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On the Number of Nonisomorphic Models in $L_{\infty,\kappa}$ When κ is Weakly Compact

SAHARON SHELAH*

In a previous paper [3] we proved that if $V = L$ then for every regular cardinal λ which is not weakly compact and any model M of cardinality λ, the number of nonisomorphic models of cardinality λ which are $L_{\infty,\lambda}$-equivalent to M is 1 or 2^λ. Here we are going to prove that the above theorem is not true for λ weakly compact.

Main Theorem *Let λ be a weakly compact cardinal. Then there exists a model M, $\|M\| = \lambda$ such that $|K_M^\lambda| = 2$, where $K_M^\lambda = \{N/\cong : N \equiv_{\infty,\lambda} M, \|N\| = \lambda\}$; moreover, we can obtain any number $\leqslant\lambda$ instead of 2.*

Proof: The theorem follows immediately from the next two lemmas.

Notation: We shall always assume that the universe of models of cardinality λ is λ and for $A \subseteq \lambda$ we denote by M_A the submodel of M whose universe is A with the relation symbols R of M of $<|A|$ places such that $R \restriction A \neq \phi$. (Note that e.g., $M \equiv_{\infty,\lambda} N$ does mean that the models have the same language whereas $M \prec_{\omega_1,\omega} N$ does not.)

Lemma 1 *Let M^1, M^2 be models with the following properties:*

(1) $M^1 \not\cong M^2$
(2) $M^1 \equiv_{\infty,\lambda} M^2$
(3) $\|M^1\| = \|M^2\| = \lambda$

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(4) *(Main Property) For every inaccessible $\mu < \lambda$ such that $M^1_\mu \prec_{\omega_1\omega_1} M^1$, the number of nonisomorphic submodels of M^1 of cardinality μ which are $L_{\infty,\mu}$-equivalent to M^1_μ is $\leqslant 2$.*

Then $|K^\lambda_{M^1}| = 2$.

Lemma 2 *Models M^1, M^2 which satisfy the requirements in Lemma 1 exist.*

Proof of Lemma 1: Suppose $\|M^3\| = \lambda$ and $M^3 \equiv_{\infty,\lambda} M^1$. We want to show that $M^3 \cong M^1$ or $M^3 \cong M^2$. We use the Hanf-Scott characterization of weakly compact cardinals, namely that λ is Π^1_1-indescribable (see [1], Theorem 77). If M^3 is not isomorphic to M^1 or to M^2, then by the Π^1_1-indescribability of λ there exists an $\alpha < \lambda$ such that $M^3_\alpha \equiv_{\infty\alpha} M^2_\alpha \equiv_{\infty\alpha} M^1_\alpha$ while $M^3_\alpha \not\cong M^1_\alpha$ and $M^3_\alpha \not\cong M^2_\alpha$. We can add to the Π^1_1 sentence witnessing the nonisomorphism that it happens not at an arbitrary ordinal α, but at an inaccessible cardinal $\mu < \lambda$. Similarly, we could get in addition that $M^1_\mu \prec_{\omega_1\omega_1} M^1$, since the set $C = \{\alpha\colon M^1_\alpha \prec_{\omega_1\omega_1} M^1\}$ is closed unbounded in λ. Thus we obtained a contradiction to property (4).

Proof of Lemma 2: We shall construct the two models M^1 and M^2, and sets of partial mappings $H^{1,2}$, $H^{2,1}$ from M^1 to M^2, and M^2 to M^1, respectively, such that the demand $M^1 \equiv_{\infty,\lambda} M^2$ will follow trivially from Karp's characterization of elementary equivalence; we shall do the construction by induction on $\alpha < \lambda$ so the demand $\|M^1\| = \|M^2\| = \lambda$ will also follow trivially. Thus the only properties we are left to check are $M^1 \not\cong M^2$ and requirement (4). We delay this to after our definition of the construction.

We will define by induction on $i < \lambda$, ordinals $\alpha_i < \lambda$, "groups" $G^l_{\alpha_i} (l = 1, 2)$ of 1-1 mappings with domain and range, a subset of α_i and families of "partial mappings" $H^{1,2}_{\alpha_i}$, $H^{2,1}_{\alpha_i}$ from α_i to α_i, with certain natural closure properties between them. The $G^l_{\alpha_i}$ will be used to build the structures. $H^{1,2}_{\alpha_i}$ and $H^{2,1}_{\alpha_i}$ will form part of the set of partial isomorphisms between the structures that guarantees $\equiv_{\infty\lambda}$. (We could actually manage with only $G^1_{\alpha_i}$ and $H^{1,2}_{\alpha_i}$ the closure is defined naturally.)

