Notre Dame Journal of Formal Logic
Volume XVII, Number 1, January 1976
NDJFAM

ON PROSLEPTIC PREMISSES

CZESŁAW LEJEWSKI

1 A prosleptic premiss is a proposition of the form

$$\text{for all } x\text{: if } \phi x \text{ then } \psi x,$$

with 'ϕx' and 'ψx' standing as abbreviations for any of the four categorical propositions, in which the bound variable 'x' may occur in the place of the subject or in that of the predicate. Thus, if we want to be more specific, we can say that a prosleptic premiss exhibits one of the following four forms:

$$\begin{array}{l}\text{for all } x\text{: if } a\ R\ x \text{ then } x\ S\ b\\ \text{for all } x\text{: if } a\ R\ x \text{ then } b\ S\ x\\ \text{for all } x\text{: if } x\ R\ a \text{ then } x\ S\ b\\ \text{for all } x\text{: if } x\ R\ a \text{ then } b\ S\ x.\end{array}$$

In these formulae 'R' and 'S' stand for any of the four functors which form categorical propositions. If we represent these functors in the traditional way with the aid of the letters 'A', 'E', 'I', and 'O', then the 64 different prosleptic premisses can be tabulated as on page 2.

According to tradition a systematic study of prosleptic premisses was initiated by Theophrastus. The position of the bound variables in a prosleptic premiss suggested to Theophrastus the division of prosleptic premisses into figures. Following the example of Aristotle he distinguished three figures, corresponding to columns I, II, and III in the table given on the following page. It is quite likely that prosleptic premisses in column IV were regarded by Theophrastus as belonging to the first figure together with the prosleptic premisses listed in column I, In fact, given the laws of the square of opposition, they can easily be shown to be converses of appropriate propositions in column I. Prosleptic premisses give rise to prosleptic syllogisms, which are inferences of the form

$$\begin{array}{l}\text{for all } x\text{: if } \phi x \text{ then } \psi x\\ \underline{\qquad\qquad\qquad \phi a \qquad\qquad}\\ \text{therefore: } \psi a,\end{array}$$

Received August 17, 1973

I

1. for all x: if a A x then x A b
2. for all x: if a A x then x E b
3. for all x: if a A x then x I b
4. for all x: if a A x then x O b
5. for all x: if a E x then x A b
6. for all x: if a E x then x E b
7. for all x: if a E x then x I b
8. for all x: if a E x then x O b
9. for all x: if a I x then x A b
10. for all x: if a I x then x E b
11. for all x: if a I x then x I b
12. for all x: if a I x then x O b
13. for all x: if a O x then x A b
14. for all x: if a O x then x E b
15. for all x: if a O x then x I b
16. for all x: if a O x then x O b

II

1. for all x: if a A x then b A x
2. for all x: if a A x then b E x
3. for all x: if a A x then b I x
4. for all x: if a A x then b O x
5. for all x: if a E x then b A x
6. for all x: if a E x then b E x
7. for all x: if a E x then b I x
8. for all x: if a E x then b O x
9. for all x: if a I x then b A x
10. for all x: if a I x then b E x
11. for all x: if a I x then b I x
12. for all x: if a I x then b O x
13. for all x: if a O x then b A x
14. for all x: if a O x then b E x
15. for all x: if a O x then b I x
16. for all x: if a O x then b O x

III

1. for all x: if x A a then x A b
2. for all x: if x A a then x E b
3. for all x: if x A a then x I b
4. for all x: if x A a then x O b
5. for all x: if x E a then x A b
6. for all x: if x E a then x E b
7. for all x: if x E a then x I b
8. for all x: if x E a then x O b
9. for all x: if x I a then x A b
10. for all x: if x I a then x E b
11. for all x: if x I a then x I b
12. for all x: if x I a then x O b
13. for all x: if x O a then x A b
14. for all x: if x O a then x E b
15. for all x: if x O a then x I b
16. for all x: if x O a then x O b

IV

1. for all x: if x A a then b A x
2. for all x: if x A a then b E x
3. for all x: if x A a then b I x
4. for all x: if x A a then b O x
5. for all x: if x E a then b A x
6. for all x: if x E a then b E x
7. for all x: if x E a then b I x
8. for all x: if x E a then b O x
9. for all x: if x I a then b A x
10. for all x: if x I a then b E x
11. for all x: if x I a then b I x
12. for all x: if x I a then b O x
13. for all x: if x O a then b A x
14. for all x: if x O a then b E x
15. for all x: if x O a then b I x
16. for all x: if x O a then b O x

'ϕx' and 'ψx' standing, as was explained earlier, for any of the four categorical propositions, in which the terms 'x' and 'a' may occur as the subject or as the predicate. It was already known to Theophrastus that prosleptic premisses III.1 and III.2 were equivalent to the categorical premisses 'a A b' and 'a E b' respectively. But is it true to say, as some ancient logicians appear to have believed it was, that all prosleptic premisses are in fact categorical propositions in disguise?

It is to this problem that a substantial part of a recent paper by William and Martha Kneale ('Prosleptic Propositions and Arguments' in *Islamic Philosophy and the Classical Tradition,* Essays presented to Richard Walzer, ed. S. M. Stern and others, University of South Carolina Press, Columbia (1972)) is devoted, and it is the results of their inquiry that I propose to examine in what follows.*

2 In trying to establish the logical relationships between prosleptic premisses and categorical propositions, William and Martha Kneale have found it convenient to divide the former into four groups. The first group consists of 26 prosleptic premisses, whose equivalence relation to categorical propositions is as follows:

I.11, II.11, II.16, III.1, III.11, and IV.11 are each equivalent to 'a A b';
I.12, III.2, III.12, and IV.2 are each equivalent to 'a A $\bar{b}$' (or to 'a E b');
I.6, II.1, II.6, III.6, III.16, and IV.6 are all equivalent to 'b A a';
I.5, III.5, III.15, and IV.15 are each equivalent to '$\bar{b}$ A a';
I.3, II.3, II.8, and IV.8 are each equivalent to 'a I b';
I.4 is equivalent to 'a I $\bar{b}$';
IV.4 is equivalent to '$\bar{a}$ I b'.

