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Illative Combinatory Logic Without Equality as a Primitive Predicate

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Introduction Combinatory logic has the simplest formal framework of any system of logic or mathematics. It has a finite set of primitive constants (or obs) which include K and S, the basic combinators. There is only one very simple formation rule:

If X and Y are obs so is (XY).

These obs include all the operators, expressions, terms, well-formed formulas, etc., of the system.

Pure combinatory logic has one predicate (=). A predicate in this sense is not an ob, it allows us to make statements about obs.

If X, Y, and Z are obs we assume the following axioms:

$$KXY = X$$
$$SXYZ = XZ(YZ)$$

as well as the rules:

(σ) If $X = Y$ then $Y = X$.
(τ) If $X = Y$ then $Y = Z$ then $X = Z$.
(μ) If $X = Y$ then $ZX = ZY$.
(θ) If $X = Y$ then $XZ = YZ$.

Illative (i.e., applied) combinatory logic has additional constant obs (which could also be part of any pure system) and it has, in its usual form, an additional predicate $\vdash$, as well as some axioms and rules involving the new obs = and $\vdash$.

Having listed the primitive concepts of a formal theory many authors would then attempt to provide a semantics, i.e., they would explain these

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concepts in terms of simpler or more basic ones. Because combinatory logic deals with very basic concepts this is difficult,[1] and the semantics is likely to be more complex than the original theory. Also, since the formation rules allow complete generality in the formation of obs, many of these will have at most a very contrived interpretation. Therefore, we will not attempt to give a semantics here.[2]

The usefulness of illative combinatory logic is easier to explain. In this logic it is possible to define propositional connectives, quantifiers, and various set theoretic and arithmetical notions and to derive their properties in a simple way. (See [5] for a summary of some of these developments.)

The formal framework of illative combinatory logic is clearly much simpler than that of other logical systems since it does not need to distinguish between various kinds of primitive objects and has only one formation rule. It is, however, more complicated in that it usually has two predicates, $\vdash$ and $=$.

Curry and Feys have shown in §7C of [9] that it is possible, in a system that has no illative rules, to replace the predicate $=$ by a constant ob Q so that

(1) $X = Y$ if and only if $\vdash QXY$.

Because it can be shown that $K \neq S$, we have that QKS is not provable and so the illative system (called $\mathcal{L}$ in [9]) is consistent.

In this paper we are interested in showing that the illative systems that have been developed with $\vdash$ and $=$ as basic predicates can be reformulated with only $\vdash$. We do this in two ways: in the first we extend $\mathcal{L}$, and in the second we extend an alternative version of $\mathcal{L}$ with illative axioms and rules.

In these new systems we cannot hope to prove (simply) both implications of (1) since formulas of the form QXY could conceivably be proved using illative rules that have no counterparts in the pure system. If we were able to prove (1), the system would be said to be Q-consistent.[3]

In general, illative combinatory logic can be set up in one of two ways: as a system mainly of rules including an introduction rule for restricted generality Ξ[4] (such systems we will call *natural deduction systems*), or as a system using mainly axioms (called a "finite formulation" by Curry, Hindley, and Seldin in [10]).

One of the strengths of combinatory logic is that it does not require the use of variables and hence avoids complicated substitution rules which arise when both bounded and free variables are involved. In natural deduction systems, however, when the introduction rule for Ξ is used, variables are unavoidable even if none appear in any theorem that is proved. In finite formulations no variables should be necessary, however in §15C of [10], $\vdash Ex$ is still postulated for a variable x and it is clearly assumed, though not stated, that the axiom schemes and rules for $=$ and $\vdash$ may involve obs containing variables. In [16] $\vdash Ex$ is also postulated and in [2], although the axiom (with $E = WQ$) is not specifically mentioned prior to the proof of the deduction theorem for Ξ, it is allowed for there. Clearly though, if variables are not needed axioms and rules involving them should also not be needed.

In this paper we prove a deduction theorem for Ξ, equivalent to the Ξ introduction rule above, in the two finite formulation systems that have only $\vdash$ as a predicate. In doing so we show, for this as well as for other systems, that if Y can be proved using a hypothesis X involving a variable u, using all the axiom schemes and rules of the system extended as though u were a primitive ob, then given a restriction on X, $\Xi(\lambda uX)(\lambda uY)$[5] can be proved using only axiom schemes and rules not involving u.

