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A More Satisfactory Description of the Semantics of Justification

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1 Introduction In [1] I developed a semantic account for a first-order language. This semantics is based on the concept of justification rather than truth. With respect to this semantic account I showed that a system of intuitionist logic is sound and complete.

I have since realized that there are certain misconceptions involved in the treatment of the semantics of justification. In [1] I said that a sentence is justified (i.e., its assertion is warranted) if it is either know to be true or follows from sentences known to be true. My mistake was in employing the concept of *following from.* It is conceivable that one sentence might follow from others even though we were incapable of knowing this (it is conceivable, but I judge it to be unlikely). A more satisfactory description has it that a sentence is justified if it is either known or *deducible* from sentences that are known. (Of course, this means deducible by *correct* deductions.) This new characterization of justification does not change the fact that a sentence can be justified without being known.

As well as affecting my informal account of justification, the mistake led me to misunderstand the role of the formal semantics presented in [1]. I thought of that semantic account as giving the significance in terms of justification of the connectives and quantifiers. But some features of the semantics were motivated by thinking in terms of the relations *following from* and *being incompatible with.* This kind of semantic account cannot capture the epistemic concept of justification.

Given a proper understanding of justification, the deductive system of [1] (the system of intuitionist logic) has a more fundamental character than the semantic account. We can determine by consulting the concept of justification—as I am presently explaining it—that the deductive system is correct (i.e., that it is sound with respect to justification). From this perspective, the semantic

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account has the character of a conjecture. This is a conjecture that the concept of justification behaves in certain ways. The fact that the deductive system turns out to be sound and complete for the semantics establishes the conjecture. These proofs do more for the semantics than they do for the deductive system—though the completeness result also shows the system to be sufficient.

As well as being misconstrued, the semantic account of [1] is more complicated than it needs to be. In what follows, I will describe the results of [1] in some detail, but from my new perspective. I will provide a simpler semantic account, and prove that it is equivalent to the account of [1]. Finally, I will compare the present account to those of Kripke and Beth.

*2 **The formal language and its semantics*** L is a conventional first-order language constructed as follows.

The symbols used in L are:

1. Punctuation: ( ), [ ], and the comma
2. Individual variables: $x, y, z, x_1, \ldots$
3. Individual constants: $a, b, c, a_1, \ldots$
4. For every $n > 0$, n-adic predicates: $F^n, G^n, H^n, F_1^n, \ldots$
5. Connectives: $\sim, \vee, \&, \supset$
6. Quantifier components: $\forall, \exists$.

Well-formed formulas (wffs):

(1) If ϕ is an n-adic predicate and $\alpha_1, \ldots, \alpha_n$ are individual symbols, then $\phi(\alpha_1, \ldots, \alpha_n)$ is a wff.
(2) If A, B are wffs, then so are $\sim A$, $[A \vee B]$, $[A \& B]$, $[A \supset B]$.
(3) If A is a wff containing free occurrences of individual variable α, then $(\forall\alpha)A$ and $(\exists\alpha)A$ are wffs.
(4) All wffs are obtained by (1)-(3).

A *sentence* of L is a wff which does not contain free occurrences of individual variables.

Although justification is relative to an individual or a group, no particular individuals will be specified. We suppose that we are dealing with some idealized community. A sentence becomes justified in time. But the language L is restricted to sentences for which justification is permanent. A sentence can fail to be justified at one time, and become justified at a later time. But once a sentence is justified, it remains justified.

To provide a semantics based on justification, we recognize three values. First, a sentence may be justified (+). To motivate the remaining two, I take incompatibility to be a primitive and irreducible semantic relation (*following from* has this same status). It is simply a fact that people are able to recognize some cases of incompatibility. If sentence A can be recognized to be incompatible with justified sentences, or if A (possibly together with justified sentences) can be used to deduce sentence B which can be recognized as being incompatible with justified sentences, then A is *strongly unjustifiable* (−). If A is neither justified nor strongly unjustifiable, then A is *weakly unjustified* (0).

The following matrix, which is discussed more fully in [1], indicates combinations of values that are possible for connectives. (When a compound expression has two values beneath it, this indicates that the values of its components are compatible with either of the two values for the compound.)

A	B	$\sim A$	$A \,\&\, B$	$A \vee B$	$A \supset B$
+	+	−	+	+	+
+	0	−	0	+	0
+	−	−	−	+	−
0	+	0, −	0	+	+
0	0	0, −	0, −	+, 0	+, 0
0	−	0, −	−	0	0, −
−	+	+	−	+	+
−	0	+	−	0	+
−	−	+	−	−	+

The symbols $\sim$, &, $\vee$ mean *not, and,* and *or*. A sentence $[A \supset B]$ has the (justification) significance that adding A to the justified sentences makes possible a deduction of B (i.e., B is deducible from $\{$Justified Sentences$\} \cup \{A\}$). This matrix is natural, given the (correctly understood) concept of justification. It does not incorporate any conjectures.

