ and $f_2 \in Z$ such that $f_1(x) = f_2(x)$, or (ii) there exist A, ξ, and η such that $\langle A, \xi, \eta, x \rangle \in W$, $V(A, \xi, \eta, x) = 1$, and $F_w(\mathsf{I}a_1a_2, A, \xi, \eta, x)$. In case (ii), $(A)_{\xi*0} = \mathsf{I}$, so $V(A, \xi, \eta, x) = \theta(\mathsf{I})(\alpha_1, \alpha_2)$, where $\alpha_i = \chi((A)_{\xi*i})(x)$ if $(A)_{\xi*i} \in F$ and $\alpha_i = \eta(\mathbf{q}(\xi*i, A))(x)$ if $(A)_{\xi*i} \in B$; $V(A, \xi, \eta, x) = 1$ implies $\alpha_1 = \alpha_2$; $F_w(\mathsf{I}a_1a_2, A, \xi, \eta, x)$ implies there exist functions $f_i \in Z$ such that $a_i \in \mathbf{w}(f_i)$ and $f_i(x) = \alpha_i$. Thus case (ii) collapses into case (i). By the Remark, there exist A, ξ, and η such that $\langle A, \xi, \eta, x \rangle \in W$, $V(A, \xi, \eta, x) = 0$, and $F_w(\mathbf{K}a_1A_1, A, \xi, \eta, x)/F_w(\mathbf{B}a_1A_1, A, \xi, \eta, x)$. $F_w(\mathbf{K}a_1A_1, A, \xi, \eta, x)/F_w(\mathbf{B}a_1A_1, A, \xi, \eta, x)$ implies $(A)_{\xi*0} = \mathbf{K}/\mathbf{B}$, $f_1(x) = \chi((A)_{\xi*1})(x)$ if $(A)_{\xi*1} \in F$, and $f_1(x) = \eta(\mathbf{q}(\xi*1, A))(x)$ if $(A)_{\xi*1} \in B$. Therefore $V(A, \xi, \eta, x) = 0$ implies $V(A, \xi*2, \eta, y) = 0$ for some $y \in X$ such that $\langle x, y \rangle \in R_{k/b}(f_2)$. Then $\langle \mu(x), \mu(y) \rangle \in S_{k/b}(a_2)$ and $F_w(A_1, A, \xi*2, \eta, y)$, so $\mathbf{N}A_1 \in \mu(y)$.

(m2). Suppose $\mathbf{NN}A_1 \in \mu(x)$. By the Remark, there exist A, ξ, and η such that $\langle A, \xi, \eta, x \rangle \in W$, $V(A, \xi, \eta, x) = 0$, and $F_w(\mathbf{N}A_1, A, \xi, \eta, x)$. But since $(A)_{\xi*0} = \mathbf{N}$, we have $\langle A, \xi*1, \eta, x \rangle \in W$, $V(A, \xi*1, \eta, x) = 1$, and $F_w(A_1, A, \xi*1, \eta, x)$. Therefore $A_1 \in \mu(x)$.

(m3). Suppose $\mathbf{C}A_1A_2 \in \mu(x)$. There exist A, ξ, and η such that $\langle A, \xi, \eta, x \rangle \in W$, $V(A, \xi, \eta, x) = 1$, and $F_w(\mathbf{C}A_1A_2, A, \xi, \eta, x)$. Since $(A)_{\xi*0} = \mathbf{C}$, $\langle A, \xi*1, \eta, x \rangle \in W$, $\langle A, \xi*2, \eta, x \rangle \in W$, $V(A, \xi*1, \eta, x) = 0$ or $V(A, \xi*2, \eta, x) = 1$, $F_w(A_1, A, \xi*1, \eta, x)$, and $F_w(A_2, A, \xi*2, \eta, x)$. Therefore $\mathbf{N}A_1 \in \mu(x)$ or $A_2 \in \mu(x)$.

(m4). Suppose $\mathbf{NC}A_1A_2 \in \mu(x)$. By the Remark, there exist A, ξ, and η such that $\langle A, \xi, \eta, x \rangle \in W$, $V(A, \xi, \eta, x) = 0$, and $F_w(\mathbf{C}A_1A_2, A, \xi, \eta, x)$. Since $(A)_{\xi*0} = \mathbf{C}$, $\langle A, \xi*1, \eta, x \rangle \in W$, $\langle A, \xi*2, \eta, x \rangle \in W$, $V(A, \xi*1, \eta, x) = 1$, $V(A, \xi*2, \eta, x) = 0$, $F_w(A_1, A, \xi*1, \eta, x)$, and $F_w(A_2, A, \xi*2, \eta, x)$. Therefore $A_1 \in \mu(x)$ and $\mathbf{N}A_2 \in \mu(x)$.

(m5). Suppose $\mathsf{E}xA_1 \in \mu(x)$. There exist A, ξ, and η such that $\langle A, \xi, \eta, x\rangle \in W$, $V(A, \xi, \eta, x) = 1$, and $F_{\mathsf{w}}(\mathsf{E}xA_1, A, \xi, \eta, x)$. Since $(A)_{\xi*0} = \mathsf{E}$ and $V(A, \xi, \eta, x) = 1$, we have $V(A, \xi*2, \nu, x) = 1$ for some extension ν of η to ξ such that $\nu(\xi) \in Z$ and $\nu(\xi)(x) \in \psi(x)$. If $a \in \mathbf{w}(\nu(\xi))$, then $\mathsf{P}a \in \mu(x)$ and $F_{\mathsf{w}}(A_1(a/x), A, \xi*2, \nu, x)$, so $A_1(a/x) \in \mu(x)$.

(m6). Suppose $\mathsf{NE}xA_1 \in \mu(x)$ and $\mathsf{P}a \in \mu(x)$. By the Remark, there exist A, ξ, and η such that $\langle A, \xi, \eta, x\rangle \in W$, $V(A, \xi, \eta, x) = 0$, and $F_{\mathsf{w}}(\mathsf{E}xA_1, A, \xi, \eta, x)$. Suppose there exist A', ξ', and η' such that $\langle A', \xi', \eta', x\rangle \in W$, $V(A', \xi', \eta', x) = 1$, and $F_{\mathsf{w}}(\mathsf{P}a, A', \xi', \eta', x)$. Since $(A')_{\xi'*0} = \mathsf{P}$, $V(A', \xi', \eta', x) = \theta(\mathsf{P})(\alpha)$, where $\alpha = \chi((A')_{\xi'*1})(x)$ if $(A')_{\xi'*1} \in F$ and $\alpha = \eta'(\mathbf{q}(\xi'*1, A'))(x)$ if $(A')_{\xi'*1} \in B$. $V(A', \xi', \eta', x) = 1$ implies $\alpha \in \psi(x)$. Since $F_{\mathsf{w}}(\mathsf{P}a, A', \xi', \eta', x)$, $a \in \mathbf{w}(f)$ for some $f \in Z$ such that $f(x) = \alpha$. Thus if ν is the extension of η to ξ defined by $\nu(\xi) = f$, $\nu(\xi)(x) \in \psi(x)$. Therefore $\langle A, \xi*2, \nu, x\rangle \in W$, $V(A, \xi*2, \nu, x) = 0$, and $F_{\mathsf{w}}(A_1(a/x), A, \xi*2, \nu, x)$. Consequently, $\mathsf{N}A_1(a/x) \in \mu(x)$.

(m7). Suppose $a \in \mathbf{v}(\mu(x))$. Then if $a \in \mathbf{w}(f)$, $f(x) = f(x)$, so $\mathsf{I}aa \in \mu(x)$.

(m8). Suppose $\mathsf{I}a_1a_2 \in \mu(x)$. As in the proof of (k3)/(b3), there exist functions $f_i \in Z$ such that $a_i \in \mathbf{w}(f_i)$ and $f_1(x) = f_2(x)$.