Specifically, we want $G^l_{\alpha_i}$ to be closed under restriction of the domain, composition, and inverse, and to contain the identity map on α_i. $H^{1,2}_{\alpha_i}$ should be closed under restriction of the domain, and if $g \in G^1_{\alpha_i}$ and $h \in H^{1,2}_{\alpha_i}$, then $h \circ g \in H^{1,2}_{\alpha_i}$, while if $g \in G^2_{\alpha_i}$ then $g \circ h \in H^{1,2}_{\alpha_i}$. $H^{2,1}_{\alpha_i}$ should have the analogous closure properties, and the inverse of a function in $H^{1,2}_{\alpha_i}$ should be in $H^{2,1}_{\alpha_i}$, and vice versa.

We will also choose α_i strictly increasing so that $\alpha_{i+1} \leqslant \alpha_i + \alpha_i$ and take $\alpha_i = \bigcup_{j<i} \alpha_j$ for limit i. It follows easily that for any inaccessible i, $\alpha_i = i$.

Now we just have to carry out the construction and fulfill the requirements.

There will be three cases in the construction. It is understood that in each case we also do whatever is necessary to get the closure properties described above, e.g., add the identity function.

Case A. i is a limit ordinal. Here we simply define $G^1_{\alpha_i} = \bigcup_{j<i} G^1_{\alpha_j} \cup \{\mathrm{id}_{\alpha_i}\}$, and similarly for the others.

Case B. $i = j + 1$ where j is not inaccessible.

Among the triples (I, f, α) such that $I \in \{G^1_{\alpha_j}, G^2_{\alpha_j}, H^{1,2}_{\alpha_j}, H^{2,1}_{\alpha_j}\}$, $f \in I$, dom $f \subseteq \alpha$, range $f \subseteq \alpha$, but f has no extension with domain including α in I, choose one with minimal α, say f^j. We extend f^j to a function $f_j\colon \alpha_j \xrightarrow{\text{onto}}$ range $f \cup [\alpha_j, \alpha_i)$. (One can easily check by induction that this will always be possible.) Since λ is strongly inaccessible, each $f \in G^1_{\alpha_j}$ and $f \in H^{1,2}_{\alpha_j}$ will eventually be extended in this way. Now $G^l_{\alpha_i}$, $H^{1,2}_{\alpha_i}$ and $H^{2,1}_{\alpha_i}$ are defined naturally by adding f and closing.

Case C. $i = j + 1$ where j is inaccessible. Here we always take $\alpha_i = \alpha_j + \alpha_j$. We will have $G^1_{\alpha_i} \supseteq G^1_{\alpha_j}$ defined as follows: First, for all $\xi < \alpha_j$ define f_ξ with domain ξ such that for all $\zeta < \xi$, $f_\xi(\zeta) = \alpha_j + \zeta$. Next, define $G^1_{\alpha_1}$ as the closure of the set of the elements $\{f_\xi\colon \xi < \alpha_j\} \cup G^1_{\alpha_j}$ and the identity function on α_i. $G^2_{\alpha_i}$ is defined similarly.

To define $H^{1,2}_{\alpha_i}$, define first functions g^1 and g^2 such that $g^1(\zeta) = \alpha_j + \zeta$, and $g^2(\alpha_j + \zeta) = \zeta$, for all $\zeta < \alpha_j$. $H^{1,2}_{\alpha_i}$ is the closure of $H^{1,2}_{\alpha_j} \cup \{g^1, g^2\}$. $H^{2,1}_{\alpha_i}$ is defined similarly.

It is not difficult to verify that the construction, as described above, can be carried out, and that the following holds:

(*) *Suppose $j < i < \lambda$. Then there are no new mappings in any of $G^1_{\alpha_i}$, $G^2_{\alpha_i}$, $H^{1,2}_{\alpha_i}$, $H^{2,1}_{\alpha_i}$ both of whose domain and range are included in α_j. If $f \in G^1_{\alpha_i}$, then $f \cap (\alpha_j \times \alpha_j) \in G^1_{\alpha_j}$, and similarly for the others.*

Now define

$$G^l = \bigcup_{i<\lambda} G^l_{\alpha_i} \ (= 1, 2)$$

and

$$H^{1,2} = \bigcup_{i<\lambda} H^{1,2}_{\alpha_i},\ H^{2,1} = \bigcup_{i<\lambda} H^{2,1}_{\alpha_i}.$$