The distinguishing characteristic of the first group of prosleptic premisses is the method by which they can be proved to be equivalent to appropriate categorical propositions. As the Kneales put it the method involves the use of the *modus ponens,* the *modus tollens*, *substitution* for bound variables, *conditionalization, generalization,* and of course the use of *syllogistic.* This means that if we were to derive the equivalences in the object language of syllogistic, we would require a system of the logic of propositions including the usual rules for operating with the universal quantifier. The required derivations present no logical problems of any complexity, and we have no need to go into details here. Suffice it to note that by syllogistic the Kneales appear to understand a theory which besides the syllogisms comprises the laws of the square of opposition, the laws of conversion, the laws of obversion, and such 'tautologies' as

S1. *For all a:* a A a.
S2. *For all a: it is not the case that* $\bar{a}$ I a.

*On the subject of prosleptic premisses and prosleptic syllogisms see also: William and Martha Kneale, *The Development of Logic,* Clarendon Press, Oxford (1971) (fifth impression), pp. 106-109, and C. Lejewski, "On prosleptic syllogisms," *Notre Dame Journal of Formal Logic,* vol. II (1961), pp. 158-176.

And it is in this sense that the terms 'syllogistic' and 'traditional syllogistic' will be used in the present essay.

3 The second group of prosleptic propositions consists of ten members. Their equivalence relation to categorical propositions is as follows:

> III.3 and IV.3 are each equivalent to 'a A b';
> III.4 is equivalent to 'a A $\bar{b}$' (or to 'a E b');
> I.7, II.4, II.7, III.7, and IV.7 are each equivalent to '$\bar{a}$ A b';
> I.8 and III.8 are each equivalent to 'b A a'.

However, in no instance can the equivalence relation be established by means which have been found to be sufficient in the case of the prosleptic premisses of the first group. Consequently, the Kneales have recourse to an informal but very ingenious argument. They rightly point out that in traditional syllogistic neither empty nor universal terms are admissible as substituends for the variables. Now if in, say, I.7 we substitute for 'x' the compound term '$\bar{a} \cap \bar{b}$' assuming that it is not empty then I.7 turns out to be false. Thus the truth of I.7 appears to imply the inadmissibility of '$\bar{a} \cap \bar{b}$' as a substituend, which can only mean that '$\bar{a} \cap \bar{b}$' is empty. This entails '$\bar{a}$ E $\bar{b}$', from which we conclude that $\bar{a}$ A b. Again if in II.4 we substitute for 'x' the compound term '$a \cup b$' then II.4 turns out to be false. Thus its truth implies the inadmissibility of '$a \cup b$' as a substituend. Since neither 'a' nor 'b' is empty, '$a \cup b$', being inadmissible, must be universal, and the universality of '$a \cup b$' implies that $\bar{a}$ A b.

In order to appreciate the nature of these two arguments let us reconstruct, in more detail and as close to the Kneales' line of thought as possible, the proof that I.7 implies '$\bar{a}$ A b'. And let us begin our reconstruction by stating explicitly all the required presuppositions other than those belonging to the logic of propositions or to syllogistic as defined in the previous section. We presuppose that

M1. *For all a and b: if '$\bar{a} \cap \bar{b}$' is not admissible then $\bar{a}$* E *$\bar{b}$.*
M2. *For all a and b: if '$\bar{a} \cap \bar{b}$' is admissible then a* E *$\bar{a} \cap \bar{b}$.*
M3. *For all a and b: if '$\bar{a} \cap \bar{b}$' is admissible then $\bar{a} \cap \bar{b}$* A *$\bar{b}$.*

We now go on to prove:

M4. *For all a and b: if (for all x: if a* E *x then x* I *b) and '$\bar{a} \cap \bar{b}$' is admissible then '$\bar{a} \cap \bar{b}$' is not admissible.*

Proof: For all a and b:

if (1) for all x: if a E x then x I b	
and (2) '$\bar{a} \cap \bar{b}$' is admissible then	
(3) a E $\bar{a} \cap \bar{b}$	(M2, 2)
and (4) if a E $\bar{a} \cap \bar{b}$ then $\bar{a} \cap \bar{b}$ I b	(1, 2)
and (5) $\bar{a} \cap \bar{b}$ I b	(4, 3)
and (6) $\bar{a} \cap \bar{b}$ A $\bar{b}$	(M3, 2)
and (7) $\bar{b}$ I b	(Disamis, 5, 6)
and, finally, '$\bar{a} \cap \bar{b}$' is not admissible.	(7, S2)

M5. *For all a and b: if (for all x: if a* E *x* then *x* I *b) then '$\bar{a} \cap \bar{b}$' is not admissible.* (M4)

M6. *For all a and b: if (for all x: if a* E *x then x* I *b) then $\bar{a}$* A *b.* (M5, M1, obversion)

If my reconstruction of the proof of M6 is correct then it makes it obvious in what direction syllogistic is presupposed to have been extended. Presuppositions M1, M2, and M3 belong to metasyllogistic as the notion of admissibility does not actually occur in the object language, which is used for the purpose of formulating theses of syllogistic. Extending syllogistic into a system of metasyllogistic is a worthwhile project, but is it necessary to go as far as this in order to prove the implication under consideration? It would seem that '$\bar{a} \cap \bar{b}$' is admissible if and only if $\bar{a}$ I $\bar{b}$, whatever *a* and *b* may be. This being so, it would appear that M1, M2, and M3 could be replaced, respectively, by

S3. *For all a and b: if it is not the case that $\bar{a}$* I *$\bar{b}$ then $\bar{a}$* E *$\bar{b}$.*