1. Suppose $\mathsf{P}a_1 \in \mu(x)$. As in the proof of (m6), there is an $f \in Z$ such that $a_1 \in \mathbf{w}(f)$ and $f(x) \in \psi(x)$. Thus $f = f_1$, so $f_2(x) \in \psi(x)$ and $\mathsf{P}a_2 \in \mu(x)$.
2. Suppose $\mathsf{I}a_1'a_2' \in \mu(x)$, where $a_1' = a_1$. As in the proof of (k3)/(b3), there exist $f_i' \in Z$ such that $a_i' \in \mathbf{w}(f_i')$ and $f_1'(x) = f_2'(x)$. Thus $f_1' = f_1$, so $f_2'(x) = f_1'(x) = f_1(x) = f_2(x)$, and so $\mathsf{I}a_2a_2' \in \mu(x)$. The case where $a_2' = a_1$ is similar.
3. Suppose $fa_1' \ldots a_n' \in \mu(x)$, where $f \neq \mathsf{P}$, $f \neq \mathsf{I}$, and $a_k' = a_1$ for some k such that $1 \leqslant k \leqslant n$. There exist A, ξ, and η such that $\langle A, \xi, \eta, x\rangle \in W$, $V(A, \xi, \eta, x) = 1$, and $F_{\mathsf{w}}(fa_1' \ldots a_n', A, \xi, \eta, x)$. $(A)_{\xi*0} = f$, so $V(A, \xi, \eta, x) = \theta(f)(\alpha_1, \ldots, \alpha_n)$, where $\alpha_i = \chi((A)_{\xi*i})(x)$ if $(A)_{\xi*i} \in F$ and $\alpha_i = \eta(\mathbf{q}(\xi*i, A))(x)$ if $(A)_{\xi*i} \in B$. $F_{\mathsf{w}}(fa_1' \ldots a_n', A, \xi, \eta, x)$ implies there exist $f_i' \in Z$ such that $a_i' \in \mathbf{w}(f_i')$ and $f_i'(x) = \alpha_i$. Thus $f_1 = f_k'$, so $f_2(x) = f_1(x) = f_k'(x) = \alpha_k$. Therefore $F_{\mathsf{w}}(fa_1' \ldots a_{k-1}'a_2a_{k+1}' \ldots a_n', A, \xi, \eta, x)$, so $fa_1' \ldots a_{k-1}'a_2a_{k+1}' \ldots a_n' \in \mu(x)$.

(m1). Suppose $A_1 \in \mu(x)$ and $\mathsf{N}A_1 \in \mu(x)$. The reductio is by induction on the complexity of A_1.

1. $\mathbf{c}(A_1) = 1$.

a. $A_1 = p$. There exist A, A', ξ, ξ', η, and η' such that $\langle A, \xi, \eta, x\rangle \in W$, $\langle A', \xi', \eta', x\rangle \in W$, $V(A, \xi, \eta, x) = 1$, $V(A', \xi', \eta', x) = 0$, $F_{\mathsf{w}}(p, A, \xi, \eta, x)$, and $F_{\mathsf{w}}(p, A', \xi', \eta', x)$. But since $(A)_\xi = (A')_{\xi'} = p$, $V(A, \xi, \eta, x) = V(A', \xi', \eta', x) = \phi(p, x)$, RAA.
b. $A_1 = \mathsf{P}a$. As in the proof of (m6), there is an $f \in Z$ such that $a \in \mathbf{w}(f)$ and $f(x) \in \psi(x)$. By the Remark, there exist A, ξ, and η such that $\langle A, \xi, \eta, x\rangle \in W$, $V(A, \xi, \eta, x) = 0$, and $F_{\mathsf{w}}(\mathsf{P}a, A, \xi, \eta, x)$. Since $(A)_{\xi*0} = \mathsf{P}$, $V(A, \xi, \eta, x) = \theta(\mathsf{P})(\alpha)$, where $\alpha = \chi((A)_{\xi*1})(x)$ if $(A)_{\xi*1} \in F$ and $\alpha = \eta(\mathbf{q}(\xi*1, A))(x)$ if $(A)_{\xi*1} \in B$. $F_{\mathsf{w}}(\mathsf{P}a, A, \xi, \eta, x)$ implies there exists some $g \in Z$ such that $a \in \mathbf{w}(g)$ and $g(x) = \alpha$. Thus $g = f$. But $f(x) \in \psi(x)$ implies $\theta(\mathsf{P})(\alpha) = 1$, RAA.
c. $A_1 = \mathsf{I}a_1a_2$. As in the proof of (k3)/(b3), there exists $f_i \in Z$ such that

$a_i \in \mathsf{w}(f_i)$ and $f_1(x) = f_2(x)$. By the Remark, there exist A, ξ, and η such that $\langle A, \xi, \eta, x\rangle \in W$, $V(A, \xi, \eta, x) = 0$, and $F_{\mathsf{w}}(\mathsf{I}a_1a_2, A, \xi, \eta, x)$. $(A)_{\xi*0} = \mathsf{I}$, so $V(A, \xi, \eta, x) = \theta(\mathsf{I})(\alpha_1, \alpha_2)$, where $\alpha_i = \chi((A)_{\xi*i})(x)$ if $(A)_{\xi*i} \in F$ and $\alpha_i = \eta(\mathsf{q}(\xi*i, A))(x)$ if $(A)_{\xi*i} \in B$. $F_{\mathsf{w}}(\mathsf{I}a_1a_2, A, \xi, \eta, x)$ implies $f_i(x) = \alpha_i$. But then $\theta(\mathsf{I})(\alpha_1, \alpha_2) = 1$, RAA.

d. $A_1 = fa_1 \ldots a_n$, where $f \neq \mathsf{P}$ and $f \neq \mathsf{I}$. There exist A, A', ξ, ξ', η, and η' such that $\langle A, \xi, \eta, x\rangle \in W$, $\langle A', \xi', \eta', x\rangle \in W$, $V(A, \xi, \eta, x) = 1$, $V(A', \xi', \eta', x) = 0$, $F_{\mathsf{w}}(fa_1 \ldots a_n, A, \xi, \eta, x)$, and $F_{\mathsf{w}}(fa_1 \ldots a_n, A', \xi', \eta', x)$. Since $(A)_{\xi*0} = f$, $V(A, \xi, \eta, x) = \theta(f)(\alpha_1, \ldots, \alpha_n)$, where $\alpha_i = \chi((A)_{\xi*i})(x)$ if $(A)_{\xi*i} \in F$ and $\alpha_i = \eta(\mathsf{q}(\xi*i, A))(x)$ if $(A)_{\xi*i} \in B$. Since $F_{\mathsf{w}}(fa_1 \ldots a_n, A, \xi, \eta, x)$, there exist $f_i \in Z$ such that $a_i \in \mathsf{w}(f_i)$ and $f_i(x) = \alpha_i$. Since $(A')_{\xi'*0} = f$, $V(A', \xi', \eta', x) = \theta(f)(\beta_1, \ldots, \beta_n)$, where $\beta_i = \chi((A')_{\xi'*i})(x)$ if $(A')_{\xi'*n} \in F$ and $\beta_i = \eta'(\mathsf{q}(\xi'*i, A'))(x)$ if $(A')_{\xi'*i} \in B$. $F_{\mathsf{w}}(fa_1 \ldots a_n, A', \xi', \eta', x)$ implies there exist $f_i' \in Z$ such that $a_i \in \mathsf{w}(f_i')$ and $f_i'(x) = \beta_i$. Therefore $f_i = f_i'$, so $\alpha_i = \beta_i$. But then $V(A, \xi, \eta, x) = V(A', \xi', \eta', x)$, RAA.

2. $\mathsf{c}(A_1) > 1$. We assume that if $\mathsf{c}(A_2) < \mathsf{c}(A_1)$, then $\mathsf{N}A_2 \in \mu(y)$ implies $A_2 \notin \mu(y)$ for each $y \in X$.

a. $A_1 = \mathsf{N}A_2$. By (m2), $\mathsf{N}A_1 \in \mu(x)$ implies $A_2 \in \mu(x)$. Thus $A_1 \notin \mu(x)$ by the induction hypothesis, RAA.

b. $A_1 = \mathsf{C}A_2A_3$. By (m4), $\mathsf{N}A_1 \in \mu(x)$ implies $A_2 \in \mu(x)$ and $\mathsf{N}A_3 \in \mu(x)$. By (m3), $A_1 \in \mu(x)$ implies $\mathsf{N}A_2 \in \mu(x)$ or $A_3 \in \mu(x)$, neither of which is possible by the induction hypothesis, RAA.

c. $A_1 = \mathsf{E}xA_2$. By (m5), $A_1 \in \mu(x)$ implies $A_2(a/x) \in \mu(x)$ and $\mathsf{P}a \in \mu(x)$ for some a. By (m6), $\mathsf{N}A_1 \in \mu(x)$ implies $\mathsf{N}A_2(a/x) \in \mu(x)$, contradicting the induction hypothesis, RAA.

d. $A_1 = \mathsf{K}aA_2/\mathsf{B}aA_2$. By (k3)/(b3) and (m7), there is some $\mu(y) \in \Omega$ such that $\langle \mu(x), \mu(y)\rangle \in S_{k/b}(a)$ and $\mathsf{N}A_2 \in \mu(y)$. Since $\mu(y) \in A(\mu(x), \Omega, S_{k/b})(a)$, $A_2 \in \mu(y)$ by (k2)/(b2), contradicting the induction hypothesis, RAA. Q.E.D.

5 *Alternative Notion of Defensibility* To facilitate the proofs of the Validity and Completeness Theorems, we introduce here an equivalent notion of defensibility. This concept is also closer formally to that given by Hintikka [1], and we shall conclude this section with a brief comparison.