The deduction theorem (for either system) we get here is exactly that of [2] so the work (such as that listed in [5]) which has been developed on the basis of the deduction theorem in [2], can also be developed from the work in this paper.

The axioms and rules for = in [2] are of course replaced here by axioms and rules for Q. Some of the remaining axioms in [2] are altered slightly to make the proof of the deduction theorem completely independent of the choice of L, the class of obs over which the theorem can be carried out. This has major advantages when we wish to consider higher-order logics (as in [7]).

Finally in this paper we compare several versions of the system in which the deduction theorem holds and also compare these with some other systems, particularly those of Church [8] and Goodman [12].

The altered system $\mathcal{Q}$ We assume that we have at least Q, K, and S as primitive constants and that we have the following axiom schemes and rules:

Axiom Schemes

(K$_1$) $\vdash QX(KXY)$
(K$_2$) $\vdash Q(KXY)X$
(S$_1$) $\vdash Q(XZ(YZ))(SXYZ)$
(S$_2$) $\vdash Q(SXYZ)(XZ(YZ))$

Rules

Eq $QXY, X \vdash Y$
(μ) $QXY \vdash Q(UX)(UY)$

where X, Y, Z, and U are obs formed from the primitive constants by application. (The names of the axioms and rules are similar to those of rules in [9].)

We now prove:

Theorem 1

(i) $QXY, QZX \vdash QZY$
(ii) $\vdash QXX$
(iii) $QXY \vdash QYX$.

Proof: (i) By (μ), $QXY \vdash Q(QZX)(QZY)$
so by Eq $QXY, QZX \vdash QZY$.
(ii) By (K$_1$), (K$_2$) and (i).
(iii) By (S$_1$) $\vdash Q[QX(KXX)][SQ(KX)X]$,
by (K$_1$) $\vdash QX(KXX)$
so by Eq $\vdash SQ(KX)X$,
and then by (μ) and Eq $QXY \vdash SQ(KX)Y$.
Now by (S$_2$) $\vdash Q(SQ(KX)Y)(QY(KXY))$,
so by Eq $QXY \vdash QY(KXY)$.
By (K$_2$) $\vdash Q(KXY)X$,
so by (i) $QXY \vdash QYX$.

All substitution properties for equality follow for Q from Eq and (μ).

We have simplified the system of [9] in that we now only have two rules instead of seven, but we have complicated it as we now have four axiom schemes instead of only one ($\vdash QXX$ in [9]). This is unavoidable even if we have restricted generality (Ξ) and attempt to write, say, (K_1) as an axiom, i.e.,

$$\vdash WQx \supset_x WQy \supset_y Qx(Kxy)^6.$$

Written formally this is:

$$\vdash \Xi(WQ)\{S[K(\Xi(WQ))][S(S(KB)Q)K]\}$$

where W and B should also be replaced by their definitions in terms of K and S.

To obtain Rule (K_1) from this, all of (K_1), (K_2), (S_1), and (S_2) are needed.

The deduction theorem In addition to the primitives K, S, and Q, we will take Ξ with the elimination rule:

Rule Ξ $\quad \Xi XY, XU \vdash YU$,

and the extra axioms:

Axiom 1 $\quad \vdash Lx \supset_x . \, xu \supset_u xu$

Axiom 2 $\quad \vdash Lx \supset_x . \, y \supset_y (xu \supset_u y)$

Axiom 3 $\quad \vdash Lx \supset_x .\!\cdot\!. \, WQt \supset_t : (xu \supset_u yu(tu)) \supset_y . \, (xu \supset_u (yuv \supset_v zuv)) \supset_z (xu \supset_u zu(tu))$

Axiom 4 $\quad \vdash Lx \supset_x : (xu \supset_u zu) \supset_z . \, (xu \supset_u Q(zu)(yu)) \supset_y (xu \supset_u yu)$

