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A NOTE ON THE AXIOM OF CHOICE IN LEŚNIEWSKI'S ONTOLOGY

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This paper generalizes the results of [1] and hence a familiarity with [1] is presupposed. In [1] it was shown that the formulae:

$\mathbf{AC}^{\varepsilon}$ $[\exists f]::[Aa]:A\,\varepsilon\,a\,.\supset.\,f\lfloor a\rfloor\;\varepsilon\,a$
ACH $[\exists R]\therefore\rightrightarrows\{R\}:[Aa]:A\,\varepsilon\,a\,.\supset.\,[\exists B]\,.\,B\,\varepsilon\,a\,.\,R\{aB\}$

are inferentially equivalent in the field of Leśniewski's Ontology, and further that they are equivalent to standard forms of the axiom of choice. Both $\mathbf{AC}^{\varepsilon}$ and **ACH** are stated using the primitive epsilon of ontology, a functor belonging to the semantical category S/nn.

Here the equivalence result will be explicitly extended to cover generalizations of these two formulae, stated using so-called higher epsilons, functors analogous to the primitive epsilon but belonging to categories of the form $S/\alpha\alpha$, where α is an arbitrary semantical category.

The paper divides naturally into four parts. Section **1** (**2**) introduces the general form of the definition of the generalized epsilon for nominal (propositional) categories and shows that a thesis having the same structural form as the primitive axiom for Ontology is derivable. Section **3** (**4**) presents the demonstration of the equivalence of $\mathbf{AC}^{\varepsilon}_{\alpha}$ and $\mathbf{ACH}_{\alpha}$, where α is a nominal (propositional) category.

If α is an arbitrary nominal (propositional) category and ϕ is a functor belonging to category α, then $\phi\lfloor\nu_1 \ldots \nu_n\rfloor$ $(\phi\{\nu_1 \ldots \nu_n\})$ will stand for the expression which belongs to the category n(S) that has as its first word the functor ϕ and that contains the variables $\nu_1, \ldots, \nu_n$, in that order. Representing formulae this way greatly simplifies the treatment given to them, since the exact structure of the parenthemes associated with a formula plays no role in the demonstrations to follow. Of course the "proofs" that incorporate such formulae are not proofs at all; rather they are proof schema which allow for the construction of genuine proofs for formulae of any determinate category.

The applicability of the proof schema presupposes the availability of certain definitions and theses. In particular the previous definition of

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functors analogous to equality which take as arguments variables belonging to those categories in use is presupposed. For example, if the expression $\phi[\nu_1 \ldots \nu_n]$ occurs in the course of a demonstration, it is guaranteed that for each of the variables, ν_i, if ν_i belongs to the semantical category β_i, then a definition of the form '$[\mu\theta]$: ________ $.\equiv. \underset{i}{\circ}\{\mu\underset{i}{\theta}\}$' is available, where '$\circ$' in '$\underset{i}{\circ}\{\mu\underset{i}{\theta}\}$' is a functor analogous to equality in the category $S/\beta_i\beta_i$. In addition it will be assumed that theses of the form

$$\textbf{E}.\quad [\varphi\nu_1 \ldots \nu_n\mu_1 \ldots \mu_n] : \varphi\langle\mu_1 \ldots \mu_n\rangle . \underset{1}{\circ}\{\mu_1\underset{1}{\nu_1}\} . \;\ldots\; . \underset{n}{\circ}\{\mu_n\underset{n}{\nu_n}\} .\supset. \varphi\langle\nu_1 \ldots \nu_n\rangle$$

are available whenever necessary.

1 In this section the general form of the definition of the higher epsilons for nominal categories is introduced and it is shown that a formula analogous to the primitive axiom can be derived.

$$D1.1\quad [fg] :: [\exists A\nu_1 \ldots \nu_n] : A\,\varepsilon\, f[\nu_1 \ldots \nu_n] . A\,\varepsilon\, g[\nu_1 \ldots \nu_n] : [B\mu_1 \ldots \mu_n C\eta_1 \ldots \eta_n] : B\,\varepsilon\, f[\mu_1 \ldots \mu_n] . C\,\varepsilon\, f[\eta_1 \ldots \eta_n] .\supset. B\,\varepsilon\, C . \underset{1}{\circ}\{\mu_1\underset{1}{\eta_1}\} . \;\ldots\; . \underset{n}{\circ}\{\mu_n\underset{n}{\eta_n}\} \therefore\equiv. \varepsilon\{fg\}$$

$A1\quad [fg] : \varepsilon\{fg\} .\supset. \varepsilon\{ff\}$ [D1.1]

$A2\quad [fgh] : \varepsilon\{fg\} . \varepsilon\{hf\} .\supset. \varepsilon\{fh\}$

PR $\quad [fgh] :: \mathrm{Hp}(2) .\supset\therefore$

3. $\quad [B\mu_1 \ldots \mu_n C\eta_1 \ldots \eta_n] : B\,\varepsilon\, f[\mu_1 \ldots \mu_n] . C\,\varepsilon f[\eta_1 \ldots \eta_n] .\supset. B\,\varepsilon\, C . \underset{1}{\circ}\{\mu_1\underset{1}{\eta_1}\} . \;\ldots\; . \underset{n}{\circ}\{\mu_n\underset{n}{\eta_n}\} :$ [1, D1.1]

$\quad [\exists A\nu_1 \ldots \nu_n] :$

4. $\quad A\,\varepsilon\, f[\nu_1 \ldots \nu_n] .$

5. $\quad A\,\varepsilon\, h[\nu_1 \ldots \nu_n] \therefore$ [2, D1.1]

$\quad \varepsilon\{fh\}$ [3, 4, 5, D1.1]

$$D1.2\quad [Af\nu_1 \ldots \nu_n] : A\,\varepsilon\, f[\nu_1 \ldots \nu_n] .\equiv. *\{Af\}\langle\nu_1 \ldots \nu_n\rangle$$