N1. *For all a and b: if $\bar{a}$* I *$\bar{b}$ then a* E *$\bar{a} \cap \bar{b}$.*
N2. *For all a and b: if $\bar{a}$* I *$\bar{b}$ then $\bar{a} \cap \bar{b}$* A *$\bar{b}$.*

On these assumptions the proof that I.7 implies '$\bar{a}$ A *b*' could be recast as follows:

N3. *For all a and b: if (for all x: if a* E *x then x* I *b) and $\bar{a}$* I *$\bar{b}$ then it is not the case that $\bar{a}$* I *$\bar{b}$.*

Proof: For all *a* and *b*:

if (1) for all *x*: if *a* E *x* then *x* I *b*
and (2) $\bar{a}$ I $\bar{b}$ then
(3) *a* E $\bar{a} \cap \bar{b}$ (N1, 2)
and (4) $\bar{a} \cap \bar{b}$ I *b* (1, 3)
and (5) $\bar{a} \cap \bar{b}$ A $\bar{b}$ (N2, 2)
and (6) $\bar{b}$ I *b* (Disamis, 4, 5)
and, finally, it is not the case that $\bar{a}$ I $\bar{b}$. (6, S2)

N4. *For all a and b: if (for all x: if a* E *x then x* I *b) then it is not the case that $\bar{a}$* I *$\bar{b}$.* (N3)

N5. *For all a and b: if (for all x: if a* E *x then x* I *b) then $\bar{a}$* A *b.* (N4, S3, obv.)

In the proof just given no use is made of the notion of admissibility or non-admissibility, and, speaking generally, no recourse is had to metasyllogistic. However, the use of the compound term '$\bar{a} \cap \bar{b}$' raises some awkward problems. It would appear that presuppositions N1 and N2, if treated as additional axioms of syllogistic or as consequences of some additional axioms, make the use of '$\bar{a} \cap \bar{b}$' legitimate. This unfortunately is not so. For while N1 and N2 are unobjectionable, provided their antecedents are true, they turn out to be meaningless if the antecedents are false, as in that case the compound term '$\bar{a} \cap \bar{b}$', being empty, is altogether inadmissible in the language of traditional syllogistic. And if we cannot guarantee the availability of '$\bar{a} \cap \bar{b}$' on an axiomatic basis then we cannot introduce it with the aid of a definition. For a definition is nothing else but a single axiom added to the system in virtue of a general rule. I shall return to this problem in the next section, and in the meantime I shall try to prove, without resorting to compound terms, that I.7 implies '$\bar{a}$ A b'. All that is needed for the proof is the license to use a certain rule of inference, which was discovered by Aristotle himself and used by him in alternative proofs of some of his syllogisms. I am referring to inference by ecthesis, which, given a proposition 'a I b', allows us to infer that for some c: c A a and c A b. However, instead of making use of a new rule of inference, we can add to the presuppositions of syllogistic a proposition which lends validity to ecthesis. And this is what we shall do. We shall assume

S4. *For all a and b: if a I b then for some c: c A a and c A b.*

We shall also assume the usual rules for operating with the particular quantifier.

Within the framework of syllogistic, extended in this way, the new proof takes the following form:

S5. *For all a and b: if (for all x: if a E x then x I b) and it is not the case that $\bar{a}$ A b then $\bar{a}$ A b.*

Proof: For all a and b:

if (1)	for all x: if a E x then x I b	
and (2)	it is not the case that $\bar{a}$ A b then	
(3)	$\bar{a}$ O b	(square of opposition, 2)
and (4)	$\bar{a}$ I $\bar{b}$	(obv., 3)
and	for some c: (5) c A $\bar{a}$ and (6) c A $\bar{b}$	(S4, 4)
and (7)	c E b	(obv., 6)
and (8)	c E a	(obv., 5)
and (9)	a E c	(conversion, 8)
and (10)	c I b	(1, 9)
and (11)	it is not the case that c E b	(sq. of opp., 10)
and, finally,	$\bar{a}$ A b.	(7, 11)

S6. *For all a and b: if (for all x: if a E x then x I b) then $\bar{a}$ A b.* (S5)

The proof that, conversely, '$\bar{a}$ A b' implies I.7 is as follows:

S7. *For all a, b, and x: if $\bar{a}$ A b and a E x then x I b.*

Proof: For all a, b, and x:

if (1)	$\bar{a}$ A b	
and (2)	a E x then	
(3)	x E a	(conv., 2)
and (4)	x A $\bar{a}$	(obv., 3)
and (5)	x A b	(Barbara, 1, 4)
and, finally,	x I b.	(opp., 5)

S8. *For all a and b: if $\bar{a}$* A *b then for all x: if a* E *x then x* I *b.* (S7)

The Kneales' proof that II.4 implies '$\bar{a}$ A b', refers to the inadmissibility of a compound term '$a \cup b$'. In this case, again, we can dispense with the compound term and prove the required implication with the aid of S4. And the same applies to the remaining prosleptic premisses of the second group.

4 The third group of prosleptic premisses, in the scheme worked out by the Kneales, consists of the following twelve propositions: I.1, I.2, I.15, I.16, II.2, II.5, II.12, II.15, IV.1, IV.5, IV.12, and IV.16. Before we examine the conclusions arrived at by the Kneales, let us first note that:

I.2, II.2, II.5, and IV.5 are each equivalent to I.1 with 'b' replaced by '$\bar{b}$';

IV.1 is equivalent to I.1 with 'a' and 'b' replaced by '$\bar{a}$' and '$\bar{b}$' respectively;

IV.16 is equivalent to the converse of I.1;

I.15, II.12, II.15, and IV.12 are each equivalent to the converse of I.1 with 'a' and 'b' replaced by '$\bar{a}$';

I.16 is equivalent to the converse of I.1 with 'a' and 'b' replaced by '$\bar{a}$' and '$\bar{b}$' respectively.

The equivalences listed above can be established within the framework of syllogistic.