Axiom 5 $\quad \vdash Lx \supset_x .\!\cdot\!. \, WQv \supset_v : WQw \supset_w . \, (xu \supset_u Q(vu)(wu)) \supset_z (xu \supset_u Q((zu)(vu))((zu)(wu)))$

Axiom 6 $\quad \vdash Lx \supset_x .\!\cdot\!. \, WQy \supset_y : WQz \supset_z . \, xu \supset_u Q(yu)(K(yu)(zu))$

Axiom 7 $\quad \vdash WQy \supset_y .\!\cdot\!. \, WQz \supset_z : Lx \supset_x . \, xu \supset_u Q(K(yu)(zu))(yu)$

Axiom 8 $\quad \vdash WQy \supset_y :: WQz \supset_z .\!\cdot\!. \, WQw \supset_w : Lx \supset_x . \, xu \supset_u Q[(yu)(wu)((zu)(wu))][S(yu)(zu)(wu)]$

Axiom 9 $\quad \vdash WQy \supset_y :: WQz \supset_z .\!\cdot\!. \, WQw \supset_w : Lx \supset_x . \, xu \supset_u Q[S(yu)(zu)(wu)][(yu)(wu)((zu)(wu))]$.

Of these, Axioms 1 and 3 are identical to those in [2], however the alternative form of Axiom 2 means that Axioms 6 and 7 of [2]: $\vdash x \supset_x Hx$ and $\vdash LH$, where 'HX' stands for 'X is a proposition', do not need to be assumed. Also there is no need to make any assumptions regarding the definition of L. (In [2] L was FAH, but it was shown later that the deduction theorem could be extended to hold if $L = FUH$ for various values of U. In later publications it was also shown that the use of $\vdash LH$ could be avoided in various ways.)

Before proving the deduction theorem using these axioms, we need to introduce some extra notation. We will assume that the free variables introduced in a proof using m hypotheses are $u_1, u_2, \ldots, u_m$ and that the hypothesis involving u_m is the first to be eliminated.

If the set of obs is extended to include $u_1, u_2, \ldots, u_m$ and others formed using the primitive constants and these by application, we will denote the corresponding extended axiom schemes and rules by $(K_1)_m$, $(K_2)_m$, $(S_1)_m$, $(S_2)_m$, and $(Eq)_m$ and $(\mu)_m$.

If Δ is the set of all the hypotheses in a deduction of an ob Y using these extended axioms and rules as well as Rule Ξ and Axioms 1-9, we will write:

$$\Delta \vdash_{\overline{m}} Y.$$

We can now state and prove the deduction theorem in the following form:

The Deduction Theorem for Ξ *If Δ, $X \vdash_{\overline{m}} Y$, where u_m is not in Δ, then Δ, $L(\lambda u_m X) \vdash_{\overline{m-1}} X \supset_{u_m} Y$.*

Proof: Let there be n steps $Y_1, Y_2, \ldots, Y_n = Y$ in the proof of Y from Δ and X. We show by induction on k that:

$$\Delta \vdash_{\overline{m-1}} X \supset_{u_m} Y_k.$$

We consider six cases, the first three of which constitute the $k = 1$ case. All six cases apply when we have assumed $\Delta \vdash_{\overline{m-1}} X \supset_{u_m} Y_i$ for $1 \leqslant i \leqslant k - 1$ and attempt to prove this for $i = k$.

Case 1. Y_k is X.
By Axiom 1, $L(\lambda u_m X) \vdash X \supset_{u_m} X$,
so obviously $L(\lambda u_m X) \vdash_{\overline{m-1}} X \supset_{u_m} X$.

Case 2. Y_k is an axiom or is in Δ and has no free u_m in it.
By Axiom 2, $L(\lambda u_m X), Y_k \vdash X \supset_{u_m} Y_k$,
so as in this case $\Delta \vdash_{\overline{m-1}} Y_k$
we have $\Delta, L(\lambda u_m X) \vdash_{\overline{m-1}} X \supset_{u_m} Y_k$.