Let Ω be a countable set of model sets and let R_k and R_b be elements of $(P(\Omega^2))^{\mathsf{v}(\Sigma\Omega)}$. $\langle \Omega, R_k\rangle$ is a *k'-model system* provided:

k'1. If $\mathsf{K}aA \in \mu \in \Omega$, then $A \in \mu$.
k'2. If $\mathsf{K}a_1A \in \mu \in \Omega$ and $\mathsf{I}a_1a_2 \in \mu$, then $\mathsf{K}a_2A \in \mu$.
k'3. If $\mathsf{NK}a_1A \in \mu \in \Omega$ and $\mathsf{I}a_1a_2 \in \mu$, then $\mathsf{NK}a_2A \in \mu$.
k'4. If $\mathsf{K}aA \in \mu \in \Omega$ and $\langle \mu, \nu\rangle \in R_k(a)$, then $\mathsf{K}aA \in \nu$.
k'5. If $\mathsf{NK}aA \in \mu \in \Omega$, then there exist some $\nu \in \Omega$ such that $\langle \mu, \nu\rangle \in R_k(a)$ and $\mathsf{N}A \in \nu$.

$\langle \Omega, R_b\rangle$ is a *b'-model system* provided:

b'1. If $\mathsf{B}aA \in \mu \in \Omega$, then there is some $\nu \in \Omega$ such that $\langle \mu, \nu\rangle \in R_b(a)$.
b'2. If $\mathsf{B}a_1A \in \mu \in \Omega$ and $\mathsf{I}a_1a_2 \in \mu$, then $\mathsf{B}a_2A \in \mu$.

b′3. If $\mathbf{NB}a_1A \in \mu \in \Omega$ and $\mathsf{I}a_1a_2 \in \mu$, then $\mathbf{NB}a_2A \in \mu$.
b′4. If $\mathbf{B}aA \in \mu \in \Omega$ and $\langle \mu, \nu \rangle \in R_b(a)$, then $\mathbf{B}aA \in \nu$.
b′5. If $\mathbf{B}aA \in \mu \in \Omega$ and $\langle \mu, \nu \rangle \in R_b(a)$, then $A \in \nu$.
b′6. If $\mathbf{NB}aA \in \mu \in \Omega$, then there exists some $\nu \in \Omega$ such that $\langle \mu, \nu \rangle \in R_b(a)$ and $\mathbf{N}A \in \nu$.

$\langle \Omega, R_k, R_b \rangle$ is a *kb′-model system* provided:

kb′1. $\langle \Omega, R_k \rangle$ is a k′-model system.
kb′2. $\langle \Omega, R_b \rangle$ is a b′-model system.
kb′3. If $\mathbf{K}aA \in \mu \in \Omega$ and $\langle \mu, \nu \rangle \in R_b(a)$, then $\mathbf{K}aA \in \nu$.

As before we shall sometimes omit reference to the relation functions R_k and R_b and refer to Ω as a k′-, b′-, or kb′-model system. A set λ of formulae is *k′/b′/kb′-defensible* iff there exists a k′/b′/kb′-model system Ω such that $\lambda \subset \mu$ for some $\mu \in \Omega$.

Theorem 3 *If λ is kb′-defensible, then λ is kb-defensible.*

Proof: Suppose $\langle \Omega, R_k, R_b \rangle$ is a kb′-model system such that $\lambda \subset \mu$ for some $\mu \in \Omega$. If $a \in \mathbf{v}(\Sigma\Omega)$, define $S_b(a) = R_b(a) \cup \{\langle \mu, \mu \rangle \mid - \exists A(\mathbf{B}aA \in \mu)\}$ and $S_k(a) = R_k(a) \cup S_b(a) \cup \{\langle \mu, \mu \rangle \mid \mu \in \Omega\}$. We claim that $\langle \Omega, S_k, S_b \rangle$ is a kb-model system.
(k1), (b1), and (kb3) are satisfied by construction (plus (b′1) in the case of (b1)).
(k3)/(b3). Suppose $\mathbf{NK}a_1A/\mathbf{NB}a_1A \in \mu$ and $\mathsf{I}a_1a_2 \in \mu$. By (k′3)/(b′3), $\mathbf{NK}a_2A/\mathbf{NB}a_2A \in \mu$, and by (k′5)/(b′6), there exists some $\nu \in \Omega$ such that $\langle \mu, \nu \rangle \in R_{k/b}(a_2)$ and $\mathbf{N}A \in \nu$. But $R_{k/b}(a_2) \subset S_{k/b}(a_2)$ by construction.
(b2). Suppose $\mathbf{B}aA \in \mu$ and $\nu \in A(\mu, \Omega, S_b)(a)$. Then there are sequences $\langle \eta_0, \ldots, \eta_{n+1} \rangle$ and $\langle a_0, \ldots, a_{n+1} \rangle$ such that $\mu = \eta_0$, $\nu = \eta_{n+1}$, $a = a_0$, and for $0 \leqslant i \leqslant n$, $\mathsf{I}a_ia_{i+1} \in \eta_i$ and $\langle \eta_i, \eta_{i+1} \rangle \in S_b(a_{i+1})$. We show by induction on i that $\mathbf{B}a_{i+1}A \in \eta_i$; we may assume that $\langle \eta_i, \eta_{i+1} \rangle \in R_b(a_{i+1})$. $\mathbf{B}a_0A \in \eta_0$ and $\mathsf{I}a_0a_1 \in \mu$, so by (b′2), $\mathbf{B}a_1A \in \eta_0$. Assume that $\mathbf{B}a_{i+1}A \in \eta_i$. $\langle \eta_i, \eta_{i+1} \rangle \in R_b(a_{i+1})$, so by (b′4), $\mathbf{B}a_{i+1}A \in \eta_{i+1}$. $\mathsf{I}a_{i+1}a_{i+2} \in \eta_{i+1}$, so by (b′2), $\mathbf{B}a_{i+2}A \in \eta_{i+1}$. Therefore $\mathbf{B}a_{n+1}A \in \eta_n$. But $\langle \eta_n, \eta_{n+1} \rangle \in R_b(a_{n+1})$, so by (b′5), $A \in \eta_{n+1} = \nu$.
(k2). Suppose $\mathbf{K}aA \in \mu$ and $\nu \in A(\mu, \Omega, S_k)(a)$. Then there are sequences $\langle \eta_0, \ldots, \eta_{n+1} \rangle$ and $\langle a_0, \ldots, a_{n+1} \rangle$ such that $\mu = \eta_0$, $\nu = \eta_{n+1}$, $a = a_0$, and for $0 \leqslant i \leqslant n$, $\mathsf{I}a_ia_{i+1} \in \eta_i$ and $\langle \eta_i, \eta_{i+1} \rangle \in S_k(a_{i+1})$. We show by induction on i that $\mathbf{K}a_iA \in \eta_i$; we may assume that $\langle \eta_i, \eta_{i+1} \rangle \in R_k(a_{i+1})$ or $\langle \eta_i, \eta_{i+1} \rangle \in R_b(a_{i+1})$. $\mathbf{K}a_0A \in \eta_0$. Assume that $\mathbf{K}a_iA \in \eta_i$. $\mathsf{I}a_ia_{i+1} \in \eta_i$, so by (k′2), $\mathbf{K}a_{i+1}A \in \eta_i$. If $\langle \eta_i, \eta_{i+1} \rangle \in R_k(a_{i+1})$, then $\mathbf{K}a_{i+1}A \in \eta_{i+1}$ by (k′4); if $\langle \eta_i, \eta_{i+1} \rangle \in R_b(a_{i+1})$, then $\mathbf{K}a_{i+1}A \in \eta_{n+1}$ by (kb′3). Therefore $\mathbf{K}a_{i+1}A \in \eta_{n+1}$, so $A \in \eta_{n+1} = \nu$, by (k′1).Q.E.D.

Theorem 4 *If λ is kb-defensible, then λ is kb′-defensible.*

Proof: Suppose (Ω, R_k, R_b) is a kb-model system such that $\lambda \subset \mu$ for some $\mu \in \Omega$. We may assume that if $a \in \mathbf{v}(\Sigma\Omega)$ and $\nu \in \Omega$, then $\mathsf{I}aa \in \nu$.