$A3$[1] $\quad [\varphi\nu_1 \ldots \nu_n\mu_1 \ldots \mu_n] : \underset{1}{\circ}\{\nu_1\underset{1}{\mu_1}\} . \;\ldots\; . \underset{n}{\circ}\{\nu_n\underset{n}{\mu_n}\} . \varphi\langle\nu_1 \ldots \nu_n\rangle .\supset. \varphi\langle\mu_1 \ldots \mu_n\rangle$

$A4\quad [fgh] : \varepsilon\{fg\} . \varepsilon\{hf\} .\supset \varepsilon\{hg\}$

PR $\quad [fgh] :: \mathrm{Hp}(2) .\supset\therefore$

$\quad [\exists A\nu_1 \ldots \nu_n] \therefore$

3. $\quad A\,\varepsilon\, f[\nu_1 \ldots \nu_n]$

4. $\quad A\,\varepsilon g[\nu_1 \ldots \nu_n] :$

5. $\quad [BC\mu_1 \ldots \mu_n\eta_1 \ldots \eta_n] : B\,\varepsilon\, f[\mu_1 \ldots \mu_n] . C\,\varepsilon\, f[\eta_1 \ldots \eta_n] .\supset. B\,\varepsilon\, C . \underset{1}{\circ}\{\mu_1\underset{1}{\eta_1}\} . \;\ldots\; . \underset{n}{\circ}\{\mu_n\underset{n}{\eta_n}\} :$ [1, D1.1]

$\quad [\exists B\theta_1 \ldots \theta_n] \therefore$

1. This is a formula of type **E**, mentioned in the introductory remarks.

6. $B\,\varepsilon\, h\,[\theta_1 \ldots \theta_n]\,.$
7. $B\,\varepsilon\, f\,[\theta_1 \ldots \theta_n]\,:$
8. $[CD\mu_1 \ldots \mu_n\eta_1 \ldots \eta_n] : C\,\varepsilon\, h\,[\mu_1 \ldots \mu_n]\,.\, D\,\varepsilon\, h\,[\eta_1 \ldots \eta_n]\,.\supset.\, B\,\varepsilon\, C\,.\circ \underset{1}{\{}\mu_1\eta_1\underset{1}{\}}\,.\;\ldots\;.\circ \underset{n}{\{}\mu_n\eta_n\underset{n}{\}}\,:$ [2, *D1.1*]
9. $B\,\varepsilon A\,.$ [3, 7, 5]
10. $B\,\varepsilon\, g\,[\nu_1 \ldots \nu_n]\,.$ [4, 9]
11. $*\text{⋲}Bg\text{⋺}\langle\nu_1 \ldots \nu_n\rangle\,.$ [10, *D1.2*]
12. $*\text{⋲}Bg\text{⋺}\langle\theta_1 \ldots \theta_n\rangle\,.$ [3, 5, 7, 11, *A3*]
13. $B\,\varepsilon\, g\,[\theta_1 \ldots \theta_n]\,\therefore$ [12, *D1.2*]

$\varepsilon\{hg\}$ [6, 8, 13, *D1.1*]

A5 $[fghi] : \varepsilon\{fg\}\,.\,\varepsilon\{hf\}\,.\,\varepsilon\{if\}\,.\supset.\,\varepsilon\{hi\}$

PR $[fghi] :$ Hp(3) $.\supset.$

4. $\varepsilon\{fi\}\,.$ [1, 3, *A2*]

$\varepsilon\{hi\}$ [2, 4, *A4*]

D1.3 $[AB\nu_1 \ldots \nu_n\mu_1 \ldots \mu_n] : A\,\varepsilon B\,.\,B\,\varepsilon A\,.\circ \underset{1}{\{}\nu_1\mu_1\underset{1}{\}}\,.\;\ldots\;.\circ \underset{n}{\{}\nu_n\mu_n\underset{n}{\}}\,.\equiv.$
$A\,\varepsilon\,\phi_\delta\text{⋲}B\nu_1 \ldots \nu_n\text{⋺}\,[\mu_1 \ldots \mu_n]$

A6 $[A\nu_1 \ldots \nu_n] : A\,\varepsilon A\,.\supset.\, A\,\varepsilon\,\phi_\delta\text{⋲}A\nu_1 \ldots \nu_n\text{⋺}\,[\nu_1 \ldots \nu_n]$ [*D1.3*]

A7 $[A\nu_1 \ldots \nu_n B\mu_1 \ldots \mu_n C\eta_1 \ldots \eta_n] : B\,\varepsilon\,\phi_\delta\text{⋲}A\nu_1 \ldots \nu_n\text{⋺}\,[\mu_1 \ldots \mu_n]\,.$
$C\,\varepsilon\,\phi_\delta\text{⋲}A\nu_1 \ldots \nu_n\text{⋺}\,[\eta_1 \ldots \eta_n]\,.\supset.\, B\,\varepsilon\, C\,.\circ \underset{1}{\{}\eta_1\mu_1\underset{1}{\}}\,.\;\ldots\;.\circ \underset{n}{\{}\eta_n\mu_n\underset{n}{\}}$
[*D1.3*]

A8 $[Af\nu_1 \ldots \nu_n] : A\,\varepsilon\, f\,[\nu_1 \ldots \nu_n]\,.\supset.\,\varepsilon\{\phi_\delta\text{⋲}A\nu_1 \ldots \nu_n\text{⋺}f\}$
[*A6, A7, D1.1*]

A9 $[fB\nu_1 \ldots \nu_n C\mu_1 \ldots \mu_n] \therefore [jk] : \varepsilon\{jf\}\,.\,\varepsilon\{kf\}\,.\supset.\,\varepsilon\{jk\} :$
$B\,\varepsilon\, f\,[\nu_1 \ldots \nu_n]\,.\, C\,\varepsilon\, f\,[\mu_1 \ldots \mu_n] :\supset.\, B\,\varepsilon\, C\,.\circ \underset{1}{\{}\nu_1\mu_1\underset{1}{\}}\,.\;\ldots\;.$
$\circ \underset{n}{\{}\nu_n\mu_n\underset{n}{\}}$