Case 3. Y_k is an instance of $(K_1)_m$, $(K_2)_m$, $(S_1)_m$, or $(S_2)_m$ that involves u_m. In the first case let $Y_k = QV(KVZ)$, then $\lambda u_m V$ and $\lambda u_m Z$ have at most $u_1, u_2, \ldots, u_{m-1}$ as free variables so $\vdash_{\overline{m-1}} WQ(\lambda u_m V)$ and $\vdash_{\overline{m-1}} WQ(\lambda u_m Z)$ and hence by Axiom 6 we have:

$$L(\lambda u_m X) \vdash_{\overline{m-1}} X \supset_{u_m} QV(KVZ).$$

Similarly the results for $(K_2)_m$, $(S_1)_m$, and $(S_2)_m$ follow from Axioms 7, 8, and 9.

Case 4. Y_k is obtained from Y_i and Y_j by Rule $(Eq)_m$. By the inductive hypothesis we have: $\Delta, L(\lambda u_m X) \vdash_{\overline{m-1}} X \supset_{u_m} QZY_k$ and $\Delta, L(\lambda u_m X) \vdash_{\overline{m-1}} X \supset_{u_m} Z$, for some Z, so by Axiom 4 we have:

$$\Delta, L(\lambda u_m X) \vdash_{\overline{m-1}} X \supset_{u_m} Y_k.$$

Case 5. Y_k is obtained from Y_i by Rule $(\mu)_m$. By the inductive hypothesis if $Y_k = Q(ZV)(ZW)$ and $Y_i = QVW$, we have

$$\Delta, L(\lambda u_m X) \vdash_{\overline{m-1}} X \supset_{u_m} QVW$$

so by Axiom 5 as $\vdash_{\overline{m-1}} TQ(\lambda u_m V)$ and $\vdash_{\overline{m-1}} TQ(\lambda u_m W)$ we have

$$\Delta, L(\lambda u_m X) \vdash_{\overline{m-1}} X \supset_{u_m} Q(ZV)(ZW).$$

Case 6. Y_k is obtained from Y_i and Y_j by Rule Ξ. By the inductive hypothesis, if Y_j is $Zv \supset_v Wv$, Y_i is ZV and Y_k is WV,

$$\Delta, L(\lambda u_m X) \vdash_{\overline{m-1}} X \supset_{u_m} (Zv \supset_v Wv)$$

and Δ, $L(\lambda u_m X) \vdash_{\overline{m-1}} X \supset_{u_m} ZV$. Thus by Axiom 3 and $\vdash_{\overline{m-1}} TQ(\lambda u_m V)$ we have:

$$\Delta, L(\lambda u_m X) \vdash_{\overline{m-1}} X \supset_{u_m} WV.$$

Thus the theorem holds in all cases.

It follows easily from this that:

Corollary 1 *If $X_1u_1, X_2u_2, \ldots, X_mu_m \vdash_{\overline{m}} Y$ where each u_i is not in X_j for $j \leqslant i$, then*

$$\vdash X_1u_1 \supset_{u_1}: X_2u_2 \supset_{u_2} \ldots X_mu_m \supset_{u_m} Y.$$

Thus all uses of hypotheses and all uses of the versions of (K_1), (K_2), (S_1), (S_2), Eq, and (μ) involving variables can be eliminated from a proof.

This theorem (and its corollary) could equally well be proved if we had the rules of [9] replacing (K_1), (K_2), (S_1), and (S_2) (and also if we had extra (constant) axioms). To allow for these rules, we would need instead of Axioms 6, 7, 8, and 9:

Axiom 6′ $\vdash WQy \supset_y \therefore WQz \supset_z: Lx \supset_x (xu \supset_u wu(K(yu)(zu)))$
$\supset_w (xu \supset_u wu(yu))$

Axiom 7′ $\vdash WQy \supset_y \therefore WQz \supset_z: Lx \supset_x (xu \supset_u wu(yu))$
$\supset_w (xu \supset_u wu(K(yu)(zu)))$

Axiom 8′ $\vdash WQy \supset_y :: WQz \supset_z \therefore WQv \supset_v: Lx \supset_x.$
$[xu \supset_u wu(yu(vu)(zu(vu)))] \supset_w [xu \supset_u wu(S(yu)(zu)(vu))]$

Axiom 9′ $\vdash WQy \supset_y :: WQz \supset_z \therefore WQv \supset_v: Lx \supset_x.$
$[xu \supset_u wu(S(yu)(zu)(vu))] \supset_w [xu \supset_u wu(yu(vu)(zu(vu)))]$

Axiom 10′ $\vdash Lx \supset_x (xu \supset_u yu) \supset (xu \supset_u I(yu))$

Axiom 11′ $\vdash Lx \supset_x (xu \supset_u I(yu)) \supset (xu \supset_u yu).$

(Here we are assuming that I is defined by SKK. In [9] it is a primitive, but these two axioms are required in either case.)