If X is a set of sets of formulae, $x \in X$, and $R \in (\mathcal{P}(X^2))^{\mathbf{v}(\Sigma\Omega)}$, define $k_0(x, X, R) = \{\mathbf{K}a_2A \mid \exists y \exists a_1[(x \in A(y, X, R)(a_1, a_2)) \ \& \ (\mathbf{K}a_1A \in y)]\}$ and $b_0(x, X, R) = \{\mathbf{B}a_2A \mid \exists y \exists a_1[(x \in A(y, X, R)(a_1, a_2)) \ \& \ (\mathbf{B}a_1A \in y)]\}$. If x is a set of formulae, define $k_1(x) = \{\mathbf{NK}a_2A \mid \exists a_1[(\mathbf{NK}a_1A \in x) \ \& \ (\mathbf{I}a_1a_2 \in x)]\}$ and $b_1(x) = \{\mathbf{NB}a_2A \mid \exists a_1[(\mathbf{NB}a_1A \in x) \ \& \ (\mathbf{I}a_1a_2 \in x)]\} \cup \{\mathbf{B}a_2A \mid \exists a_1[(\mathbf{B}a_1A \in x) \ \& \ (\mathbf{I}a_1a_2 \in x)]\}$.

If $\mu \in \Omega$, let $\mu^0 = \mu \cup k_0(\mu, \Omega, R_k) \cup b_0(\mu, \Omega, R_b)$ and $\mu^1 = \mu^0 \cup k_1(\mu^0) \cup b_1(\mu^0)$. Define $R_k^1(a)$ and $R_b^1(a)$ for $a \in \mathbf{v}(\Sigma\Omega)$ on $\Omega^1 = \{\mu^1 \mid \mu \in \Omega\}$ by $\langle \mu^1, \nu^1 \rangle \in R^1_{k/b}(a)$ iff $\langle \mu, \nu \rangle \in R_{k/b}(a)$. We claim that $\langle \Omega^1, R_k^1, R_b^1 \rangle$ is a kb′-model system.

Remark: $\langle \Omega^1, R_k^1, R_b^1 \rangle$ satisfies (k2) and (b2).

(k2). Suppose $\mathbf{K}aA \in \mu^1$ and $\nu^1 \in A(\mu^1, \Omega^1, R_k^1)(a)$. Then $\mathbf{K}aA \in \mu^0$ and $\nu \in A(\mu, \Omega, R_k)(a)$. Since $\mathbf{I}aa \in \mu$ and $\langle \mu, \mu \rangle \in R_k(a)$, there exist η and a_1 such that $\mu \in A(\eta, \Omega, R_k)(a_1, a)$ and $\mathbf{K}a_1A \in \eta$. But then $\nu \in A(\eta, \Omega, R_k)(a_1)$, so $A \in \nu \subset \nu^1$ by (k2).

(b2). Suppose $\mathbf{B}aA \in \mu^1$ and $\nu^1 \in A(\mu^1, \Omega^1, R_b^1)(a)$. Then $\nu \in A(\mu, \Omega, R_b)(a)$ and since $\mathbf{I}aa \in \mu$, $\mathbf{B}a_1A \in \mu^0$ for some a_1 such that $\mathbf{I}a_1a \in \mu$. If $\mathbf{B}a_1A \in \mu$, then $A \in \nu \subset \nu^1$ by (b2); otherwise there exist η and a_2 such that $\mu \in A(\eta, \Omega, R_b)(a_2, a_1)$ and $\mathbf{B}a_2A \in \eta$, in which case $\nu \in A(\eta, \Omega, R_b)(a_2)$, so $A \in \nu \subset \nu^1$ by (b2).

(k′1). Suppose $\mathbf{K}aA \in \mu^1$. Since $\mathbf{I}aa \in \mu$ and $\langle \mu^1, \mu^1 \rangle \in R_k^1(a)$, $\mu^1 \in A(\mu^1, \Omega^1, R_k^1)(a)$, so $A \in \mu^1$ by the Remark.

(b′1) follows trivially from (b1).

(k′2). Suppose $\mathbf{K}a_1A \in \mu^1$ and $\mathbf{I}a_1a_2 \in \mu^1$. Then $\mathbf{I}a_1a_2 \in \mu$ and $\mathbf{K}a_1A \in \mu^0$. Since $\mathbf{I}a_1a_1 \in \mu$ and $\langle \mu, \mu \rangle \in R_k(a_1)$, there exist η and a_3 such that $\mu \in A(\eta, \Omega, R_k)(a_3, a_1)$ and $\mathbf{K}a_3A \in \eta$. But then $\mu \in A(\eta, \Omega, R_k)(a_3, a_2)$, so $\mathbf{K}a_2A \in \mu^0 \subset \mu^1$.

(b′2). Suppose $\mathbf{B}a_1A \in \mu^1$ and $\mathbf{I}a_1a_2 \in \mu^1$. Then $\mathbf{I}a_1a_2 \in \mu$ and since $\mathbf{I}a_1a_1 \in \mu$, $\mathbf{B}a_3A \in \mu^0$ for some a_3 such that $\mathbf{I}a_3a_1 \in \mu$. But then by (m8), $\mathbf{I}a_3a_2 \in \mu$, so $\mathbf{B}a_2A \in \mu^1$.

(k′3)/(b′3). Suppose $\mathbf{NK}a_1A/\mathbf{NB}a_1A \in \mu^1$ and $\mathbf{I}a_1a_2 \in \mu^1$. Then $\mathbf{I}a_1a_2 \in \mu$. Since $\mathbf{I}a_1a_1 \in \mu$, $\mathbf{NK}a_3A/\mathbf{NB}a_3A \in \mu^0$ for some a_3 such that $\mathbf{I}a_3a_1 \in \mu$; but then by (m8), $\mathbf{I}a_3a_2 \in \mu$, so $\mathbf{NK}a_2A/\mathbf{NB}a_2A \in \mu^1$.

(k′4). Suppose $\mathbf{K}aA \in \mu^1$ and $\langle \mu^1, \nu^1 \rangle \in R_k^1(a)$. Then $\mathbf{K}aA \in \mu^0$ and $\langle \mu, \nu \rangle \in R_k(a)$. Since $\mathbf{I}aa \in \mu$ and $\langle \mu, \mu \rangle \in R_k(a)$, there exist η and a_1 such that $\mu \in A(\eta, \Omega, R_k)(a_1, a)$ and $\mathbf{K}a_1A \in \eta$. But then $\nu \in A(\eta, \Omega, R_k)(a_1, a)$, so $\mathbf{K}aA \in \nu^0 \subset \nu^1$.

(b′4). Suppose $\mathbf{B}aA \in \mu^1$ and $\langle \mu^1, \nu^1 \rangle \in R_b^1(a)$. Then $\langle \mu, \nu \rangle \in R_b(a)$, and since $\mathbf{I}aa \in \mu$, $\mathbf{B}a_1A \in \mu^0$ for some a_1 such that $\mathbf{I}a_1a \in \mu$. If $\mathbf{B}a_1A \in \mu$, then $\nu \in A(\mu, \Omega, R_b)(a_1, a)$, so $\mathbf{B}aA \in \nu^0$. Otherwise there exist η and a_2 such that $\mu \in A(\eta, \Omega, R_b)(a_2, a_1)$ and $\mathbf{B}a_2A \in \eta$, in which case $\nu \in A(\eta, \Omega, R_b)(a_2, a)$, so $\mathbf{B}aA \in \nu^0$.

(b′5). Suppose $\mathbf{B}aA \in \mu^1$ and $\langle \mu^1, \nu^1 \rangle \in R_b^1(a)$. Since $\mathbf{I}aa \in \mu$, $\nu^1 \in A(\mu^1, \Omega^1, R_b^1)(a)$, so $A \in \nu^1$ by the Remark.

(k′5)/(b′6). Suppose $\mathbf{NK}aA/\mathbf{NB}aA \in \mu^1$. Since $\mathbf{I}aa \in \mu$, $\mathbf{NK}a_1A/\mathbf{NB}a_1A \in \mu$ for some a_1 such that $\mathbf{I}a_1a \in \mu$. By (k3)/(b3), there exists some $\nu \in \Omega$ such that $\langle \mu, \nu \rangle \in R_{k/b}(a)$ and $\mathbf{N}A \in \nu$; thus there exists some $\nu^1 \in \Omega^1$ such that $\langle \mu^1, \nu^1 \rangle \in R^1_{k/b}(a)$ and $\mathbf{N}A \in \nu^1$.

(kb′3). Suppose $\mathbf{K}aA \in \mu^1$ and $\langle \mu^1, \nu^1 \rangle \in R_b^1(a)$. Then $\mathbf{K}aA \in \mu^0$ and $\langle \mu, \nu \rangle \in R_b(a)$. By (kb3), $\langle \mu, \nu \rangle \in R_k(a)$, so $\langle \mu^1, \nu^1 \rangle \in R_k^1(a)$ and the result follows by (k′4).

We easily verify that (m2)-(m8) continue to hold for elements of Ω^1.