PR $[fB\nu_1 \ldots \nu_n C\mu_1 \ldots \mu_n] ::$ Hp(3) $:\supset\therefore$

4. $\varepsilon\{\phi_\delta\text{⋲}B\nu_1 \ldots \nu_n\text{⋺}f\}\,.$ [2, *A8*]
5. $\varepsilon\{\phi_\delta\text{⋲}C\mu_1 \ldots \mu_n\text{⋺}f\}\,.$ [3, *A8*]
6. $\varepsilon\{\phi_\delta\text{⋲}B\nu_1 \ldots \nu_n\text{⋺}\,\phi_\delta\text{⋲}C\mu_1 \ldots \mu_n\text{⋺}\} :$ [1, 4, 5]

$[\exists A\eta_1 \ldots \eta_n] :$

7. $A\,\varepsilon\,\phi_\delta\text{⋲}B\nu_1 \ldots \nu_n\text{⋺}\,[\eta_1 \ldots \eta_n]$
8. $A\,\varepsilon\,\phi_\delta\text{⋲}C\mu_1 \ldots \mu_n\text{⋺}\,[\eta_1 \ldots \eta_n]$ [6, *D1.1*]
9. $A\,\varepsilon\, C\,.$ [8, *D1.3*]
10. $B\,\varepsilon A\,.$ [7, *D1.3*]
11. $\circ \underset{1}{\{}\nu_1\eta_1\underset{1}{\}}\,.\;\ldots\;.\circ \underset{n}{\{}\nu_n\eta_n\underset{n}{\}}\,.$ [7, *D1.3*]
12. $\circ \underset{1}{\{}\mu_1\eta_1\underset{1}{\}}\,.\;\ldots\;.\circ \{\mu_n\eta_n\}\,:$ [8, *D1.3*]
13. $\circ \underset{1}{\{}\nu_1\mu_1\underset{1}{\}}\,.\;\ldots\;.\circ \underset{n}{\{}\nu_n\mu_n\underset{n}{\}}\,.$ [11, 12]

$B\,\varepsilon\, C\,.\circ \underset{1}{\{}\nu_1\mu_1\underset{1}{\}}\,.\;\ldots\;.\circ \underset{n}{\{}\nu_n\mu_n\underset{n}{\}}$ [9, 10, 13]

A10 $[fgh] :: \varepsilon\{hf\} : [i] : \varepsilon\{if\}\,.\supset.\,\varepsilon\{ig\} : [jk] : \varepsilon\{jf\}\,.\,\varepsilon\{kf\}\,.\supset.$
$\varepsilon\{jk\}\,\therefore\supset.\,\varepsilon\{fg\}$

PR $[fgh] \because$ Hp(3) $\therefore\supset::$

4. $\varepsilon\{hg\} \therefore$ [1, 2]

$[\exists A\nu_1 \ldots \nu_n] \therefore$

5. $A\,\varepsilon\, h[\nu_1 \ldots \nu_n]\,.$

6. $A\,\varepsilon\, f[\nu_1 \ldots \nu_n]:$

7. $[BC\eta_1 \ldots \eta_n\mu_1 \ldots \mu_n] : B\,\varepsilon\, h[\eta_1 \ldots \eta_n]\,.\, C\,\varepsilon\, h[\mu_1 \ldots \mu_n] \,.\!\supset\!.\, B\,\varepsilon\, C\,.\circ\underset{1}{[}\eta_1\mu_1\underset{1}{]}\,. \ldots .\underset{n}{\{}\eta_n\mu_n\underset{n}{\}}:$ [1, *D1.1*] (5–7)

$[\exists D\theta_1 \ldots \theta_n]:$

8. $D\,\varepsilon\, h[\theta_1 \ldots \theta_n]\,.$

9. $D\,\varepsilon\, g[\theta_1 \ldots \theta_n]\,.$ [4, *D1.1*] (8–9)

10. $A\,\varepsilon\, D\,.$ [5, 7, 8]

11. $A\,\varepsilon\, g[\theta_1 \ldots \theta_n]\,.$ [9, 10]

12. $\circ\underset{1}{\{}\theta_1\nu_1\underset{1}{\}}\,. \ldots .\circ\underset{n}{\{}\theta_n\nu_n\underset{n}{\}}\,.$ [5, 7, 8]

13. $\ast\langle\!\langle Ag\rangle\!\rangle\langle\theta_1 \ldots \theta_n\rangle:$ [11, *D1.2*]

14. $\ast\langle\!\langle Ag\rangle\!\rangle\langle\nu_1 \ldots \nu_n\rangle:$ [12, 13, *A3*]

15. $[BC\mu_1 \ldots \mu_n\eta_1 \ldots \eta_n] : B\,\varepsilon\, f[\mu_1 \ldots \mu_n]\,.\, C\,\varepsilon\, f\{\eta_1 \ldots \eta_n\} \,.\!\supset\!.\, B\,\varepsilon\, C\,.\circ\underset{1}{\{}\mu_1\eta_1\underset{1}{\}}\,. \ldots .\circ\underset{n}{\{}\mu_n\nu_n\underset{n}{\}} \therefore$ [3, *A9*]

$\varepsilon\{fg\}$ [6, 15, 16, *D1.1*]

A11 $[fg] :: \varepsilon\{fg\} \,.\!\equiv\!\therefore [\exists h] : \varepsilon\{hf\} : [i] : \varepsilon\{if\} \,.\!\supset\!.\, \varepsilon\{ig\} : [jk] : \varepsilon\{jf\}\,.\, \varepsilon\{kf\} \,.\!\supset\!.\, \varepsilon\{jk\}$ [*A1, A4, A5, A10*]