The definition of the models The universe of each M^l will be λ. Let S_α denote the set of all sequences of elements of λ of length less than α. Given f, a 1-1 function whose domain includes the range of $\bar{\alpha} \in S_\alpha$, we write $f(\bar{\alpha})$ to denote the element of S_α obtained by applying f pointwise to the elements of the sequence $\bar{a}$. Each G^l induces a natural equivalence relation on S_λ as follows: for $\bar{a}, \bar{b} \in S_\lambda$, $\bar{a} \sim_l \bar{b} \iff \exists f \in G[f(\bar{a}) = \bar{b}]$. Since λ is weakly compact, $\lambda^{<\lambda} = \lambda$, and so there are λ equivalence classes modulo $\sim_l$.

Let R^l_t, $t < \lambda$ enumerate the equivalence classes modulo $\sim_l$. Let R^1_t and R^2_s correspond provided there are $\bar{a}, \bar{b} \in S_\lambda$ such that $R^1_t(\bar{a}), R^2_s(\bar{b})$ and $f(\bar{a}) = \bar{b}$ for some $f \in H^{1,2}$. The closure properties required above guarantee that this is a 1-1 correspondence. M^1 will have relation symbols, $\underline{R}_t$, $t < \lambda$, interpreted by R^1_t respectively. M^2 will have $\underline{R}_t$ interpreted by that R^2_s corresponding to R^1_t (rather than by R^2_t itself.) Thus, M^l has relations of all lengths less than λ, and λ relations altogether.

By exploiting stage B of the construction sufficiently often (and since $\lambda^{<\lambda} = \lambda$) it can be arranged so that $f \in H^{1,2}$ and $f(\bar{a}) = \bar{b}$ actually implies $tp_{L_{\infty\lambda}}(\bar{a}, M^1) = tp_{L_{\infty\lambda}}(\bar{b}, M^2)$, and similarly inside each model (using $f \in G^1$ or $f \in G^2$, respectively) so that our models will have elimination of quantifiers. Of course, this immediately gives requirement (2). In addition, because of the closure properties, whenever $\langle M^1, a_i\rangle_{i<\mu} \equiv_{\infty\lambda} \langle M^2, b_i\rangle_{i<\mu}$, for some $\mu < \lambda$, then there will be some $f \in H^{1,2}$ such that $h(a_i) = b_i$ for $i < \mu$.

We just have to fulfill our requirements (4) and (1); for (4) it suffices to prove the following:

Claim *If $\chi < \lambda$ is an inaccessible cardinal such that $M^1_\chi \prec_{\omega_1\omega_1} M^1$, $A \subseteq M^1$, $|A| = \chi$, and $M^1_A \equiv_{\infty,\chi} M^1_\chi$, then either $M^1_A \cong M^1_\chi$ or $M^1_A \cong M^2_\chi$.* (Notice that by definition of $H^{1,2}$ it follows that $M^2_\chi \cong M^1_{[\chi,\chi+\chi)}$.)

Remark: Remember that M^1_A is the restriction of M^1 to A using relations with $<|A|$ places only.

Proof of the claim: We will show by induction on $i < \lambda$ that there is no $A \subseteq \alpha_i$ violating the claim.

For the base of the induction we start with $\alpha_\chi = \chi$ since this is the first case in which A could be a subset of α_i. If $A = \chi$ we are done and so we assume the contrary. We shall assume $M^1_A \equiv_{\infty,\chi} M^1_\chi$ and derive a contradiction.

First we choose $\zeta \in \chi\backslash A$. Then we define by induction on $n < \omega$, $i(n) < \lambda$ and $\zeta(n) \in A$ such that $i(n) < i(n+1)$ and $\zeta(n) \in [\alpha_{i(n)}, \alpha_{i(n+1)}) \cap A$, and $\zeta(n) > \zeta$. Since $|A| = \chi$ is inaccessible there is no problem defining such a sequence.

We now claim:

(**) *For each $j < \lambda$ there does not exist $f \in G^1_{\alpha_j}$ such that $f(\zeta) \neq \zeta$, but for each $n \in \omega$, $f(\zeta_n) = \zeta_n$.*

In fact, we prove even more:

If $g \in G^1_{\alpha_j}$, and $g(\zeta_n) = \zeta_n$ for all $n \in \omega$, then g restricted to $\bigcup_{n<\omega} i(n)$ is the identity function on $\bigcup_{n<\omega} i(n)$.