In the fourth group of prosleptic premisses the Kneales have included the following: I.9, I.10, I.13, I.14, II.9, II.10, II.13, II.14, III.9, III.10, III.13, III.14, IV.9, IV.10, IV.13, and IV.14, sixteen propositions altogether. Again, let us note that, within the framework of syllogistic,

I.14; II.9, II.14, and IV.9 are each equivalent to I.13 with 'b' replaced by '$\bar{b}$';

IV.13 is equivalent to the converse of I.13.

Now, if we apply ecthesis as made available by S4 then we can prove that the prosleptic premiss I.13, which in the classification suggested by the Kneales belongs to the fourth group of prosleptic premisses, is equivalent to the converse of I.1 with 'a' replaced by '$\bar{a}$'. Here is the proof:

S9. *For all a, b, and y: if (for all x: if b* A *x then x* A *$\bar{a}$) and a* O *y and it is not the case that y* A *b then y* A *b.*

Proof: For all a, b, and y:

if (1) for all x: if b A x then x A $\bar{a}$
and (2) a O y
and (3) it is not the case that y A b then
(4) y O b (opp., 3)
and (5) y I $\bar{b}$ (obv., 4)
and, for some z:
(6) z A y (S4, 5)
and (7) z A $\bar{b}$

and (8) z E b (obv., 7)
and (9) a O z (Baroco, 6, 2)
and (10) it is not the case that a A z (opp., 9)
and (11) it is not the case that a E $\bar{z}$ (obv., 10)
and (12) it is not the case that $\bar{z}$ E a (conv., 11)
and (13) it is not the case that $\bar{z}$ A $\bar{a}$ (obv., 12)
and (14) it is not the case that b A $\bar{z}$ (1, 13)
and (15) it is not the case that b E z (obv., 14)
and (16) it is not the case that z E b (conv., 15)
and, finally, y A b. (8, 16)

S10. *For all a, b, and y: if (for all x: if b* A *x then x* A $\bar{a}$*) and a* O *y then y* A *b.* (S9)

S11. *For all a, b, and y: if (for all x: if a* O *x then x* A *b) and b* A *y then y* A $\bar{a}$*.*

Proof: For all a, b, and y:

if (1) for all x: if a O x then x A b
and (2) b A y then
(3) it is not the case that b O y (opp., 2)
and (4) it is not the case that b I $\bar{y}$ (obv., 3)
and (5) it is not the case that $\bar{y}$ A b (conv., 4)
and (6) it is not the case that a O $\bar{y}$ (1, 5)
and (7) it is not the case that a I y (obv., 6)
and (8) a E y (opp., 7)
and (9) y E a (conv., 8)
and, finally, y A $\bar{a}$. (obv., 9)

S12. *For all a and b: (for all x: if a* O *x then x* A *b) if and only if (for all x: if b* A *x then x* A $\bar{a}$*).* (S10, S11)

It is evident from S12 that I.13 is equivalent to the converse of I.1 with 'a' replaced by '$\bar{a}$'.

The Kneales construe premiss I.1 as equivalent to '$\vee$ A b' with '$\vee$' meaning the same as 'entity' or as short for '$a \cup \bar{a}$', '$b \cup \bar{b}$', etc. Their analysis of I.1 can be reconstructed as follows: Suppose that

(1) for all x: if a A x then x A b;

from (1) we conclude, by substitution, that

(2) if a A $\vee$ then $\vee$ A b,

which yields

(3) $\vee$ A b

by modus ponens.

The last step in the deduction shows that among their presuppositions the Kneales have

K1. *For all* a: a A $\vee$,

which is also needed for the purpose of proving that '$\vee$ A b' implies I.1. This latter proof is as follows: Suppose that

(1) $\vee$ A b;

in accordance with K1 we have

(2) x A $\vee$;

by applying the syllogism Barbara to (1) and (2) we get

(3) x A b,

from which we conclude that

(4) if a A x then x A b;

by generalising (4) we derive I.1, that is to say,

(5) for all x: if a A x then x A b.

One could argue that K1 is intuitively sound as it simply means that every a is an entity, whatever a may be. However, the term '$\vee$', being universal, is inadmissible in traditional syllogistic, and of this the Kneales are well aware. Now, if the term '$\vee$' is inadmissible then so is K1. Moreover, when we describe K1 as inadmissible, we do not mean to say that it is independent of the ordinary presuppositions of syllogistic. K1 cannot in fact be added to the latter as a further presupposition. For if it were added to the presuppositions of syllogistic then we could infer from it that $\bar{\vee}$ A $\vee$, which in virtue of the square of oppositions implies that $\bar{\vee}$ I $\vee$. This result contradicts S2, and shows that within the framework of syllogistic K1 is either meaningless, and hence cannot serve as a presupposition in a proof, or false. And if it is false then, of course, it makes it possible to derive the equivalence of I.1 to '$\vee$ A b'. It also makes it possible to derive the non-equivalence of I.1 to '$\vee$ A b'. In proving that K1 is either meaningless or false, we made use of a negated term, and referred to S2 as a thesis of syllogistic. We must not, therefore, exclude the possibility that it is the way in which negated terms are used in traditional syllogistic that calls for revision while K1 remains unexceptionable. Be this as it may, the outcome of our discussion so far seems to be that any investigation of prosleptic premisses should be related to a set of unequivocally determined presuppositions.

With this in mind let us now analyse I.1 and establish an equivalence relationship which may throw some light on the meaning of that prosleptic premiss. If we can do this, we shall have solved the problem of the remaining premisses of the third group and also the problem of the six prosleptic premisses of the fourth group, each of which we have already said to be equivalent to a substitutional variant of I.1 or of its converse.