(m1). The supposition that $A_1 \in \mu^1$ and $\mathbf{N}A_1 \in \mu^1$ leads immediately to a contradiction except where $A_1 = \mathbf{K}aA/\mathbf{B}aA$. Accordingly, suppose that $\mathbf{K}aA/\mathbf{B}aA \in \mu^1$ and $\mathbf{NK}aA/\mathbf{NB}aA \in \mu^1$, and assume (m1) holds for formulae A_1 with

$\mathsf{c}(A_1) < \mathsf{c}(\mathsf{K}aA)/\mathsf{c}(\mathsf{B}aA)$. By (k′5)/(b′6), there exists some ν^1 such that $\langle \mu^1, \nu^1 \rangle \in R^1_{k/b}(a)$ and $\mathsf{N}A \in \nu^1$. But by (k′4), (k′1)/(b′5), $A \in \nu^1$, RAA. Q.E.D.

Let us now compare our notion of kb′-defensibility with that in [1]. Evidently, our notation differs somewhat from Hintikka's. We have written $\mathsf{I}\alpha_1\alpha_2$ for his $\alpha_1 = \alpha_2$, $\mathsf{N}A$ for his $\sim A$, $\mathsf{C}A_1A_2$ for his $A_1 \supset A_2$, $\mathsf{K}\alpha A$ for his $\mathsf{K}_\alpha A$, and $\mathsf{B}\alpha A$ for his $\mathsf{B}_\alpha A$. The 1-ary functional constant P does not appear in [1], but its role is assumed there by formulae of the form $\mathsf{E}x\mathsf{I}ax$, to which $\mathsf{P}a$ is "virtually equivalent". Hintikka's operators **C**, **P**, &, and **U** may be regarded as defined symbols and have accordingly been omitted in $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$.

More significant are the differences in the notions of model set and model system. We have retained (C.KK*) in (k′4) rather than bothering with its qualified form. The more important departure is that (m5) and (m6) replace less general but more complicated conditions dealing with quantifiers. None of Hintikka's rules apply to quantifications over believers or knowers (over 'subscripts', as he puts it), whereas (m5) and (m6) contain no such restriction. Hintikka explicitly rejects ($\mathrm{C.E_0}$) and ($\mathrm{C.U_0}$)—the partial corresponds of (m5) and (m6)—in favor of (108) and (109), then ($\mathrm{C.E_{ep}}$) and ($\mathrm{C.U_{ep}}$), et al. He wishes, he says, to block inferences like $\mathsf{E}x\mathsf{K}a\mathsf{I}xa_1$ from $\mathsf{K}a\mathsf{I}a_1a_1$. However, ($\mathrm{C.E_0}$) and ($\mathrm{C.U_0}$) are entirely adequate for this purpose. It seems rather his desire to read $\mathsf{E}x\mathsf{K}a\mathsf{I}xa_1$ as ⌜a knows who a_1 is⌝ that leads him to reject ($\mathrm{C.E_0}$) and ($\mathrm{C.U_0}$). Of course, it is of interest to know whether the system can handle such locations. However, it seems a tactical error to forego an investigation of how well it supports the conventional reading ⌜there is something—call it x—such that x actually exists and a knows that x is a_1⌝ in favor of attacking a more specialized and probably more difficult problem.

It is easy to verify that the differences are significant. For example, (1) $\mathsf{E}x\mathsf{I}xa_1$ "virtually implies" $\mathsf{E}x\mathsf{K}a\mathsf{I}xa_1$ in $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ but not in [1], while (2) $\mathsf{E}x\mathsf{K}aA$ "virtually implies" $\mathsf{K}a\mathsf{E}xA$ in [1] but not in $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$. It can be argued that the conventional reading of the quantifier supports $\mathcal{J}\langle \mathsf{K}, \mathsf{B}\rangle$ rather than Hintikka here. For (1), consider that one knows of each thing that it is self-identical, whatever else one knows or fails to know of it; if a_1 exists, then there does exist something (namely a_1) known by a to be a_1, since a knows a_1 is a_1. As for (2), imagine that one knows of something through a work of literature one considers fictional, while in fact that thing actually exists (something answers to the concept one has through reading the work); of course, we must assume that fictionality is not part of one's concept of the thing, but this does not seem unreasonable.

Theorems 1-4 show that the semantics of section 3 are implicit in the notion of kb′-defensibility. The universe of discourse is a domain of possible individuals; it is intended that membership be restricted to entities capable of knowing and believing, although this condition could quite easily be liberalized. As usual, the predicates are true or false of these individuals. Essentially, with each possible world x are associated two sets of possible individuals: those of which there is a concept in x (afforded by the use in x of some referring expression) and a subset thereof comprising those actually existing in x. In general, if x and y are distinct possible worlds and r is a referring expression used in both x and y, then the possible individual to which r refers in x will be distinct from the possible individual to which it refers in y. Accordingly, to make the situation in x bear upon the situation in y, each possible individual of which there is a concept in y must be connected to some possible individual of which there exists a concept in x; this is accomplished by making each a 'part' (an 'aspect') of some transworld personality.

The supposed difficulty about 'identifying' individuals across possible worlds which some have found in Hintikka [2] does not appear to arise here. It can be posed as follows: how are the transworld personalities to be constructed, i.e., how are we to decide which possible individual in world x corresponds to a given possible individual in world y? But here it is merely a matter of defining the appropriate function. There may be practical difficulties in making the connections that show a given set of formulae to be satisfiable, but they seem entirely comparable to those encountered in ordinary first-order logic and not to require any excursions into essentialism. While kb$'$-defensibility is not, as we have seen, quite faithful to Hintikka, it seems unlikely that the semantic basis of a completely faithful formalization would introduce any additional difficulties of this sort.

We turn now to some peripheral matters. We have taken account of the shifting of reference involved in moving from one possible world to another by altering the reference of names while keeping that of predicates fixed. That is, a predicate f is assigned an extension in the domain of possible individuals, and the case in which f is true of a in world x and false of a in world y is handled by letting a refer to different possible individuals. An apparently equivalent approach would seem to be to keep fixed the reference of names while changing that of predicates, that is, to define θ on pairs $\langle f, x\rangle$. However, this does not work. A set which ought to be satisfiable (and which is defensible) is $\lambda = \{\mathsf{I}a_1a_2, \mathsf{K}afa_1, \mathsf{NK}afa_2\}$; under the proposed change we would presumably require $\theta(\mathsf{I}, x_0)(\alpha_1, \alpha_2) = 1$ iff $\alpha_1 = \alpha_2$, but in this case there is no interpretation of f which satisfies λ.

The substantive starting point of this investigation has, of course, been kb$'$-defensibility. We have introduced kb-defensibility only because the nonrecursive character of the conditions defining kb$'$-defensibility makes a truth definition difficult. In view of Theorems 3 and 4, kb-defensibility has the same force as kb$'$-defensibility, the rather strange rules (k2), (b2), (k3), and (b3) notwithstanding. However, since the consequent of (k3)/(b3) is not the negation of the consequent of (k2)/(b2), the evaluation function V may be undefined at certain elements of W. I do not know whether kb-defensibility can be formulated in a tidier manner, but it may be of interest to record some formulations which do not work.

1. kb$_1$-defensibility is defined like kb-defensibility, except that (k3)/(b3) is weakened to: if $\mathsf{NK}aA/\mathsf{NB}aA \in \mu$, then there is some $\nu \in A(\mu, \Omega, R_{k/b})(a)$

such that $\mathsf{N}A \in \nu$. The interpretation rules can now be stated more cleanly, and the analogs of Theorems 1, 2, and 3 can be proved. However, if λ is kb$_1$-defensible, then λ need not be kb$'$-defensible. For example, it is easy to verify that $\lambda = \{\mathsf{NK}a_1\mathsf{K}a_2\mathsf{I}a_1a_2, \mathsf{K}a_1\mathsf{I}a_1a_2\}$ is not kb$'$-defensible. But if $\mu_1 = \{\mathsf{NK}a_1\mathsf{K}a_2\mathsf{I}a_1a_2, \mathsf{K}a_1\mathsf{I}a_1a_2, \mathsf{I}a_1a_2, \mathsf{I}a_2a_1, \mathsf{I}a_1a_1, \mathsf{I}a_2a_2\}$, $\mu_2 = \{\mathsf{I}a_1a_1, \mathsf{I}a_2a_2, \mathsf{I}a_3a_3, \mathsf{I}a_1a_2, \mathsf{I}a_1a_3, \mathsf{I}a_2a_1, \mathsf{I}a_2a_3, \mathsf{I}a_3a_1, \mathsf{I}a_3a_2\}$, $\mu_3 = \{\mathsf{NK}a_2\mathsf{I}a_1a_2, \mathsf{I}a_1a_2, \mathsf{I}a_2a_1, \mathsf{I}a_1a_1, \mathsf{I}a_2a_2\}$, $\mu_4 = \{\mathsf{NI}a_1a_2, \mathsf{I}a_1a_1, \mathsf{I}a_2a_2\}$, $\Omega = \{\mu_1, \mu_2, \mu_3, \mu_4\}$, $R_b(a_1) = R_b(a_2) = R_b(a_3) = R_k(a_1) = \{\langle\mu_1, \mu_1\rangle, \langle\mu_2, \mu_2\rangle, \langle\mu_3, \mu_3\rangle, \langle\mu_4, \mu_4\rangle\}$, $R_k(a_2) = R_b(a_2) \cup \{\langle\mu_1, \mu_2\rangle, \langle\mu_3, \mu_4\rangle\}$, and $R_k(a_3) = R_b(a_3) \cup \{\langle\mu_2, \mu_3\rangle\}$, then $\langle\Omega, R_k, R_b\rangle$ is a kb$_1$-model system. Consequently kb$_1$-defensibility is too weak.