2 This section duplicates the results of the preceding section for propositional categories.

D2.1 $[\varphi\psi] :: [\exists\nu_1 \ldots \nu_n] : \varphi\{\nu_1 \ldots \nu_n\}\,.\,\psi\{\nu_1 \ldots \nu_n\} : [\mu_1 \ldots \mu_n\eta_1 \ldots \eta_n] : \varphi\{\mu_1 \ldots \mu_n\}\,.\,\varphi\{\eta_1 \ldots \eta_n\} \,.\!\supset\!.\, \circ\underset{1}{\{}\mu_1\eta_1\underset{1}{\}}\,. \ldots .\circ\underset{n}{\{}\mu_n\eta_n\underset{n}{\}} \therefore\!\equiv\!.\, \varepsilon\{\varphi\psi\}$

B1 $[\varphi\psi] : \varepsilon\{\varphi\psi\} \,.\!\supset\!.\, \varepsilon\{\varphi\varphi\}$ [*D2.1*]

B2 $[\varphi\psi\theta] : \varepsilon\{\varphi\psi\}\,.\,\varepsilon\{\theta\varphi\} \,.\!\supset\!.\, \varepsilon\{\varphi\theta\}\,.$

PR $[\varphi\psi\theta] : \mathrm{Hp}(2) \,.\!\supset\!.$

3. $[\mu_1 \ldots \mu_n\eta_1 \ldots \eta_n] : \varphi\{\mu_1 \ldots \mu_n\}\,.\,\varphi\{\eta_1 \ldots \eta_n\} \,.\!\supset\!.\, \circ\underset{1}{\{}\mu_1\eta_1\underset{1}{\}}\,. \ldots .\circ\underset{n}{\{}\mu_n\eta_n\underset{n}{\}}:$ [1, *D2.1*]

$[\exists\nu_1 \ldots \nu_n]:$

4. $\varphi\{\nu_1 \ldots \nu_n\}\,.$

5. $\theta\{\nu_1 \ldots \nu_n\}:$ [2, *D2.1*] (4–5)

$\varepsilon\{\varphi\theta\}$ [3, 4, 5, *D2.1*]

B3 $[\varphi\nu_1 \ldots \nu_n\mu_1 \ldots \mu_n] : \circ\underset{1}{\{}\mu_1\nu_1\underset{1}{\}}\,. \ldots .\circ\underset{n}{\{}\mu_n\nu_n\underset{n}{\}}\,.\,\varphi\{\mu_1 \ldots \mu_n\} \,.\!\supset\!.\, \varphi\{\nu_1 \ldots \nu_n\}$ [see *A3*]

B4 $[\varphi\psi\theta] : \varepsilon\{\varphi\psi\}\,.\,\varepsilon\{\theta\varphi\} \,.\!\supset\!.\, \varepsilon\{\theta\psi\}$

PR $[\varphi\psi\theta] \therefore \mathrm{Hp}(2) \,.\!\supset\!::$

3. $[\mu_1 \ldots \mu_n\eta_1 \ldots \eta_n] : \varphi\{\mu_1 \ldots \mu_n\}\,.\,\varphi\{\eta_1 \ldots \eta_n\} \,.\!\supset\!.\, \circ\underset{1}{\{}\mu_1\eta_1\underset{1}{\}}\,. \ldots .\circ\underset{n}{\{}\mu_n\eta_n\underset{n}{\}}:$ [1, *D2.1*]

4. $[\mu_1 \ldots \mu_n \eta_1 \ldots \eta_n] : \theta\{\mu_1 \ldots \mu_n\} . \theta\{\eta_1 \ldots \eta_n\} . \supset . \circ\{\underset{1}{\mu_1}\underset{1}{\eta_1}\} . \ldots .$
$\circ\{\underset{n}{\mu_n}\underset{n}{\eta_n}\} \therefore$ [2, *D2.1*]

$[\exists \nu_1 \ldots \nu_n] \therefore$

5. $\varphi\{\nu_1 \ldots \nu_n\}$ [1, *D2.1*]
6. $\psi\{\nu_1 \ldots \nu_n\} :$

$[\exists \gamma_1 \ldots \gamma_n] :$

7. $\theta\{\gamma_1 \ldots \gamma_n\} .$ [2, *D2.1*]
8. $\varphi\{\gamma_1 \ldots \gamma_n\} .$
9. $\circ\{\underset{1}{\gamma_1}\underset{1}{\nu_1}\} . \ldots . \circ\{\underset{n}{\gamma_n}\underset{n}{\nu_n}\} :$ [5, 8, 3]
10. $\theta\{\nu_1 \ldots \nu_n\} \therefore$ [7, 9, *B3*]

$\varepsilon\{\theta\psi\}$ [6, 10, 4, *D2.1*]

B5 $[\phi\varphi\psi\theta] : \varepsilon\{\phi\varphi\} . \varepsilon\{\psi\phi\} . \varepsilon\{\theta\phi\} . \supset . \varepsilon\{\psi\theta\}$

PR $[\phi\varphi\psi\theta] : \mathrm{Hp}(3) . \supset .$

4. $\varepsilon\{\phi\theta\} .$ [1, 3, *B2*]

$\varepsilon\{\psi\theta\}$ [2, 4, *B4*]

D2.2 $[\nu_1 \ldots \nu_n \mu_1 \ldots \mu_n] : \circ\{\underset{1}{\nu_1}\underset{1}{\mu_1}\} . \ldots . \circ\{\underset{n}{\nu_n}\underset{n}{\mu_n}\} . \equiv . \phi_K \leftarrow \nu_1 \ldots \nu_n \rightarrow$
$\{\mu_1 \ldots \mu_n\}$

B6 $[\nu_1 \ldots \nu_n] : \phi_K \leftarrow \nu_1 \ldots \nu_n \rightarrow \{\nu_1 \ldots \nu_n\}$ [*D2.2*]

B7 $[\nu_1 \ldots \nu_n \mu_1 \ldots \mu_n \eta_1 \ldots \eta_n] : \phi_K \leftarrow \nu_1 \ldots \nu_n \rightarrow \{\mu_1 \ldots \mu_n\} .$
$\phi_K \leftarrow \nu_1 \ldots \nu_n \rightarrow \{\eta_1 \ldots \eta_n\} . \supset . \circ\{\underset{1}{\mu_1}\underset{1}{\eta_1}\} . \ldots . \circ\{\underset{n}{\mu_n}\underset{n}{\eta_n}\}$ [*D2.2*]