A comparison between various systems The work above gives rise to four systems in addition to that of [2]. First we could consider $\mathcal{L}$ extended by Rule Ξ and the deduction theorem (DTΞ), we will call this $\mathcal{L}_e$; then we can have another natural deduction system $\mathcal{L}_a$, which is the altered version of $\mathcal{L}$ also with Rules Ξ and DTΞ. Finally we have the two axiomatic systems that correspond to these, we will call them $\mathcal{L}_e^+$ and $\mathcal{L}_a^+$. Clearly these are at least as strong as $\mathcal{L}_e$ and $\mathcal{L}_a$, respectively.

The rules of $\mathcal{L}_e$ (and $\mathcal{L}_e^+$) can be proved in $\mathcal{L}_a$ (and $\mathcal{L}_a^+$) and vice versa, but the axioms in each of [2], $\mathcal{L}_e^+$ and $\mathcal{L}_a^+$ cannot be proved in any of the other systems unless we have extra axioms such as:

(2) $\vdash L(WQ)$

(3) $\vdash LL$

(4) $\vdash Lx \supset_x L(\Xi x)$,

etc.

If $L = FAH$, as it is in [2], that system becomes inconsistent and so will those based on $\mathcal{L}$ with appropriate axioms or rules for H, so it is best not to assume (3). If however, as in [7], we allow (the axioms and hence) DTΞ to hold for several values of L a version of (3) and (4) with different L's could be sufficient for what is required and it is possible that the four $\mathcal{Q}$-based systems could be equivalent.

Some other systems of combinatory logic, or systems that use it such as those of Fitch [11] and Aczel [1], simply postulate a deduction rule and so have to have variables in their proofs, as our natural deduction systems have.

The system [8] of Church[7] must contain free variables in its formation rules and bound variables in the system because it is based on λ-calculus rather than combinatory logic. A careful reading of the proof of his deduction theorem shows that it proves, as ours does, that free variables are not necessary in a proof.

The system [12] of Goodman is especially interesting since it also has $\vdash$ as the only predicate; moreover an equality $\equiv$ somewhat similar to our Q is present in the same system. The differences lie in the facts that Goodman only allows terms of the form $X \equiv Y$, where X and Y are our obs ($\equiv$ is not an ob) and that $\equiv$ does not have all the rules that Q has. For example, $\vdash X \equiv X$ is not provable for all X and a term c can only be substituted for a variable in an equation if $\vdash c \equiv c$.

This substitution rule is one of Goodman's 13 basic postulates; 5 others involve variables. These postulates can however be rewritten to involve no variables. Instead of postulates X–XIII, we write down all the possible instances obtainable by means of the substitution rule (III), i.e.:

$$\begin{aligned} a \equiv a, b \equiv b &\vdash Kab \equiv a \\ a \equiv a, b \equiv b &\vdash Sab \equiv Sab \\ ac(bc) \equiv ac(bc) &\vdash Sabc \equiv ac(bc) \\ Sabc \equiv Sabc &\vdash Sabc \equiv ac(bc). \end{aligned}$$

The other postulate (IV) involving a variable, $\vdash X \equiv X$, can now be left out as can (III), the substitution rule.

Despite the fact that this system does not contain Ξ and hence does not have a P (for implication) defined in terms of it, the following deduction theorem for implication (defined in terms of K and S) can be derived:[8]

If a is a proposition and $\Delta, a \equiv T \vdash b \equiv T$ then $\Delta, b \equiv b \vdash a \supset b \equiv T$.