We prove this by induction on j. First, for $j < \bigcup_{n<\omega} i(n)$, if $g \in G^1_{\alpha_j}$, then the domain of g is too small. If $j = \bigcup_{n<\omega} i(n)$, then each $g \in G^1_{\alpha_i}$ is either the identity on α_j or already in $G^1_{\alpha_k}$, for some $k < j$. The interesting case is $j > \bigcup_{n<\omega} i(n)$. If j is a limit ordinal, then the result follows easily by induction.

If $j = i + 1$ for i not inaccessible, then if we let $h = g \cap (\alpha_j \times \alpha_j)$, by (*) $h \in G_{\alpha_j}$ and $h(\zeta_n) = \zeta_n$ for all $n \in \omega$, whence, by our induction hypothesis, h restricted to $\bigcup_{n<\omega} i(n)$ is the identity on $\bigcup_{n<\omega} i(n)$, and so also for g.

Finally, if $j = i + 1$ for i inaccessible, if $g \in G^1_{\alpha_j}$, then by the construction, either $g \in G^1_{\alpha_i}$, g is the identity, or, if not, and if $\zeta(n) \in$ domain g, then $g(\zeta(n)) \geqslant \alpha_i$ and so $g(\zeta(n)) \neq \zeta(n)$.

Now, let S and R be the relations satisfied by $\langle \zeta(0), \zeta(1), \ldots\rangle$ and $\langle \zeta, \zeta(0), \zeta(1), \ldots\rangle$ respectively. Then because of (**), and using the "string elimination of quantifiers",

(+) $\quad M^1 \models \forall y_0 y_1 \ldots [\underline{S}(y_0, y_1, \ldots) \to \exists! y \underline{R}(y, y_0, y_1, \ldots)]$.

Now, since $M^1_\chi \prec_{\omega_1\omega_1} M^1$ and $M^1_\chi \equiv_{\infty\chi} M_A$, we have $M_A \models \forall y_0 y_1 \ldots [\underline{S}(y_0, y_1, \ldots) \to \exists! y \underline{R}(y, y_0, y_1, \ldots)]$. Thus, in particular $M_A \models \underline{R}(\zeta', y(0),$

$\zeta(1), \ldots)$ for some $\zeta' \in A$. Now, since $\zeta \notin A$, $\zeta' \neq \zeta$, and this violates the uniqueness in (†). This proves the case for the base of the induction of the proof of the claim.

We now consider each case in the construction.

Case A. $i > \chi$ is a limit ordinal. We may assume that $A \subseteq \alpha_i$ but that $A \nsubseteq \alpha_j$ for $j < i$ or else the result follows by the inductive assumption. We can then define by induction on $n \in \omega$, $i(n)$ and a_n such that $i(n) < i(n+1) < i$ and $a_n \in A \cap [\alpha_{i(n)}, \alpha_{i(n+1)})$. Then, by using (*), we can show that there is no $f \in G^1$ such that $\operatorname{dom} f \subseteq M_\chi$ and range $f = \{a_0, a_1, \ldots\}$. But this means that the type of $\langle a_0, a_1, \ldots \rangle$ is not realized in M_χ, and this contradicts the assumption that $M_\chi^1 \equiv_{\infty\lambda} M_A$.

Case B. $i = j + 1$, where j is not inaccessible. Again we may assume $A \nsubseteq \alpha_j$. Let f^j and f_j be as in the construction. Next, if $A \subseteq$ range f_j, then by applying f_j^{-1} to A we find an isomorphic copy as a subset of α_j. This is just the first case again. Thus, we may assume that there are $a \in (A \cap \alpha_j)\backslash$range f_j and $b \in A \cap [\alpha_j, \alpha_i)$. As above, it is sufficient to show that there is no $f \in G^1$, and $a', b' \in M_\chi$ such that $f(a') = a$, $f(b') = b$. We show there is no such f in any $G^1_{\alpha_\zeta}$. For $\zeta < i$ this is clearly impossible. The only interesting case is $G^1_{\alpha_i}$, since for $\zeta > i$ we can appeal to (*). Recall however that $G^1_{\alpha_i}$ is obtained from $G^1_{\alpha_j}$ by adding f_j and closing to form a "group". Now, by the choice of a and b, we see that there can be no a', b', and $f \in G^1_{\alpha_2}$ as above.

Case C. $i = j + 1$ where j is inaccessible. Just as in Case B we find $\alpha \in A \cap \alpha_j$ and $b \in A \cap [\alpha_j, \alpha_i)$. But now, no such f can exist because any $f \in G^1_{\alpha_i}$ other than the identity whose range intersects $[\alpha_j, \alpha_i)$ has its range actually included in $[\alpha_j, \alpha_i)$.