B8 $[\varphi\nu_1 \ldots \nu_n] : \varphi\{\nu_1 \ldots \nu_n\} . \supset . \varepsilon\{\phi_K \leftarrow \nu_1 \ldots \nu_n \rightarrow \varphi\}$ [*B6, B7, D2.1*]

B9 $[\phi\nu_1 \ldots \nu_n \mu_1 \ldots \mu_n] \therefore [\varphi\psi] : \varepsilon\{\varphi\phi\} . \varepsilon\{\psi\phi\} . \supset . \varepsilon\{\varphi\psi\} :$
$\phi\{\nu_1 \ldots \nu_n\} . \phi\{\mu_1 \ldots \mu_n\} : \supset . \circ\{\underset{1}{\nu_1}\underset{1}{\mu_1}\} . \ldots . \circ\{\underset{n}{\nu_n}\underset{n}{\mu_n}\}$

PR $[\phi\nu_1 \ldots \nu_n \mu_1 \ldots \mu_n] :: \mathrm{Hp}(3) : \supset \therefore$

4. $\varepsilon\{\phi_K \leftarrow \nu_1 \ldots \nu_n \rightarrow \phi\}$ [2, *B8*]
5. $\varepsilon\{\phi_K \leftarrow \mu_1 \ldots \mu_n \rightarrow \phi\} .$ [3, *B8*]
6. $\varepsilon\{\phi_K \leftarrow \nu_1 \ldots \nu_n \rightarrow \phi_K \leftarrow \mu_1 \ldots \mu_n \rightarrow\} .$ [1, 4, 5]

$[\exists \eta_1 \ldots \eta_n] :$

7. $\phi_K \leftarrow \nu_1 \ldots \nu_n \rightarrow \{\eta_1 \ldots \eta_n\} .$ [6, *D2.1*]
8. $\phi_K \leftarrow \mu_1 \ldots \mu_n \rightarrow \{\eta_1 \ldots \eta_n\} \therefore$

$\circ\{\underset{1}{\nu_1}\underset{1}{\mu_1}\} . \ldots . \circ\{\underset{n}{\nu_n}\underset{n}{\mu_n}\}$ [7, 8, *D2.2*]

B10 $[\phi\varphi\psi] :: \varepsilon\{\psi\phi\} : [\theta] : \varepsilon\{\theta\phi\} . \supset . \varepsilon\{\theta\varphi\} : [\theta\lambda] : \varepsilon\{\theta\phi\} . \varepsilon\{\lambda\phi\} . \supset .$
$\varepsilon\{\theta\lambda\} \therefore \supset . \varepsilon\{\phi\varphi\}$

PR $[\phi\varphi\psi] :: \mathrm{Hp}(3) \therefore \supset \therefore$

4. $\varepsilon\{\psi\phi\} .$

$[\exists \nu_1 \ldots \nu_n] \therefore$

5. $\psi\{\nu_1 \ldots \nu_n\} .$ [1, *D2.1*]
6. $\phi\{\nu_1 \ldots \nu_n\} .$
7. $[\mu_1 \ldots \mu_n \eta_1 \ldots \eta_n] : \psi\{\mu_1 \ldots \mu_n\} .$
$\psi\{\eta_1 \ldots \eta_n\} . \supset . \circ\{\underset{1}{\mu_1}\underset{1}{\eta_1}\} . \ldots . \circ\{\underset{n}{\mu_n}\underset{n}{\eta_n}\} :$

$[\exists \gamma_1 \ldots \gamma_n] :$

8. $\psi\{\gamma_1 \ldots \gamma_n\} .$ [4, *D2.1*]
9. $\varphi\{\gamma_1 \ldots \gamma_n\} .$
10. $\circ\{\underset{1}{\gamma_1}\underset{1}{\nu_1}\} . \ldots . \circ\{\underset{n}{\gamma_n}\underset{n}{\nu_n}\} :$ [5, 7, 8]

11. $\varphi\{\nu_1 \ldots \nu_n\}$. [9, 10, *B3*]
12. $[\mu_1 \ldots \mu_n \eta_1 \ldots \eta_n] : \phi\{\mu_1 \ldots \mu_n\} . \phi\{\eta_1 \ldots \eta_n\} . \supset . \circ\{\underset{1}{\mu_1}\underset{1}{\eta_1}\} . \ldots . \circ\{\underset{n}{\mu_n}\underset{n}{\eta_n}\} \therefore$ [3, *B9*]

$\varepsilon\{\phi\varphi\}$ [6, 11, 12, *D2.1*]

B11 $[\phi\varphi] :: \varepsilon\{\phi\varphi\} . \equiv \therefore [\exists\psi] . \varepsilon\{\psi\phi\} : [\psi] : \varepsilon\{\psi\phi\} . \supset . \varepsilon\{\psi\varphi\} : [\psi\theta] :$ $\varepsilon\{\psi\phi\} . \varepsilon\{\theta\phi\} . \supset . \varepsilon\{\psi\theta\}$ [*B1, B4, B5, B10*]

3 In this section, $\mathbf{AC}^{\varepsilon}_{\alpha}$ is shown to be equivalent to $\mathbf{ACH}_{\alpha}$ where α is a nominal category.