2. kb$_2$-defensibility is defined by (k1), (b1), (k$'$2), (b$'$2), (k$'$3), (b$'$3), (k$'$5), (b$'$5), (kb3), plus the following conditions: (a) $R_{k/b}(a)$ is transitive, and (b) if $\mathsf{K}aA/\mathsf{B}aA \in \mu$ and $\langle\mu, \nu\rangle \in R_{k/b}(a)$, then $A \in \nu$. Then the analogs of Theorems 1, 2, and 3 can be proved. However, the presence of (k$'$2), (b$'$2), (k$'$3), and (b$'$3) gives rise to the same aesthetic difficulties in defining interpretations as does kb-defensibility. Furthermore, if λ is kb$_2$-defensible, then λ need not be kb$'$-defensible. For example, it is easy to verify that $\lambda = \{\mathsf{K}a_1 p, \mathsf{NK}a_1\mathsf{CI}a_1a_2\mathsf{K}a_1\mathsf{K}a_2 p\}$ is not kb$'$-defensible. But if $\mu_1 = \{\mathsf{K}a_1 p, p, \mathsf{I}a_1a_1, \mathsf{I}a_2a_2, \mathsf{NK}a_1\mathsf{CI}a_1a_2\mathsf{K}a_1\mathsf{K}a_2 p\}$, $\mu_2 = \{\mathsf{NCI}a_1a_2\mathsf{K}a_1\mathsf{K}a_2 p, p, \mathsf{I}a_1a_2, \mathsf{I}a_2a_1, \mathsf{I}a_1a_1, \mathsf{I}a_2a_2, \mathsf{NK}a_1\mathsf{K}a_2 p, \mathsf{NK}a_2\mathsf{K}a_2 p\}$, $\mu_3 = \{\mathsf{NK}a_2 p, p, \mathsf{I}a_2a_2\}$, $\mu_4 = \{\mathsf{NK}a_2 p, \mathsf{I}a_2a_2\}$, $\mu_5 = \{\mathsf{N}p\}$, $\Omega = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$, $R_b(a_1) = R_b(a_2) = \{\langle\mu_1, \mu_1\rangle, \langle\mu_2, \mu_2\rangle, \langle\mu_3, \mu_3\rangle, \langle\mu_4, \mu_4\rangle, \langle\mu_5, \mu_5\rangle\}$, $R_k(a_1) = R_b(a_1) \cup \{\langle\mu_1, \mu_2\rangle, \langle\mu_2, \mu_3\rangle, \langle\mu_1, \mu_3\rangle\}$, and $R_k(a_2) = R_b(a_2) \cup \{\langle\mu_2, \mu_4\rangle, \langle\mu_4, \mu_5\rangle, \langle\mu_2, \mu_5\rangle, \langle\mu_3, \mu_5\rangle\}$, then $\langle\Omega, R_k, R_b\rangle$ is a kb$_2$-model system. Consequently kb$_2$-defensibility is too weak.

3. kb$_3$-defensibility is defined like kb$_2$-defensibility, except that (k$'$2)/(b$'$2) and (k$'$3)/(b$'$3) are replaced by the following condition: if $\mathsf{I}a_1a_2 \in \mu$ and $\langle\mu, \nu\rangle \in R_{k/b}(a_1)$, then $\langle\mu, \nu\rangle \in R_{k/b}(a_2)$. Then the interpretation rules can be stated more cleanly, and the analogs of Theorems 1, 2, and 4 can be proved. However, if λ is kb$'$-defensible, λ need not be kb$_3$-defensible. For example, it is easy to verify that $\lambda = \{\mathsf{K}a_1 p, \mathsf{I}a_1a_2, \mathsf{NK}a_1\mathsf{K}a_2\mathsf{K}a_1 p\}$ is not kb$_3$-defensible. But if $\mu_1 = \{\mathsf{K}a_1 p, \mathsf{I}a_1a_2, \mathsf{NK}a_1\mathsf{K}a_2\mathsf{K}a_1 p, \mathsf{K}a_2 p, p, \mathsf{I}a_2a_1, \mathsf{I}a_1a_1, \mathsf{I}a_2a_2, \mathsf{NK}a_2\mathsf{K}a_2\mathsf{K}a_1 p\}$, $\mu_2 = \{\mathsf{NK}a_2\mathsf{K}a_1 p, \mathsf{K}a_1 p, \mathsf{K}a_2 p, p, \mathsf{I}a_1a_1, \mathsf{I}a_2a_2\}$, $\mu_3 = \{\mathsf{NK}a_1 p, \mathsf{K}a_2 p, p\}$, $\mu_4 = \{\mathsf{N}p\}$, $\Omega = \{\mu_1, \mu_2, \mu_3, \mu_4\}$, $R_b(a_1) = R_b(a_2) = \{\langle\mu_1, \mu_1\rangle, \langle\mu_2, \mu_2\rangle, \langle\mu_3, \mu_3\rangle, \langle\mu_4, \mu_4\rangle\}$, $R_k(a_1) = R_b(a_1) \cup \{\langle\mu_1, \mu_2\rangle, \langle\mu_3, \mu_4\rangle\}$, and $R_k(a_2) = R_b(a_2) \cup \{\langle\mu_1, \mu_2\rangle, \langle\mu_2, \mu_3\rangle\}$, then $\langle\Omega, R_k, R_b\rangle$ is a kb$'$-model system. Consequently, kb$_3$-defensibility is too strong.

6 *Provability* The *logical axioms* of $\mathcal{J}\langle\mathsf{K}, \mathsf{B}\rangle$ are sequents of the form $\Gamma, A \rightarrow A, \Delta$. The *identity axioms* of $\mathcal{J}\langle\mathsf{K}, \mathsf{B}\rangle$ are sequents of the form $\Gamma \rightarrow \mathsf{I}aa, \Delta$. A sequent is an *axiom* of $\mathcal{J}\langle\mathsf{K}, \mathsf{B}\rangle$ iff it is a logical axiom or an identity axiom. The *rules of inference* of $\mathcal{J}\langle\mathsf{K}, \mathsf{B}\rangle$ are the following:

Propositional rules:

$$\mathrm{N}_0.\quad \frac{\Gamma \rightarrow A, \Delta}{\Gamma, \mathsf{N}A \rightarrow \Delta}$$

$$\mathrm{N}_1.\quad \frac{\Gamma, A \rightarrow \Delta}{\Gamma \rightarrow \mathsf{N}A, \Delta}$$

C_0. $\dfrac{\Gamma \rightarrow A_1, \Delta \quad \Gamma, A_2 \rightarrow \Delta}{\Gamma, \mathsf{C}A_1A_2 \rightarrow \Delta}$

C_1. $\dfrac{\Gamma, A_1 \rightarrow A_2, \Delta}{\Gamma \rightarrow \mathsf{C}A_1A_2, \Delta}$

Quantifier rules:

E_0. $\dfrac{\Gamma, A(a/x), \mathsf{P}a \rightarrow \Delta}{\Gamma, \mathsf{E}xA \rightarrow \Delta}$ if a does not appear in $\Gamma, \mathsf{E}xA \rightarrow \Delta$

E_1. $\dfrac{\Gamma \rightarrow A(a/x), \Delta \quad \Gamma \rightarrow \mathsf{P}a, \Delta}{\Gamma \rightarrow \mathsf{E}xA, \Delta}$

Identity rules:

I_0. $\dfrac{\Gamma, \mathsf{I}a_1a_2, \mathsf{I}a_2a_1 \rightarrow \Delta}{\Gamma, \mathsf{I}a_1a_2 \rightarrow \Delta}$

IF_0. $\dfrac{\Gamma, \mathsf{I}a_ka_k', fa_1 \ldots a_n, fa_1 \ldots a_{k-1}a_k'a_{k+1} \ldots a_n \rightarrow \Delta}{\Gamma, \mathsf{I}a_ka_k', fa_1 \ldots a_n \rightarrow \Delta}$, where f is an n-ary functional constant or variable and $1 \leqslant k \leqslant n$