Remark: The reader will observe that the purpose of the special definition for inaccessibles is really to get over the case $i = \chi + 1$.

To finish the entire proof we need only show the following:

Claim $M^1 \not\cong M^2$.

Proof of Claim: Suppose to the contrary that f is an isomorphism from M^1 onto M^2. Choose by induction increasing sequences $i(n)$ and $\zeta(n)$ such that $\alpha_{i(n)} < \zeta(n) < \alpha_{i(n+1)}$ and $\alpha_{i(n)} < f(\zeta(n)) < \alpha_{i(n+1)}$. It is easy to see that this can be done.

We can now easily prove, by appealing to (*), that there does not exist $g \in H^{1,2}$ such that $g(\zeta(n)) = f(\zeta(n))$ for each $n < \omega$. But this means that $(M^1, \zeta(0), \zeta(1), \ldots) \not\equiv_{\infty\lambda} (M^2, f(\zeta(0)), f(\zeta(1)), \ldots)$. This, of course contradicts the assumption that f is an isomorphism, and so proves the claim, and ends the entire proof.

Concluding remarks The reader will have noticed that only relations of $\leqslant \omega$ places were used and so all others could have been omitted. (In this case, however, there is no elimination of quantifiers.) Furthermore, by coding and working in $\lambda \cup {}^\omega\lambda$ it would be possible to manage with only relations of finite length.

Open Problem Can the above results be obtained using fewer than λ symbols?

Having obtained two models in the above theorem, it is now quite easy to obtain the analogous result but with μ models instead of 2, for any $\mu \leqslant \lambda$. We indicate how this could be done. First consider the case $3 \leqslant \mu < \omega$. We would have in addition to the relations of the original example an equivalence relation. A model of power λ of the new theory would consist of $\mu-1$ equivalence classes each of power λ, and each equivalence class would carry the structure of M^1 or M^2. It is easy to see that up to isomorphism there would be exactly μ models of this complete $L_{\infty\lambda}$ theory. For $\omega \leqslant \mu \leqslant \lambda$ we first choose $\kappa = \aleph_\mu \leqslant \lambda$ (since λ inaccessible) and do the same construction, but with κ equivalence classes.

Alternatively, for $\kappa < \lambda$, we can construct the M^i, $i < \kappa$, simultaneously, replacing '2' in condition (4) of Lemma 1 by 'κ'. For $\kappa = \lambda$ the result can then be obtained by defining M^i_α, $i < \alpha < \lambda$, by induction on α, and again replacing '2' in condition (4) of Lemma 1 by 'κ'. More importantly this applies also to the case $\lambda < \kappa < 2^\lambda$ provided that λ is supercompact. Then, by a theorem of Menas [2], there is a normal fine ultrafilter $\mathscr{D}$ on $\mathcal{P}_{<\lambda}(\kappa) = \{A \subseteq \kappa: |A| < \lambda\}$, such that for some $f: \lambda \to \lambda$, the set $\{A \in \mathcal{P}_{<\lambda}(\kappa): A \cap \lambda$ is an ordinal and A has order type $f(A \cap \lambda)\}$ belongs to $\mathscr{D}$. Now, we can repeat the proof replacing '2' in condition (4) of Lemma 1 by '$f(\kappa)$', and in the proof of Lemma 1 use the normality of $\mathscr{D}$ instead of weak compactness. Specifically, we can choose $f_\alpha: \lambda \to \lambda$ for $\alpha \leqslant \kappa$ such that $f_\alpha(j) < 2^{|j|}$ and $\{A \in \mathcal{P}_{<\lambda}(\kappa): A \cap \lambda$ is an ordinal and $A \cap \alpha$ has order type $f_\alpha(A \cap \lambda)\}$ belongs to $\mathscr{D}$. Now, in the proof of Lemma 2, we define by induction on $i < \lambda$, $M^\alpha_i (\alpha < \kappa)$, such that M^α_i depends on $f_\alpha \restriction (i+1)$ only.

The hypothesis of supercompactness is strong, but as far as we know, the consistency of ZFC + "κ weakly compact + $2^\kappa > \kappa^+$" is known to follow only from supercompactness.

Open Problem Finally, we should like to point out that the question of the number of models of power λ $\infty\lambda$-equivalent to a model of power λ for λ singular and of cofinality greater than ω is still open.

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The Institute for Advanced Studies
The Hebrew University
Jerusulem, Israel