I am unable to offer a satisfactory analysis of I.1 within the framework of traditional syllogistic even if the latter were extended to provide for ecthesis. Ecthesis will be needed to be sure, but in addition a further extension of syllogistic appears to be required, an extension which does not, however, run the risk of inadvertently introducing inadmissible expressions into the system. The weakest thesis known to me which is strong enough for our purpose is this:

S13. *For all a, b, c, d, and e: if (for all f: if f* A *c then for some g: (g* A *f and (g* A *a or g* A *$\bar{b}$))) and e* A *d and for all g: if g* A *e then ((it is not the case that g* A *a) and (it is not the case that g* A *$\bar{b}$)) then for some j: for all h: h* A *j if and only if for all f: if f* A *h then for some g: (g* A *f and (g* A *a or g* A *$\bar{b}$)).*

In order to grasp the meaning of this lengthy proposition let us note that it is an instance of a more general but simpler thesis, which can be expressed as follows:

S14. *For all a, b, c, and ϕ: if (for all d: if d* A *a then ϕd) and c* A *b and it is not the case that ϕc then for some f: for all d: d* A *f if and only if for all e: if e* A *d then ϕe.*

S14 amounts to saying that if (i) there is a class a, whose every subclass satisfies a certain condition ϕ, and (ii) there is a class b, a certain subclass of which does not satisfy the condition ϕ, then (iii) there is a class f such that any class d is a subclass of f if and only if every subclass of d satisfies ϕ.

The reader will probably have noticed that S13 and S14, which is a generalisation of S13, have a certain affinity to axioms of reducibility, and hence to definitions. They are weaker than the corresponding definitions but while the definitions in question involve the use of inadmissible terms, no such terms occur in S13 or S14. However, in both these theses we have occurrences of the particular quantifier, which has already been used in connection with ecthesis.

We can now turn to prosleptic premiss I.1 and offer our analysis of it. This will consist in the deduction of the following theses:

S15. *For all a and b: if (for all x: if a* A *x then x* A *b) then a* A *b.* (S1)

S16. *For all a and b: if (for all x: if a* A *x then x* A *b) and it is not the case that b* A *a then b* A *a.*

Proof: For all a and b:

if (1) for all x: if a A x then x A b
and (2) it is not the case that b A a then
(3) for all c: if c A a then for some d: (d A c and (d A a or d A $\bar{b}$)) (S1)
and (4) for all c: if c A $\bar{b}$ then for some d: (d A c and (d A a or d A $\bar{b}$)) (S1)
and (5) b O a (opp., 2)
and (6) b I $\bar{a}$ (obv., 5)
and for some c:
(7) c A b
and (8) c A $\bar{a}$ (S4, 6)
and (9) c E a (obv., 8)

and (10) c E $\bar{b}$ (obv., 7)
and (11) it is not the case that c I a (opp., 9)
and (12) it is not the case that c I $\bar{b}$ (opp., 10)
and (13) for all e: if e A c then it is not the case that e A a (Darapti, 11)
and (14) for all e: if e A c then it is not the case that e A $\bar{b}$ (Darapti, 12)
and (15) for some d: (d A c and for all e: if e A d then ((it is not the case the e A a) and (it is not the case that e A $\bar{b}$))) (S1, 13, 14)
and for some f:
(16) for all e: e A f if and only if for all c: if c A e then for some d: (d A c and (d A a or d A $\bar{b}$)) (S13, 3, 15)
and (17) a A f (16, 3)
and (18) $\bar{b}$ A f (16, 4)
and (19) f A b (1, 17)
and (20) $\bar{b}$ A b (Barbara, 19, 18)
and (21) $\bar{b}$ I b (opp., 20)
and, finally, b A a. (21, S2)

S17. *For all a and b: if (for all x: if a* A *x then x* A *b) then b* A *a.* (S16)

S18. *For all a, b, and y: if (for all x: if a* A *x then x* A *b) and (it is not the case that* $\bar{b}$ A *y) and (it is not the case that* $\bar{b}$ A $\bar{y}$*) then* ($\bar{b}$ A *y or* $\bar{b}$ A $\bar{y}$).

Proof: For all a, b, and y:

if (1) for all x: if a A x then x A b
and (2) it is not the case that $\bar{b}$ A y
and (3) it is not the case that $\bar{b}$ A $\bar{y}$ then
(4) a A b (1, S1)
and (5) $\bar{b}$ O y (opp., 2)
and (6) $\bar{b}$ I $\bar{y}$ (obv., 5)
and (7) $\bar{b}$ O $\bar{y}$ (opp., 3)
and (8) $\bar{b}$ I y (obv., 7)
and for some c:
(9) c A $\bar{b}$
and (10) c A $\bar{y}$ (S4, 6)
and (11) c E b (obv., 9)
and (12) c E a (Camestres, 4, 11)
and (13) it is not the case that c I a (opp., 12)
and (14) c E y (obv., 10)
and for some d:
(15) d A $\bar{b}$
and (16) d A y (S4, 8)
and (17) c E d (Camestres, 16, 14)
and (18) it is not the case that c I d (opp., 17)
and (19) for all e: if e A a then for some f: (f A e and (f A a or f A d)) (S1)
and (20) for all e: if e A d then for some f: (f A e and (f A a or f A d)) (S1)

and (21) for all f: if f A c then it is not the case that f A a (Darapti, 13)
and (22) for all f: if f A c then it is not the case that f A d (Darapti, 18)
and (23) for some e: (e A c and for all f: if f A e then ((it is not the case that f A a) and (it is not the case that f A d))) (S1, 21, 22)
and for some h:
(24) for all g: g A h if and only if for all e: if e A g then for some f: (f A e and (f A a or f A d)) (S13, 19, 23)
and (25) a A h (24, 19)
and (26) d A h (24, 20)
and (27) h A b (1, 25)
and (28) d A b (Barbara, 27, 26)
and (29) d E $\bar{b}$ (obv., 28)
and (30) it is not the case that d A $\bar{b}$ (opp., 29)
and, finally, $\bar{b}$ A y or $\bar{b}$ A $\bar{y}$. (15, 30)

S19. *For all a, b, and y: if (for all x: if a* A *x then x* A *b) then* ($\bar{b}$ A y *or* $\bar{b}$ A $\bar{y}$). (S18)