D3.1 $[fg] : [A\nu_1 \ldots \nu_n] : A\,\varepsilon\, f\{\nu_1 \ldots \nu_n\} . \equiv . A\,\varepsilon\, g\{\nu_1 \ldots \nu_n\} : \equiv . \circ\{fg\}$
D3.2 $[R] : [fgh] : R\{fg\} . R\{fh\} . \supset . \circ\{hg\} : \equiv . \rightrightarrows\{R\}$
D3.3 $[fA\nu_1 \ldots \nu_n] : A\,\varepsilon\, A . \sim(A\,\varepsilon\, f\{\nu_1 \ldots \nu_n\}) . \equiv . A\,\varepsilon\,\mathsf{N}\{f\}\{\nu_1 \ldots \nu_n\}$
D3.4 $[fg\varphi] : \varepsilon\{ff\} . \circ\{\varphi\{g\}f\} . \equiv . \mathbf{Val}\{\varphi\}\{gf\}$

C1 $[\varphi fgh] : \mathbf{Val}\{\varphi\}\{gf\} . \mathbf{Val}\{\varphi\}\{gh\} . \supset . \circ\{fh\}$ [*D3.1, D3.4*]
C2 $[\varphi] : \rightrightarrows\{\mathbf{Val}\{\varphi\}\}$ [*C1, D3.2*]
C3 $[\varphi fg] \therefore [hk] : \varepsilon\{hk\} . \supset . \varepsilon\{\varphi\{k\}k\} : \varepsilon\{fg\} : \supset .$ $[\exists k] . \varepsilon\{kg\} . \mathbf{Val}\{\varphi\}\{gk\}$
PR $[\varphi fg] \therefore \mathrm{Hp}(2) : \supset .$
3. $\varepsilon\{\varphi\{g\}g\}$. [1, 2]
4. $\varepsilon\{\varphi\{g\}\varphi\{g\}\}$. [3, *A1*]
5. $\mathbf{Val}\{\varphi\}\{g\varphi\{g\}\}$. [4, *D3.1, D3.4*]
$[\exists k] . \varepsilon\{kg\} . \mathbf{Val}\{\varphi\}\{gk\}$ [3, 5]

C4 $[\exists\varphi] \therefore [fg] : \varepsilon\{fg\} . \supset . \varepsilon\{\varphi\{g\}g\} \therefore \supset \therefore [\exists R] \therefore \rightrightarrows\{R\} : [fg] :$ $\varepsilon\{fg\} . \supset . [\exists k] . \varepsilon\{kg\} . R\{gk\}$ [*C2, C3*]

At this point it would be desirable to introduce an "ontological definition" of the form:

Df $[fgR] : R\{fg\} . \varepsilon\{gg\} . \equiv . \varepsilon\{g\,\mathbf{Func}\{\underset{1}{R}\}\{\underset{1}{f}\}\}$

However, the rule of definition allows definitions of this form only if the epsilon involved is the primitive epsilon. In order to meet this requirement we introduce the following definition of '$\mathbf{Func}\{\underset{1}{R}\}\{\underset{1}{f}\}$' and derive **Df** as a consequence.

D3.5 $[fRA\nu_1 \ldots \nu_n] : [\exists g] . R\{fg\} . \varepsilon\{gg\} . A\,\varepsilon\, g\{\nu_1 \ldots \nu_n\} . \equiv .$ $A\,\varepsilon\,\mathbf{Func}\{\underset{1}{R}\}\{\underset{1}{f}\}\{\nu_1 \ldots \nu_n\}$

C5 $[AfRg\nu_1 \ldots \nu_n] : \varepsilon\{gg\} . R\{fg\} . A\,\varepsilon\, g\{\nu_1 \ldots \nu_n\} . \supset .$ $A\,\varepsilon\,\mathbf{Func}\{\underset{1}{R}\}\{\underset{1}{f}\}\{\nu_1 \ldots \nu_n\}$ [*D3.5*]

C6 $[Rfg] : \varepsilon\{gg\} . R\{fg\} . \supset . \varepsilon\{g\,\mathbf{Func}\{\underset{1}{R}\}\{\underset{1}{f}\}\}$
PR $[Rfg] : \mathrm{Hp}(2) . \supset .$
$[\exists A\nu_1 \ldots \nu_n]$.
3. $A\,\varepsilon\, g\{\nu_1 \ldots \nu_n\}$. [1, *D1.1*]
4. $A\,\varepsilon\,\mathbf{Func}\{R\}\{f\}\{\nu_1 \ldots \nu_n\}$. [1, 2, 3, *C5*]

5. $[BC\mu_1 \ldots \mu_n\eta_1 \ldots \eta_n] : B\,\varepsilon\, g\,[\mu_1 \ldots \mu_n]\,.$
$C\,\varepsilon\, g\,[\eta_1 \ldots \eta_n]\ .\supset.\ B\,\varepsilon\, C\,.$
$\circ\underset{1}{\{}\mu_1\eta_1\underset{1}{\}}\,.\ \ldots\ .\circ\underset{n}{\{}\mu_n\eta_n\underset{n}{\}} :$ [1, *D1.1*]

$\varepsilon\{g\ \mathbf{Func}\underset{1}{\{}R\underset{1}{\}}\ \{f\}\}$ [3, 4, 5, *D1.1*]

D3.6 $[Rfg] : R\{fg\}\ .\equiv.\ \chi_\alpha\{\!\!*\,RF\,*\!\!\}\{g\}$

E3.1 $[fg] \therefore [A\nu_1 \ldots \nu_n] : A\,\varepsilon\, f\,[\nu_1 \ldots \nu_n]\ .\equiv.\ A\,\varepsilon\, g\,[\nu_1 \ldots \nu_n]\ :\equiv:$
$[\varphi] : \varphi\{f\}\ .\equiv.\ \varphi\{g\}$

C7 $[gRf] : \varepsilon\{g\ \mathbf{Func}\{R\}\ \{f\}\}\ .\supset.\ R\{fg\}$

PR $[gRf] :: \mathrm{Hp}(1)\ .\supset\therefore$
$[\exists A\nu_1 \ldots \nu_n] \therefore$

2. $A\,\varepsilon\, g\,[\nu_1 \ldots \nu_n]\,.$
3. $A\,\varepsilon\,\mathbf{Func}\{R\}\ \{f\}\,[\nu_1 \ldots \nu_n] :$
4. $[BC\eta_1 \ldots \eta_n\mu_1 \ldots \mu_n] : B\,\varepsilon\, g\,[\eta_1 \ldots \eta_n]\,.$
$C\,\varepsilon\, g\,[\mu_1 \ldots \mu_n]\ .\supset.\ B\,\varepsilon\, C\,.\circ\underset{1}{\{}\eta_1\mu_1\underset{1}{\}}\,.\ \ldots\ .$
$\circ\underset{n}{\{}\eta_n\ \mu_n\underset{n}{\}} :$ [1, *D1.1*]