Here '_ $\equiv T$' can be interpreted as '_ is true' so the theorem is very similar to the one derivable from our deduction theorem for Ξ:

If $\Delta, a \vdash b$ then $\Delta, Ha \vdash a \supset b$.

The differences are the extra condition on b and the fact that in Goodman's system 'a is a proposition' is defined in the metatheory and so is not an ob as is Ha.

NOTES

1. There are semantics for pure combinatory logic, for example those of Scott ([14] and [15]).

2. For more details on the difficulties in providing semantics for combinatory logic see Kearns [13].

3. This property is not essential for the system to be interesting or useful, the system *QD* of Fitch in [11] for example is not *Q*-consistent. For more details on this see [4].

4. All the propositional connectives can be defined using Ξ, as can quantification over individuals, propositions, predicates, etc. (see [3]).

5. λ-abstraction can be defined in terms of combinators by:

$$\lambda u.u = I = SKK$$
$\lambda u.X = KX$ if X is a single symbol other than u
$$\lambda u.(XY) = S(\lambda u.X)(\lambda u.Y).$$

6. ΞXY is also written as $Xu \supset_u Yu$ if u is not in X or Y.

7. This system has been proved to be inconsistent by Kleene and Rosser.

8. For details see [6].

REFERENCES

[1] Aczel, P., "Frege structures and the notions of proposition, truth and set," in *The Kleene Symposium, a Collection of Papers on Recursion Theory and Intuitionism*, eds., J. Barwise, H. J. Keisler and K. Kunen, North Holland, Amsterdam, 1980.

[2] Bunder, M. W., "A deduction theorem for restricted generality," *Notre Dame Journal of Formal Logic*, vol. 14 (1973), pp. 341-346.

[3] Bunder, M. W., "Propositional and predicate calculuses based on combinatory logic," *Notre Dame Journal of Formal Logic*, vol. 15 (1974), pp. 25-32.

[4] Bunder, M. W., "Consistency notions in illative combinatory logic," *The Journal of Symbolic Logic*, vol. 42 (1977), pp. 527-529.

[5] Bunder, M. W., "The naturalness of combinatory logic as a basis for mathematics," pp. 55-64 in *To H. B. Curry; Essays on Combinatory Logic, Lambda Calculus and Formalism*, eds., J. R. Hindley and J. P. Seldin, Academic Press, New York 1980.

[6] Bunder, M. W., "Predicate calculus and naive set theory based on pure combinatory logic," to appear in *Archiv für Mathematische Logik und Grundlagenforschung.*

[7] Bunder, M. W., "Predicate calculus of arbitrarily high finite order," to appear in *Archiv für Mathematische Logik und Grundlagenforschung.*

[8] Church, A., "A set of postulates for the foundation of logic," *Annals of Mathematics*, vol. 33 (1932), pp. 346-366 and vol. 34 (1933), pp. 839-864.

[9] Curry, H. B. and R. Feys, *Combinatory Logic*, Vol. 1, North Holland, Amsterdam, 1959.

[10] Curry, H. B., J. R. Hindley, and J. P. Seldin, *Combinatory Logic*, Vol. 2, North Holland, Amsterdam, 1972.

[11] Fitch, F. B., *Elements of Combinatory Logic*, Yale University Press, New Haven, Connecticut, 1974.

[12] Goodman, N. D., "A simplification of combinatory logic," *The Journal of Symbolic Logic*, vol. 37 (1972), pp. 225-246.

[13] Kearns, J. T., "Fully explicit deductive systems," pp. 35-54 in *To H. B. Curry, Essays on Combinatory Logic, Lambda Calculus and Formalism*, eds., J. R. Hindley and J. P. Seldin, Academic Press, New York, 1980.

[14] Scott, D. S., "Models for various type-free calculi," pp. 157-188 in *Logic, Methodology and Philosophy of Science, IV*, ed., Suppes et al., North Holland, Amsterdam, 1973.

[15] Scott, D. S., "Data types as lattices," pp. 579-657 in *Springer Lecture Notes in Mathematics*, No. 499.

[16] Seldin, J. P., *Studies in Illative Combinatory Logic*, V. R. B., Groningen, 1968.

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