S20. *For all a and b: if (for all x: if a* A *x then x* A *b) then for all x:* ($\bar{b}$ A x *or* $\bar{b}$ A $\bar{x}$). (S19)

In view of S20, I.1 implies that there exists exactly one non-b. For it is this that the proposition 'for all x: ($\bar{b}$ A x or $\bar{b}$ A $\bar{x}$)' means. In order to complete our analysis of I.1 we now go on to prove

S21. *For all a, b, and y: if b* A *a and (for all x:* ($\bar{b}$ A x *or* $\bar{b}$ A $\bar{x}$)) *and a* A *y then y* A *b.*

Proof: For all a, b, and y:

if (1) b A a
and (2) for all x: ($\bar{b}$ A x or $\bar{b}$ A $\bar{x}$)
and (3) a A y then
(4) b A y (Barbara, 3, 1)
and (5) it is not the case that $\bar{y}$ A y (opp., S2)
and (6) it is not the case that $\bar{y}$ A b (Barbara, 4, 5)
and (7) it is not the case that $\bar{y}$ E $\bar{b}$ (obv., 6)
and (8) it is not the case that $\bar{b}$ E $\bar{y}$ (conv., 7)
and (9) it is not the case that $\bar{b}$ A y (obv., 8)
and (10) $\bar{b}$ A $\bar{y}$ (2, 9)
and (11) $\bar{b}$ E y (obv., 10)
and (12) y E $\bar{b}$ (conv., 11)
and, finally, y A b. (obv., 12)

S22. *For all a and b: if b* A *a and (for all x:* ($\bar{b}$ A x *or* $\bar{b}$ A $\bar{x}$)) *then for all x: if a* A *x then x* A *b.* (S21)

S23. *For all a and b: (for all x: if a* A *x then x* A *b) if and only if (a* A *b and b* A *a and for all x:* ($\bar{b}$ A x *or* $\bar{b}$ A $\bar{x}$)). (S15, S17, S20, S22)

It is evident from S23 that I.1 is equivalent to a conjunction of three propositions two of which are categorical. Between them the three premisses say that only every a is a b and that there exists only one object which is not a b, i.e., which is non-b. They imply, of course, that non-b is the same object as non-a.

5 The following prosleptic premisses of the fourth group still remain to be considered: I.9, I.10, II.10, II.13, III.9, III.10, III.13, III.14, IV.10, and IV.14. We begin by noting that

III.9 is equivalent to I.9;

I.10, II.10, II.13, and III.10 are each equivalent to I.9 with 'b' replaced by '$\bar{b}$';

III.13 is equivalent to I.9 with 'a' replaced by '$\bar{a}$';

III.14 and IV.14 are each equivalent to the converse of I.9;

IV.10 is equivalent to the converse of I.9 with 'a' replaced by '$\bar{a}$'.

The above listed equivalences can be easily established within the framework of traditional syllogistic, given, of course, the rules for operating with the universal quantifier. We are thus left with I.9 and in order to elucidate its meaning we go on to derive the following theses:

S24. *For all a and b: if (for all x: if a I x then x A b) then a A b.* (opp., S1)

S25. *For all a and b: if (for all x: if a I x then x A b) and it is not the case that b A a then b A a.*

Proof: For all a and b:

if (1) for all x: if a I x then x A b		
and (2) it is not the case that b A a then		
(3) b O a		(opp., 2)
and (4) b I $\bar{a}$		(obv., 3)
and for some c:		
(5)	c A b	(S4, 4)
and (6)	c A $\bar{a}$	
and (7)	c E a	(obv., 6)
and (8)	a E c	(conv., 7)
and (9)	a A $\bar{c}$	(obv., 8)
and (10)	a I $\bar{c}$	(opp., 9)
and (11)	$\bar{c}$ A b	(1, 10)
and (12)	$\bar{c}$ E $\bar{b}$	(obv., 11)
and (13)	$\bar{b}$ E $\bar{c}$	(conv., 12)
and (14)	$\bar{b}$ A c	(obv., 13)
and (15) $\bar{b}$ A b		(Barbara, 5, 14)
and (16) $\bar{b}$ I b		(opp., 15)
and, finally, b A a.		(16, S2)

S26. *For all a and b: if (for all x: if a I x then x A b) then b A a.* (S25)

Incidentally, S26 enables us to prove that I.9 is equivalent to its converse but the proof need not be given in the present context. We continue by deducing

S27. *For all a, b, and y: if (for all x: if a I x then x A b) and (it is not the case that a A y) and (it is not the case that a A $\bar{y}$) then (a A y or a A $\bar{y}$).*

Proof: For all a, b, and y:

if (1) for all x: if a I x then x A b
and (2) it is not the case that a A y
and (3) it is not the case that a A $\bar{y}$ then
(4) a O y (opp., 2)
and (5) a I $\bar{y}$ (obv., 4)
and (6) $\bar{y}$ A b (1, 5)
and (7) a O $\bar{y}$ (opp., 3)
and (8) a I y (obv., 7)
and (9) y A b (1, 8)
and (10) y E $\bar{b}$ (obv., 9)
and (11) $\bar{b}$ E y (conv., 10)
and (12) $\bar{b}$ A $\bar{y}$ (obv., 11)
and (13) $\bar{b}$ A b (Barbara, 6, 12)
and (14) $\bar{b}$ I b (opp., 13)
and, finally, a A y or a A $\bar{y}$. (14, S2)

S28. *For all a, b, and y: if (for all x: if a I x then x A b) then (a A y or a A $\bar{y}$).* (S27)

S29. *For all a and b: if (for all x: if a I x then x A b) then for all x: (a A x or a A $\bar{x}$).* (S28)

S30. *For all a, b, and y: if (for all x: if a I x then x A b) and (it is not the case that $\bar{b}$ A y) and (it is not the case that $\bar{b}$ A $\bar{y}$) then ($\bar{b}$ A y or $\bar{b}$ A $\bar{y}$).*