IF_1. $\dfrac{\Gamma, \mathsf{I}a_ka_k' \rightarrow fa_1 \ldots a_n, fa_1 \ldots a_{k-1}a_k'a_{k+1} \ldots a_n, \Delta}{\Gamma, \mathsf{I}a_ka_k' \rightarrow fa_1 \ldots a_n, \Delta}$, where f is an n-ary functional constant or variable and $1 \leqslant k \leqslant n$

IK_0. $\dfrac{\Gamma, \mathsf{I}a_1a_2, \mathsf{K}a_1A, \mathsf{K}a_2A \rightarrow \Delta}{\Gamma, \mathsf{I}a_1a_2, \mathsf{K}a_1A \rightarrow \Delta}$

IK_1. $\dfrac{\Gamma, \mathsf{I}a_1a_2 \rightarrow \mathsf{K}a_1A, \mathsf{K}a_2A, \Delta}{\Gamma, \mathsf{I}a_1a_2 \rightarrow \mathsf{K}a_1A, \Delta}$

IB_0. $\dfrac{\Gamma, \mathsf{I}a_1a_2, \mathsf{B}a_1A, \mathsf{B}a_2A \rightarrow \Delta}{\Gamma, \mathsf{I}a_1a_2, \mathsf{B}a_1A \rightarrow \Delta}$

IB_1. $\dfrac{\Gamma, \mathsf{I}a_1a_2 \rightarrow \mathsf{B}a_1A, \mathsf{B}a_2A, \Delta}{\Gamma, \mathsf{I}a_1a_2 \rightarrow \mathsf{B}a_1A, \Delta}$

Operator rules:

K_0. $\dfrac{\Gamma, \mathsf{K}aA, A \rightarrow \Delta}{\Gamma, \mathsf{K}aA \rightarrow \Delta}$

K_1. $\dfrac{\mathsf{K}aA_1, \ldots, \mathsf{K}aA_n \rightarrow A}{\mathsf{K}aA_1, \ldots, \mathsf{K}aA_n \rightarrow \mathsf{K}aA}$, where n may be zero

B_0. $\dfrac{\mathsf{K}aA_1, \ldots, \mathsf{K}aA_n, \mathsf{B}aA_1', \ldots, \mathsf{B}aA_m', A_1', \ldots, A_m' \rightarrow}{\mathsf{K}aA_1, \ldots, \mathsf{K}aA_n, \mathsf{B}aA_1', \ldots, \mathsf{B}aA_m' \rightarrow}$, where n may be zero

B_1. $\dfrac{\mathsf{K}aA_1, \ldots, \mathsf{K}aA_n, \mathsf{B}aA_1', \ldots, \mathsf{B}aA_m', A_1', \ldots, A_m' \rightarrow A}{\mathsf{K}aA_1, \ldots, \mathsf{K}aA_n, \mathsf{B}aA_1', \ldots, \mathsf{B}aA_m' \rightarrow \mathsf{B}aA}$, where n or m may be zero

Enabling rules:

T_0. $\dfrac{\Gamma \to \Delta}{\Gamma, A \to \Delta}$

T_1. $\dfrac{\Gamma \to \Delta}{\Gamma \to A, \Delta}$

D_0. $\dfrac{\Gamma, A, A \to \Delta}{\Gamma, A \to \Delta}$

D_1. $\dfrac{\Gamma \to A, A, \Delta}{\Gamma \to A, \Delta}$

R_0. $\dfrac{A_1, \ldots, A_{k-1}, A_{k+1}, A_k, A_{k+2}, \ldots, A_n \to \Delta}{A_1, \ldots, A_n \to \Delta}$, where $n > 1$ and $1 \leqslant k < n$

R_1. $\dfrac{\Gamma \to A_1, \ldots, A_{k-1}, A_{k+1}, A_k, A_{k+2}, \ldots, A_n}{\Gamma \to A_1, \ldots, A_n}$, where $n > 1$ and $1 \leqslant k < n$

A finite sequence $\langle S_1, \ldots, S_n\rangle$ of sequents S_i is a *proof* (*of* S_n) iff for each i such that $1 \leqslant i \leqslant n$, either (1) S_i is an axiom, or (2) there exist integers $j, k < i$ such that S_i is inferred by (C_0) or (E_1) from S_j and S_k, or (3) there exists an integer $j < i$ such that S_i is inferred from S_j by a rule other than (C_0) or (E_1). A sequent is *provable* iff there exists a proof of it.

7 *Validity Theorem* An *interpretation* $\mathcal{I}$ *of a sequent* S is an interpretation of $|S|$; $\mathcal{I}$ gives S the *value* 0/1 as $\mathcal{I}$ does/does not simultaneously satisfy $|S|$. S is *valid* iff $|S|$ is not simultaneously satisfiable, i.e., iff each interpretation of S gives S the value 1.

The main result of this paper is that S is provable iff S is valid. The easier half is proved here.

Theorem 5 *If S is provable, then S is valid.*

Proof: In view of Theorem 2 it suffices to show that if S is provable, then $|S|$ is not kb-defensible (since $\mathbf{v}(|S|)$ is finite, the variable condition is satisfied).

We first verify that if S is an axiom, then $|S|$ is not kb-defensible. If $S = \Gamma, A \to A, \Delta$, then by (m1), $|S|$ cannot be a subset of any model set. If $S = \Gamma \to \mathsf{I}aa, \Delta$, then by (m7), $|S|$ cannot be a subset of any model set.

We now show that if S is inferred from $S_1, \ldots$ by a rule of inference of $\mathcal{J}\langle \mathbf{K}, \mathbf{B}\rangle$ and $|S_1|, \ldots$ are not kb-defensible, then $|S|$ is not kb-defensible.

(N_0). If $S = \Gamma, \mathbf{N}A \to \Delta$ and $S_1 = \Gamma \to A, \Delta$, then $|S| = |S_1|$.
(N_1). Suppose $S = \Gamma \to \mathbf{N}A, \Delta$ and $S_1 = \Gamma, A \to \Delta$. If $|S| \subset \mu$, where μ is a model set, then by (m2), $A \in \mu$, so $|S_1| \subset \mu$.
(C_0). Let $S = \Gamma, \mathbf{C}A_1A_2 \to \Delta$, $S_1 = \Gamma \to A_1, \Delta$, and $S_2 = \Gamma, A_2 \to \Delta$. If $|S| \subset \mu$, where μ is a model set, then by (m3), either $\mathbf{N}A_1 \in \mu$ or $A_2 \in \mu$, so either $|S_1| \subset \mu$ or $|S_2| \subset \mu$.
(C_1). Let $S = \Gamma \to \mathbf{C}A_1A_2, \Delta$ and $S_1 = \Gamma, A_1 \to A_2, \Delta$. If $|S| \subset \mu$, where μ is a model set, then by (m4), $A_1 \in \mu$ and $\mathbf{N}A_2 \in \mu$, so $|S_1| \subset \mu$.