$[\exists h] :$

5. $A\,\varepsilon\, h\,[\nu_1 \ldots \nu_n]\,.$
6. $\varepsilon\{hh\}\,.$ [3, *D3.5*]
7. $R\{fh\}\,.$
8. $\varepsilon\{gh\}\,.$ [2, 4, 5, *D1.1*]
9. $\varepsilon\{hg\}\,.$ [6, 8, *A2*]
10. $\circ\{hg\}\,.$ [8, 9, *D3.1*]
11. $\chi_\alpha\{\!\!*\,Rf\,*\!\!\}\{h\} \therefore$ [7, *D3.6*]
12. $\chi_\alpha\{\!\!*\,Rf\,*\!\!\}\{g\}\,.$ [10, 11, *D3.1, E3.1*]

$R\{fg\}$ [12, *D3.6*]

C8 $[fgR] : \varepsilon\{gg\}\,.\,R\{fg\}\ .\equiv.\ \varepsilon\{g\ \mathbf{Func}\{R\}\ \{f\}\}$ [*C6, C7, A1*]

C8 is the thesis which corresponds to the desired definition, **Df**.

C9 $[fghR] : \rightrightarrows\{R\}\,.\,\varepsilon\{f\ \mathbf{Func}\{R\}\ \{g\}\}\,.\,\varepsilon\{h\ \mathbf{Func}\{R\}\ \{g\}\}\ .\supset.\ \varepsilon\{fh\}$

PR $[fghR] :: \mathrm{Hp}(3)\ .\supset\therefore$

4. $\varepsilon\{ff\}\,.$ [2, *C8*]
5. $R\{gf\}\,.$ [2, *C8*]
6. $\varepsilon\{hh\}\,.$ [3, *C8*]
7. $R\{gh\}\,.$ [3, *C8*]
8. $\circ\{fh\} :$ [5, 7, 1, *D3.2*]
9. $[A\nu_1 \ldots \nu_n] : A\,\varepsilon\, f\,[\nu_1 \ldots \nu_n]\ .\equiv.\ A\,\varepsilon\, h\,[\nu_1 \ldots \nu_n] \therefore$ [8, *D3.1*]

$[\exists A\nu_1 \ldots \nu_n] \therefore$

10. $A\,\varepsilon\, f\,[\nu_1 \ldots \nu_n] :$
11. $[BC\mu_1 \ldots \mu_n\eta_1 \ldots \eta_n] : B\,\varepsilon\, f\,[\mu_1 \ldots \mu_n]\,.$
$C\,\varepsilon\, f\,[\eta_1 \ldots \eta_n]\ .\supset.\ B\,\varepsilon\, C\,.\circ\underset{1}{\{}\mu_1\eta_1\underset{1}{\}}\,.\ \ldots\ .$
$\circ\underset{n}{\{}\mu_n\eta_n\underset{n}{\}} :$ [4, *D1.1*]
12. $A\,\varepsilon\, h\,[\nu_1 \ldots \nu_n] \therefore$ [9, 10]

$\varepsilon\{fh\}$ [10, 11, 12, *D1.1*]

C10 $[fgR] : \rightrightarrows \{\!\{R\}\!\} . \varepsilon \{f \text{ Func} \{\!\{R\}\!\} \{g\}\} . \supset . \varepsilon \{\text{Func} \{\!\{R\}\!\} \{g\} f\}$
PR $[fgR] : \text{Hp}(2) . \supset .$
3. $[ghj] : \varepsilon \{h \text{ Func} \{\!\{R\}\!\} \{g\}\} . \varepsilon \{j \text{ Func} \{\!\{R\}\!\} \{g\}\} . \supset . \varepsilon \{hj\} :$ [1, *C9*]
4. $[h] : \varepsilon \{h \text{ Func} \{\!\{R\}\!\} \{g\}\} . \supset . \varepsilon \{hf\} :$ [2, 3]
$\varepsilon \{\text{Func} \{\!\{R\}\!\} \{g\} f\}$ [2, 3, 4, *A10*]
C11 $[fgR] \therefore \rightrightarrows \{\!\{R\}\!\} : [hk] : \varepsilon \{hk\} . \supset . [\exists j] . \varepsilon \{jk\} . R\{kj\} :$
$\varepsilon \{fg\} : \supset . \varepsilon \{\text{Func} \{\!\{R\}\!\} \{g\} g\}$
PR $[fgR] \therefore \text{Hp}(3) . \supset :$
$[\exists j] :$
4. $\varepsilon \{jg\} .$
5. $R\{gj\} .$ [2, 3]
6. $\varepsilon \{jj\} .$ [4, *A1*]
7. $\varepsilon \{j \text{ Func} \{\!\{R\}\!\} \{g\}\} .$ [5, 6, *C8*]
8. $\varepsilon \{\text{Func} \{\!\{R\}\!\} \{g\} j\} :$ [7, *C10*]
$\varepsilon \{\text{Func} \{\!\{R\}\!\} \{g\} g\}$ [4, 8, *A4*]
C12 $[\exists R] \therefore \rightrightarrows \{\!\{R\}\!\} : [fg] : \varepsilon \{fg\} . \supset . [\exists h] . \varepsilon \{hg\} . R\{gh\} \therefore \equiv \therefore$
$[\exists \varphi] : [fg] : \varepsilon \{fg\} . \supset . \varepsilon \{\varphi \{g\} g\}$ [*C4*, *C11*]