Proof: For all a, b, and y:

if (1) for all x: if a I x then x A b
and (2) it is not the case that $\bar{b}$ A y
and (3) it is not the case that $\bar{b}$ A $\bar{y}$ then
(4) $\bar{b}$ O y (opp., 2)
and (5) $\bar{b}$ I $\bar{y}$ (obv., 4)
and (6) $\bar{y}$ I $\bar{b}$ (conv., 5)
and (7) it is not the case that $\bar{y}$ E $\bar{b}$ (opp., 6)
and (8) it is not the case that $\bar{y}$ A b (obv., 7)
and (9) it is not the case that a I $\bar{y}$ (1, 8)
and (10) a E $\bar{y}$ (opp., 9)
and (11) $\bar{b}$ O $\bar{y}$ (opp., 3)
and (12) $\bar{b}$ I y (obv., 11)
and (13) y I $\bar{b}$ (conv., 12)
and (14) y O b (obv., 13)
and (15) it is not the case that y A b (opp., 14)
and (16) it is not the case that a I y (1, 15)
and (17) a E y (opp., 16)
and (18) a A $\bar{y}$ (obv., 17)
and (19) a E a (Cesare, 10, 18)

and (20) it is not the case that a A a (opp., 19)
and, finally, $\bar{b}$ A y or $\bar{b}$ A $\bar{y}$. (S1, 20)

S31. *For all a, b, and y: if (for all x: if a I x then x A b) then ($\bar{b}$ A y or $\bar{b}$ A $\bar{y}$).* (S30)

S32. *For all a and b: if (for all x: if a I x then x A b) then for all x: ($\bar{b}$ A x or $\bar{b}$ A $\bar{x}$).* (S31)

S33. *For all a, b, and y: if b A a and (for all x: (a A x or a A $\bar{x}$)) and (for all x: ($\bar{b}$ A x or $\bar{b}$ A $\bar{x}$)) and a I y and it is not the case that y A b then y A b.*

Proof: For all a, b, and y:

if (1) b A a
and (2) for all x: a A x or a A $\bar{x}$
and (3) for all x: $\bar{b}$ A x or $\bar{b}$ A $\bar{x}$
and (4) a I y
and (5) it is not the case that y A b then
(6) y O b (opp., 5)
and (7) y I $\bar{b}$ (obv., 6)
and (8) $\bar{b}$ I y (conv., 7)
and (9) $\bar{b}$ O $\bar{y}$ (obv., 8)
and (10) it is not the case that $\bar{b}$ A $\bar{y}$ (opp., 9)
and (11) $\bar{b}$ A y (3, 10)
and (12) a O $\bar{y}$ (obv., 4)
and (13) it is not the case that a A $\bar{y}$ (opp., 12)
and (14) a A y (2, 13)
and (15) b A y (Barbara, 14, 1)
and (16) b E $\bar{y}$ (obv., 15)
and (17) $\bar{y}$ E b (conv., 16)
and (18) $\bar{y}$ A $\bar{b}$ (obv., 17)
and (19) $\bar{y}$ A y (Barbara, 11, 18)
and (20) $\bar{y}$ I y (opp., 19)
and, finally, y A b. (20, S2)

S34. *For all a, b, and y: if b A a and (for all x: (a A x or a A $\bar{x}$)) and (for all x: ($\bar{b}$ A x or $\bar{b}$ A $\bar{x}$)) and a I y then y A b.* (S33)

S35. *For all a and b: if b A a and (for all x: (a A x or a A $\bar{x}$)) and (for all x: ($\bar{b}$ A x or $\bar{b}$ A $\bar{x}$)) then for all x: if a I x then x A b.* (S34)

S36. *For all a and b: (for all x: if a I x then x A b) if and only if (a A b and b A a and for all x: (a A x or a A $\bar{x}$) and for all x: ($\bar{b}$ A x or $\bar{b}$ A $\bar{x}$)).*
(S24, S26, S29, S32, S35)

We can see from S36 that I.9 is equivalent to a conjunction of four propositions which between them tell us that (i) only every a is a b, and that (ii) there exists only one a, and that (iii) there exists exactly one object which is not a b. In other words I.9 amounts to saying that there exist two objects only: one of them is a, or b, since a and b are one and the same object, and the other is non-a, or non-b, since non-a and non-b are again one and the same object. In a sufficiently weak system, such as traditional syllogistic extended by the inclusion of S4 and S13 among its presuppositions, I.9 does not lead to a contradiction, but it does so, for instance, if on being properly translated it is added to a system of mereology. For in mereology one can prove that if there are two different objects then there is a third object different from either of the two.

6 Within the limits of the present paper our analysis of prosleptic premisses with reference to traditional syllogistic must now be concluded. We have found, as the Kneales have done, that prosleptic premisses conveniently fall into four groups. The prosleptic premisses of the first two groups, which are the same as those distinguished by the Kneales, are each equivalent to a categorical proposition. However, in order to prove this for the prosleptic premisses of the second group we have extended traditional syllogistic by including among its presuppositions the principle of ecthesis. In this way we have avoided using inadmissible terms. We have had no recourse to inadmissible terms in connection with the analysis of the remaining prosleptic premisses. And this is why our results differ from those arrived at by the Kneales. Traditional syllogistic, which allows for the use of negated terms but does not admit terms that are empty or universal, appears to provide an appropriate framework for investigating the meaning of prosleptic premisses even though that framework has to be extended. This does not mean that it would be wrong to analyse the meaning of prosleptic premisses within the framework of a system whose language favours different restrictions as regards the range of substituends for the variables. Thus, for instance, one could examine the logical import of prosleptic premisses within the framework of a system whose language allows for the use of universal terms but does not admit empty terms or negated terms. Alternatively, one could relate one's enquiry to a system which, like that of Leśniewski's, has no inadmissible terms. Whichever strategy is adopted, the need for specifying one's presuppositions cannot be overemphasised.