(E_0). Let $S = \Gamma$, $\mathsf{E}xA \rightarrow \Delta$ and $S_1 = \Gamma$, $A(a/x)$, $\mathsf{P}a \rightarrow \Delta$, where a does not appear in S, and suppose $|S|$ is kb-defensible. By Theorem 1, $|S|$ is satisfiable; let $\mathcal{I} = \langle X, x_0, Y, \psi, Z, R_k, R_b, \chi, \theta, \phi\rangle$ be a satisfying interpretation. Let f be the first $g \in Z$ such that if ν is defined on $\{\xi_0\}$ by $\nu(\xi_0) = g$, then $V_{\mathcal{I}}(\mathsf{E}xA, 2, \nu, x_0) = 1$. Since $\mathbf{v}(|S|)$ is finite, the conditions of Theorem 2 are satisfied. If $g \in (\chi(\mathbf{v}(|S|)) - \{f\})$, let $\mathbf{w}'(g) = \mathbf{w}(g)$; if $g \in (Z - (\chi(\mathbf{v}(|S|)) \cup \{f\}))$, let $\mathbf{w}'(g) = \{a_1\}$, where a_1 is the first variable of $F - (\mathbf{v}(|S|) \cup \{a\})$ not already so assigned; if $f \in \chi(\mathbf{v}(|S|))$, let $\mathbf{w}'(f) = \mathbf{w}(f) \cup \{a\}$; and if $f \in (Z - \chi(\mathbf{v}(|S|)))$, let $\mathbf{w}'(f) = \{a\}$. If the proof of Theorem 2 is now carried out with $\mathbf{w}'$ in place of $\mathbf{w}$, we obtain a kb-model system Ω such that $|S_1| \subset \mu(x_0) \in \Omega$.
(E_1). Let $S = \Gamma \rightarrow \mathsf{E}xA$, Δ, $S_1 = \Gamma \rightarrow A(a/x)$, Δ, and $S_2 = \Gamma \rightarrow \mathsf{P}a$, Δ, and suppose that $|S| \subset \mu \in \Omega$, where Ω is a kb-model system. If $\mathsf{P}a \in \mu$, then by (m6), $\mathsf{N}A(a/x) \in \mu$, so $|S_1| \subset \mu$. If $\mathsf{P}a \notin \mu$, then $\mu' = \mu \cup \{\mathsf{N}\mathsf{P}a, \mathsf{I}aa\}$ is a model set, $\Omega' = \{\mu'\} \cup (\Omega - \{\mu\})$ is a kb-model system, and $|S_2| \subset \mu'$.
(I_0). Let $S = \Gamma$, $\mathsf{I}a_1a_2 \rightarrow \Delta$ and $S_1 = \Gamma$, $\mathsf{I}a_1a_2$, $\mathsf{I}a_2a_1 \rightarrow \Delta$. If $|S| \subset \mu$, where μ is a model set, then by (m7), $\mathsf{I}a_1a_1 \in \mu$, so by (m8), $\mathsf{I}a_2a_1 \in \mu$, so $|S_1| \subset \mu$.
(IF_0). Let $S = \Gamma$, $\mathsf{I}a_ka_k'$, $fa_1 \ldots a_n \rightarrow \Delta$ and $S_1 = \Gamma$, $\mathsf{I}a_ka_k'$, $fa_1 \ldots a_n$, $fa_1 \ldots a_{k-1}a_k'a_{k+1} \ldots a_n \rightarrow \Delta$. If $|S| \subset \mu$, where μ is a model set, then by (m8), $fa_1 \ldots a_{k-1}a_k'a_{k+1} \ldots a_n \in \mu$, so $|S_1| \subset \mu$.
(IF_1). Let $S = \Gamma$, $\mathsf{I}a_ka_k' \rightarrow fa_1 \ldots a_n$, Δ and $S_1 = \Gamma$, $\mathsf{I}a_ka_k' \rightarrow fa_1 \ldots a_n$, $fa_1 \ldots a_{k-1}a_k'a_{k+1} \ldots a_n$, Δ. If $|S| \subset \mu$, where μ is a model set, then $fa_1 \ldots a_{k-1}a_k'a_{k+1} \ldots a_n \notin \mu$, since then by (m7) and (m8), $fa_1 \ldots a_n \in \mu$, contradicting (m1). Thus $\mu' = \mu \cup \{\mathsf{N}fa_1 \ldots a_{k-1}a_k'a_{k+1} \ldots a_n\}$ is a model set, and $|S_1| \subset \mu'$.
(IK_0). Let $S = \Gamma$, $\mathsf{I}a_1a_2$, $\mathsf{K}a_1A \rightarrow \Delta$ and $S_1 = \Gamma$, $\mathsf{I}a_1a_2$, $\mathsf{K}a_1A$, $\mathsf{K}a_2A \rightarrow \Delta$. If $|S| \subset \mu \in \Omega$, where Ω is a kb'-model system, then by (k'2), $\mathsf{K}a_2A \in \mu$, so $|S_1| \subset \mu$.
(IK_1). Let $S = \Gamma$, $\mathsf{I}a_1a_2 \rightarrow \mathsf{K}a_1A$, Δ and $S_1 = \Gamma$, $\mathsf{I}a_1a_2 \rightarrow \mathsf{K}a_1A$, $\mathsf{K}a_2A$, Δ. If $|S| \subset \mu \in \Omega$, where Ω is a kb'-model system, then by (k'3), $\mathsf{NK}a_2A \in \mu$, so $|S_1| \subset \mu$.
(IB_0). Let $S = \Gamma$, $\mathsf{I}a_1a_2$, $\mathsf{B}a_1A \rightarrow \Delta$ and $S_1 = \Gamma$, $\mathsf{I}a_1a_2$, $\mathsf{B}a_1A$, $\mathsf{B}a_2A \rightarrow \Delta$. If $|S| \subset \mu \in \Omega$, where Ω is a kb'-model system, then by (b'2), $\mathsf{B}a_2A \in \mu$, so $|S_1| \subset \mu$.
(IB_1). Let $S = \Gamma$, $\mathsf{I}a_1a_2 \rightarrow \mathsf{B}a_1A$, Δ and $S_1 = \Gamma$, $\mathsf{I}a_1a_2 \rightarrow \mathsf{B}a_1A$, $\mathsf{B}a_2A$, Δ. If $|S| \subset \mu \in \Omega$, where Ω is a kb'-model system, then by (b'3), $\mathsf{NB}a_2A \in \mu$, so $|S_1| \subset \mu$.
(K_0). Let $S = \Gamma$, $\mathsf{K}aA \rightarrow \Delta$ and $S_1 = \Gamma$, $\mathsf{K}aA$, $A \rightarrow \Delta$. If $|S| \subset \mu \in \Omega$, where Ω is a kb'-model system, then by (k'1), $A \in \mu$, so $|S_1| \subset \mu$.
(K_1). Let $S = \mathsf{K}aA_1, \ldots, \mathsf{K}aA_n \rightarrow \mathsf{K}aA$ and $S_1 = \mathsf{K}aA_1, \ldots, \mathsf{K}aA_n \rightarrow A$, and suppose $|S| \subset \mu \in \Omega$, where $\langle\Omega, R_k, R_b\rangle$ is a kb'-model system. By (k'5), there is some $\nu \in \Omega$ such that $\langle\mu, \nu\rangle \in R_k(a)$ and $\mathsf{N}A \in \nu$; by (k'4), $\mathsf{K}aA_i \in \nu$ for each i, so $|S_1| \subset \nu$.
(B_0). Let $S = \mathsf{K}aA_1, \ldots, \mathsf{K}aA_n, \mathsf{B}aA_1', \ldots, \mathsf{B}aA_m' \rightarrow$ and $S_1 = \mathsf{K}aA_1, \ldots, \mathsf{K}aA_n, \mathsf{B}aA_1', \ldots, \mathsf{B}aA_m', A_1', \ldots, A_m' \rightarrow$, and suppose $|S| \subset \mu \in \Omega$, where $\langle\Omega, R_k, R_b\rangle$ is a kb'-model system. By (b'1), there is some $\nu \in \Omega$ such that $\langle\mu, \nu\rangle \in R_b(a)$; by (kb'3), $\mathsf{K}aA_i \in \nu$ for each i; by (b'4), $\mathsf{B}aA_i' \in \nu$ for each i; and by (b'5), $A_i' \in \nu$ for each i. Therefore $|S_1| \subset \nu$.
(B_1). Let $S = \mathsf{K}aA_1, \ldots, \mathsf{K}aA_n, \mathsf{B}aA_1', \ldots, \mathsf{B}aA_m' \rightarrow \mathsf{B}aA$ and $S_1 = \mathsf{K}aA_1, \ldots, \mathsf{K}aA_n, \mathsf{B}aA_1', \ldots, \mathsf{B}aA_m', A_1', \ldots, A_m' \rightarrow A$. Suppose $|S| \subset \mu \in \Omega$, where $\langle\Omega, R_k, R_b\rangle$ is a kb'-model system. By (b'6), there is some $\nu \in \Omega$ such that

$\langle \mu, \nu \rangle \in R_b(a)$ and $\mathsf{N}A \in \nu$; by (kb′3), $\mathsf{K}aA_i \in \nu$ for each i; by (b′4), $\mathsf{B}aA_i' \in \nu$ for each i; by (b′5), $A_i' \in \nu$ for each i. Therefore $|S_1| \subset \nu$.
(T_0). If $S = \Gamma, A \rightarrow \Delta$ and $S_1 = \Gamma \rightarrow \Delta$, then $|S_1| \subset |S|$.
(T_1). If $S = \Gamma \rightarrow A, \Delta$ and $S_1 = \Gamma \rightarrow \Delta$, then $|S_1| \subset |S|$.
(D_0). If $S = \Gamma, A \rightarrow \Delta$ and $S_1 = \Gamma, A, A \rightarrow \Delta$, then $|S_1| = |S|$.
(D_1). If $S = \Gamma \rightarrow A, \Delta$ and $S_1 = \Gamma \rightarrow A, A, \Delta$, then $|S_1| = |S|$.
(R_0). If $S = A_1, \ldots, A_n \rightarrow \Delta$ and $S_1 = A_1, \ldots, A_{k-1}, A_{k+1}, A_k, A_{k+2}, \ldots, A_n \rightarrow \Delta$, then $|S_1| = |S|$.
(R_1). If $S = \Gamma \rightarrow A_1, \ldots, A_n$ and $S_1 = \Gamma \rightarrow A_1, \ldots, A_{k-1}, A_{k+1}, A_k, A_{k+2}, \ldots, A_n$, then $|S_1| = |S|$. Q.E.D.

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To be continued

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