4 This section extends the results of the preceding section to the propositional categories.

D4.1 $[\varphi\psi] : [\nu_1 \ldots \nu_n] : \varphi\{\nu_1 \ldots \nu_n\} . \equiv . \psi\{\nu_1 \ldots \nu_n\} : \equiv . \circ\{\varphi\psi\}$
D4.2 $[R] : [\varphi\psi\theta] : R\{\varphi\psi\} . R\{\varphi\theta\} . \supset . \circ\{\psi\theta\} : \equiv . \rightrightarrows \{R\}$
D4.3 $[\varphi\nu_1 \ldots \nu_n] : \sim \varphi\{\nu_1 \ldots \nu_n\} . \equiv . \mathsf{N} \langle\varphi\rangle \{\nu_1 \ldots \nu_n\}$
D4.4 $[\varphi\psi\lambda] : \varepsilon\{\varphi\varphi\} . \circ\{\lambda\langle\psi\rangle\varphi\} . \equiv . \text{Val} \langle\!\langle\lambda\rangle\!\rangle \{\psi\varphi\}$

F1 $[\lambda\varphi\psi\theta] : \text{Val} \langle\!\langle\lambda\rangle\!\rangle \{\theta\varphi\} . \text{Val} \langle\!\langle\lambda\rangle\!\rangle \{\theta\psi\} . \supset . \circ\{\varphi\psi\} .$ [*D4.4*, *D4.1*]
F2 $[\lambda] . \rightrightarrows \{\text{Val} \langle\!\langle\lambda\rangle\!\rangle\}$ [*F1*, *D4.2*]
F3 $[\lambda\phi\psi] \therefore [\varphi\theta] : \varepsilon\{\varphi\theta\} . \supset . \varepsilon\{\lambda\langle\theta\rangle\theta\} : \varepsilon\{\phi\psi\} : \supset .$
$[\exists\mu] . \varepsilon\{\mu\psi\} . \text{Val} \langle\!\langle\lambda\rangle\!\rangle \{\psi\mu\}$ [*B1*, *D4.1*, *D4.4*]
F4 $[\exists\lambda] \therefore [\phi\psi] : \varepsilon\{\phi\psi\} . \supset . \varepsilon\{\lambda\langle\psi\rangle\psi\} \therefore \supset \therefore$
$[\exists R] \therefore \rightrightarrows \{R\} : [\phi\psi] : \varepsilon\{\phi\psi\} . \supset . [\exists\theta] . \varepsilon\{\theta\psi\} . R\{\psi\theta\}$ [*F2*, *F3*]

D4.5 $[\varphi R\nu_1 \ldots \nu_n] : [\exists\psi] : R\{\varphi\psi\} . \varepsilon\{\psi\psi\} . \psi\{\nu_1 \ldots \nu_n\} . \equiv .$
$\text{Func} \langle\!\langle R\rangle\!\rangle \langle\varphi\rangle \{\nu_1 \ldots \nu_n\}$

F5 $[\varphi\psi R\nu_1 \ldots \nu_n] : \varepsilon\{\psi\psi\} . R\{\varphi\psi\} . \psi\{\nu_1 \ldots \nu_n\} . \supset .$
$\text{Func} \langle\!\langle R\rangle\!\rangle \langle\varphi\rangle \{\nu_1 \ldots \nu_n\}$ [*D4.5*]
F6 $[\varphi\psi R] : \varepsilon\{\psi\psi\} . R\{\varphi\psi\} . \supset . \varepsilon\{\psi \text{ Func} \langle\!\langle R\rangle\!\rangle \langle\varphi\rangle\}$ [*D2.1*, *F5*]

D4.6 $[R\varphi\psi] : R\{\varphi\psi\} . \equiv . \chi_\beta \{\!\{R\varphi\}\!\} \{\psi\} .$

E4.1 $[\varphi\psi] \therefore [\nu_1 \ldots \nu_n] : \varphi\{\nu_1 \ldots \nu_n\} . \equiv . \psi\{\nu_1 \ldots \nu_n\} : \equiv : [\phi] : \phi\{\varphi\} . \equiv . \phi\{\psi\}$

F7 $[\varphi\psi R] : \varepsilon\{\psi \text{ Func} \langle\!\langle R\rangle\!\rangle \langle\varphi\rangle\} . \supset . R\{\varphi\psi\} .$ [*D2.1*, *D4.5*, *B2*, *D4.6*, *E4.1*]
F8 $[\varphi\psi R] : \varepsilon\{\psi\psi\} . R\{\varphi\psi\} . \equiv . \varepsilon\{\psi \text{ Func} \langle\!\langle R\rangle\!\rangle \langle\varphi\rangle\}$ [*F6*, *F7*]
F9 $[\varphi\psi\theta R] : \rightrightarrows \{R\} . \varepsilon\{\varphi \text{ Func} \langle\!\langle R\rangle\!\rangle \langle\theta\rangle\} . \varepsilon\{\psi \text{ Func} \langle\!\langle R\rangle\!\rangle \langle\theta\rangle\} . \supset . \varepsilon\{\varphi\psi\}$
[*F8*, *D4.2*, *D4.1*, *D2.1*]
F10 $[\varphi\psi R] : \rightrightarrows \{R\} . \varepsilon\{\varphi \text{ Func} \langle\!\langle R\rangle\!\rangle \langle\psi\rangle\} . \supset . \varepsilon\{\text{Func} \langle\!\langle R\rangle\!\rangle \langle\psi\rangle\varphi\}$ [*F9*, *B10*]

F11 $[\varphi\psi R] :: \rightrightarrows \{R\} : [\theta\mu] : \varepsilon\{\theta\mu\} .\supset. [\exists\phi] . \varepsilon\{\phi\mu\} . R\{\mu\phi\} : \varepsilon\{\varphi\psi\} :\supset.$
$\varepsilon\{\mathbf{Func}\langle R\rangle \langle\psi\rangle\ \psi\}$ [*B1, F8, F10, B4*]

F12 $[\exists R] \therefore \rightrightarrows \{R\} : [\theta\mu] : \varepsilon\{\theta\mu\} .\supset. [\exists\phi] . \varepsilon\{\phi\mu\} . R\{\mu\phi\} \therefore \equiv \therefore$
$[\exists\lambda] \therefore [\theta\mu] : \varepsilon\{\theta\mu\} .\supset. \varepsilon\{\lambda\langle\mu\rangle\ \mu\}$ [*F4, F11*]

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