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RMLC: Solution to a Problem Left Open by Lemmon

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A system S is *Halldén-incomplete* if and only if there are wffs A and B with no variables in common such that $\vdash_S A \vee B$ but neither $\vdash_S A$ nor $\vdash_S B$, and *strongly Halldén-incomplete* if, in addition, A and B have but one variable apiece.* Evidently, all strongly Halldén-incomplete systems are Halldén-incomplete; Lemmon [5] poses the converse as an open problem.

Consider the system *RMLC*, with detachment and adjunction as rules and, using standard conventions concerning relative binding strengths of connectives and omission of parentheses, the following axiom schemes:

R0	$A \rightarrow (A \rightarrow A)$
R1	$A \rightarrow A$
R2	$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
R3	$A \rightarrow ((A \rightarrow B) \rightarrow B)$
R4	$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
R5	$A \,\&\, B \rightarrow A$
R6	$A \,\&\, B \rightarrow B$
R7	$(A \rightarrow B) \,\&\, (A \rightarrow C) \rightarrow (A \rightarrow (B \,\&\, C))$
R8	$A \rightarrow A \vee B$
R9	$B \rightarrow A \vee B$
R10	$(A \rightarrow C) \,\&\, (B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C)$
DUMMETT	$(A \rightarrow B) \vee (B \rightarrow A)$
R11	$A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee C$
R12	$(A \rightarrow \bar{B}) \rightarrow (B \rightarrow \bar{A})$
PRE TRANS	$(A \rightarrow (\bar{B} \rightarrow A)) \rightarrow (A \rightarrow (\bar{A} \rightarrow B))$
RMLC	$(\bar{\bar{A}} \rightarrow A) \vee (B \rightarrow (C \rightarrow B))$.

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RMLC is clearly a subsystem of Dummett's *LC* [3], most of the above schemes being among those listed for *LC*-duty in [6] (pp. 316-317) and the rest easily derived, e.g., PRE TRANS from the intuitionistic $A \rightarrow (\bar{A} \rightarrow B)$ by way of $B \rightarrow (C \rightarrow B)$, and RMLC from the latter by R9. *RMLC* is also contained in the system *RM*(ingle) of [1], for R0-R12 are *RM*-axioms (p. 341), DUMMETT is RM64 (p. 397), and PRE TRANS and RMLC are readily established.

Indeed, *RM* and *LC* may be axiomatized by adding to *RMLC* (schematically) the left disjunct of RMLC for the former and the right for the latter: R0-R12 plus $\bar{\bar{A}} \rightarrow A$ suffice for *RM* according to [1] (p. 341), while R2, R4-R10, DUMMETT, R12, PRE TRANS, and $B \rightarrow (C \rightarrow B)$ give a set equivalent, with minor adjustments, to one given in [6] (p. 317) for *LC*.

A familiar, Halldén-style argument consequently completes a proof that the theorems of *RMLC* are precisely the wffs provable in both *RM* and *LC*. For assume $\vdash_{RM} C$ and $\vdash_{LC} C$. Then there must be substitution instances $A_1, \ldots, A_m$ of $\bar{\bar{A}} \rightarrow A$ and $B_1, \ldots, B_n$ of $B \rightarrow (C \rightarrow B)$ such that $A_1 \,\&\, \ldots \,\&\, A_m \vdash_{RMLC} C$ and $B_1 \,\&\, \ldots \,\&\, B_n \vdash_{RMLC} C$. It follows, by a proof similar to one in [1] (p. 302), that $(A_1 \,\&\, \ldots \,\&\, A_m) \vee (B_1 \,\&\, \ldots \,\&\, B_n) \vdash_{RMLC} C$ whence eventually, after repeated distribution moves licensed by R5-R11 (and the transitivity of $\vdash_{RMLC}$), $(A_1 \vee B_1) \,\&\, (A_1 \vee B_2) \,\&\, \ldots \,\&\, (A_m \vee B_n) \vdash_{RMLC} C$. By RMLC, however, each $A_i \vee B_j$ is available in *RMLC*, so that $\vdash_{RMLC} C$ as well, finishing the argument.[1]

For a solution to Lemmon's problem, now, let A and B have no variables in common, and just one each, and assume $\vdash_{RMLC} A \vee B$. Then $\vdash_{LC} A \vee B$ also. It is shown in [4] that the extensions (closed under substitution) of *LC* are linearly ordered, so it follows from Theorem 1 of [5] that *LC* is Halldén-complete. Thus, $\vdash_{LC} A$ or $\vdash_{LC} B$. Arbitrarily, say $\vdash_{LC} A$. Then A is a tautology of the classical, two-valued truth tables and, since these characterize the one-variable fragment of *RM* ([1], p. 413, Corollary 3.1), $\vdash_{RM} A$ as well, whereupon $\vdash_{RMLC} A$ and the latter system is thus not strongly Halldén-incomplete. Because $\bar{\bar{A}} \rightarrow A$ is scarcely in *LC*, however, and $B \rightarrow (C \rightarrow B)$ notoriously not in *RM*, neither disjunct of RMLC can be obtained in *RMLC*, so that *RMLC* is Halldén-incomplete.

NOTE

1. The problem ([1], p. 99) of axiomatizing a "constructive mingle" whose implicational fragment will be given by the implicational axiom schemes R0-R4 remains open; for

BULL $((A \rightarrow B) \rightarrow C) \rightarrow (((B \rightarrow A) \rightarrow C) \rightarrow C)$

is known from [2] to hold in *LC*, and a quick check of Parks's matrix in [1] (p. 148) shows it in RM as well. So BULL is provable in *RMLC*. But R0-R4 are intuitionistically acceptable, as BULL is not. The author suggests looking, instead, at the system *RMIC* which results when DUMMETT is deleted from *RMLC*'s axiom set and whose theorems are easily shown to be precisely those wffs provable in both *RM* and the intuitionistic sentential calculus, *IC*.

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