As is well known, Aristotle syllogistic, as axiomatised by Łukasiewicz, is based on the following presuppositions:

A1. *For all a*: a A a.
A2. *For all a*: a I a.
A3. *For all a, b, and c*: *if* b A c *and* a A b *then* a A c.
A4. *For all a, b, and c*: *if* b A c *and* b I a *then* a I c.

The directives of the system include three rules of inference, namely substitution, quantification, and detachment, and the rule of propositional definition, which makes it possible for us to introduce into the system proposition-forming functors such as 'E' and 'O'. In view of A1 and A2, and taking into consideration the usual meaning of the constant functors 'A' and 'I', it becomes obvious that empty terms are not admissible in the language of the system. However, so far there is nothing in the system to stop us from using universal terms. Thus, for instance, K1, which says that

For all a: a A $\vee$,

could be added to the presuppositions of the system. We could also introduce into the system propositions which determine the use of such compound terms as '$a \cup b$', as a term of this form can never be empty provided either 'a' or 'b' is not empty. On the other hand compound terms such as '$\bar{a}$' or '$a \cap b$' are not admissible because they may turn out to be empty, and hence, inadmissible, even if the terms substituted for the variables are themselves not empty.

The totality of theses that can be derived from A1 - A4 in virtue of the directives mentioned above can conveniently be called Aristotelian syllogistic. If, to the presuppositions of Aristotelian syllogistic, we add:

A5. *For all a and b*: a A $\bar{b}$ *if and only if it is not the case that* a I b.
A6. *For all a and b*: a I $\bar{b}$ *if and only if it is not the case that* a A b.

without changing the directives, then we obtain a system of what in this essay I have referred to as traditional syllogistic. Now, A5 and A6 make it clear that in the language of traditional syllogistic universal terms and, consequently, compound terms such as '$a \cup b$' are not admissible. For by negating a universal term we get an empty term, and among compound terms of the form '$a \cup b$' we can have some that are universal.

However, for the purpose of the present essay traditional syllogistic has turned out to be too weak. We have, therefore, extended it by adopting two further presuppositions:

A7 (=S4). *For all a and b*: *if* a I b *then for some* c: c A a *and* c A b.

A8 (=S13). *For all a, b, c, d, and e*: *if* (*for all f*: *if* f A c *then for some g*: (g A f *and* (g A a *or* g A $\bar{b}$))) *and* e A d *and for all g*: *if* g A e *then* (*it is not the case that* g A a *and it is not the case that* g A $\bar{b}$) *then for some j*: *for all h*: h A j *if and only if for all f*: *if* f A h *then for some g*: (g A f *and* (g A a *or* g A $\bar{b}$)).

Neither A7 nor A8 introduces into the language of traditional syllogistic any inadmissible terms. Moreover, the set of presuppositions consisting of A1 - A7 and including S14, which is a generalisation of A8, can be shown to be consistent if protothetic, i.e., a generalised logic of propositions, is consistent. For let us interpret the term variables of traditional syllogistic as propositional variables and the variable 'ϕ' occurring in S14 as a variable which belongs to the category of proposition-forming functors for one propositional argument; and let us interpret the constant 'A', and the constant 'I', as the functor of equivalence, i.e., as 'if and only if', and the functor '$^-$', which in traditional syllogistic forms negated terms, as the functor of propositional negation, i.e., as 'it is not the case that'. It is obvious that on this interpretation the protothetical analogues of A1 - A7 are theses of protothetic. And so is the protothetical analogue of S14, which reads as follows:

P. *For all p, q, r, and* δ: *if* ((*for all s*: *if* (*s if and only if p*) *then* δs) *and* (*r if and only if q*) *and* (*it is not the case that* δr)) *then for some s*: *for all t*: (*t if and only if s*) *if and only if for all u*: *if* (*u if and only if t*) *then* δu.

In order to convince ourselves that P is a thesis of protothetic we go on to prove

P1. *For all p, t, u, and δ: if ((for all s: if (s if and only if p) then δs) and (t if and only if p) and (u if and only if t)) then δu.*

Proof: For all p, t, u, and δ:

if (1) for all s: if (s if and only if p) then δs
and (2) t if and only if p
and (3) u if and only if t then
(4) u if and only if p (3, 2)
and, finally, δu. (1, 4)

P2. *For all p, t, and δ: if ((for all s: if (s if and only if p) then δs) and (t if and only if p)) then for all u: if (u if and only if t) then δu.* (P1)

P3. *For all p, r, t, and δ: if ((for all s: if (s if and only if p) then δs) and (it is not the case that δr) and (for all u: if (u if and only if t) then δu)) then t if and only if p.*

Proof: For all p, r, t, and δ:

if (1) for all s: if (s if and only if p) then δs
and (2) it is not the case that δr
and (3) for all u: if ((u if and only if t) then δu) then
(4) it is not the case that (r if and only if p) (1, 2)
and (5) it is not the case that (r if and only if t) (3, 2)
and, finally, t if and only if p. (4, 5)

P4. *For all p, r, t, and δ: if ((for all s: if (s if and only if p) then δs) and (it is not the case that δr)) then ((t if and only if p) if and only if for all u: if (u if and only if t) then δu).* (P2, P3)

P5. *For all p, r, and δ: if ((for all s: if (s if and only if p) then δs) and (it is not the case that δr)) then for all t: (t if and only if p) if and only if for all u: if (u if and only if t) then δu.* (P4)

P6. *For all p, r, and δ: if ((for all s: if (s if and only if p) then δs) and (it is not the case that δr)) then for some s: for all t: (t if and only if s) if and only if for all u: if (u if and only if t) then δu.* (P5)

Since P is a simple consequence of P6, the proof that our extended system of traditional syllogistic is consistent if protothetic is consistent, can now be said to have been concluded. It hardly be added that under the considered interpretation the directives of traditional syllogistic become valid directives of protothetic.

University of Manchester
Manchester, England