The matrix and the semantic account that I shall provide are general in the following sense: They give justification conditions that are correct for all subject matters and all discourses. When we deal with a particular subject matter, there will be other expressions than logical expressions which have justification conditions attached (e.g., one atomic sentence may follow from others, or be incompatible with others, when this is not a matter of logic). If we are dealing with a specific subject matter, or engaged in an investigation with a specific purpose, it can be appropriate to impose additional justification conditions on (some of) the logical expressions. We might, for example, have $[A \vee B]$ justified just in case one disjunct is.

In order to deal with quantification, we consider two domains. There is a nonempty domain $\mathscr{D}$ which contains those individuals that can be known about at present. And there is a domain ($\mathscr{D} \subseteq$) $\mathscr{D}^*$ which contains all those individuals that can ever be known about. Universally quantified sentences that are justified must be justified for every individual in $\mathscr{D}^*$. But we can require that existentially quantified sentences be justified only by individuals in $\mathscr{D}$. (This is a strong condition on $\exists$; it is not incorporated in the general semantic account.)

A generalization of the matrix for & and $\vee$ places the following constraints on $\forall$ and $\exists$ (these are the *generalized matrix conditions*):

(i) If $(\forall\alpha)A$ is a sentence which has value +, then A must have value + for every individual in $\mathscr{D}$ as value of α. If A has value − for some individual in $\mathscr{D}$ as value of α, then $(\forall\alpha)A$ has value −.

(ii) If $(\exists\alpha)A$ is a sentence which has value −, then A has value − for every individual in $\mathscr{D}$ as value of α. If A has value + for some individual in $\mathscr{D}$ as value of α, then $(\exists\alpha)A$ has value +.

In order to give a semantic account for L that is parallel to a semantic account for the propositional sublanguage of L, I will follow [1] in treating individuals as if they are constants which can be substituted for variables. If A is a wff containing free occurrences of distinct individual variables $\alpha_1, \ldots, \alpha_n$, and $\rho_1, \ldots, \rho_n$ are individuals in $\mathcal{D}$ (or $\mathcal{D}^*$), then $A[\alpha_1, \ldots, \alpha_n; \rho_1, \ldots, \rho_n]$ is the "result" of substituting $\rho_1, \ldots, \rho_n$ for $\alpha_1, \ldots, \alpha_n$ in A.

Let $\mathcal{D}$ be a nonempty domain. Let $\mathfrak{f}$ be a function which assigns individuals in $\mathcal{D}$ to individual constants of L. Then $L_{\mathcal{D}}^{\mathfrak{f}}$ is the *pseudo-language* whose elements are the following *pseudo-wffs*:

(1) A wff of L is a pseudo-wff of $L_{\mathcal{D}}^{\mathfrak{f}}$.
(2) If A is a pseudo-wff containing free occurrences of individual variable α, and $\rho \in \mathcal{D}$, then $A[\alpha; \rho]$ is a pseudo-wff of $L_{\mathcal{D}}^{\mathfrak{f}}$.
(3) All pseudo-wffs of $L_{\mathcal{D}}^{\mathfrak{f}}$ are obtained by (1) and (2).

A *pseudo-sentence* of $L_{\mathcal{D}}^{\mathfrak{f}}$ is a pseudo-wff which does not contain free occurrences of individual variables.

A *justification-valuation* (*J-valuation*) of a pseudo-language $L_{\mathcal{D}}^{\mathfrak{f}}$ is a function that assigns (exactly) one of +, 0, − to each pseudo-sentence of $L_{\mathcal{D}}^{\mathfrak{f}}$. A J-valuation $\mathcal{V}$ of $L_{\mathcal{D}}^{\mathfrak{f}}$ is *minimally acceptable* iff: (1) $\mathcal{V}$ assigns values in agreement with the matrix for the connectives, (2) $\mathcal{V}$ respects the generalized matrix conditions for the quantifiers, (3) if α is an individual constant of L, $\mathfrak{f}(\alpha) = \rho$, and A is a pseudo-sentence of $L_{\mathcal{D}}^{\mathfrak{f}}$ that contains α, then $\mathcal{V}(A) = \mathcal{V}(A[\alpha; \rho])$.

Let $L_{\mathcal{D}}^{\mathfrak{f}}$ be a pseudo-language. Let $\mathcal{V}$ be a *J*-valuation of $L_{\mathcal{D}}^{\mathfrak{f}}$. Let $\mathcal{D} \subseteq \mathcal{D}'$ and let $\mathcal{V}'$ be a J-valuation of $L_{\mathcal{D}'}^{\mathfrak{f}}$. Then $\mathcal{V}$ is *included in* $\mathcal{V}'$ iff $\mathcal{V}'$ assigns + to every pseudo-sentence assigned + by $\mathcal{V}$. ($\mathcal{V}'$ is a possible future for $\mathcal{V}$.)

The class of minimally acceptable *J*-valuations of a pseudo-language $L_{\mathcal{D}}^{\mathfrak{f}}$ contains valuations that are not satisfactory. In [1], this class was narrowed down in a step-by-step fashion to yield a class of *acceptable* *J*-valuations of $L_{\mathcal{D}}^{\mathfrak{f}}$. The definitions of [1] amount to the following:

Let $\mathcal{V}$ be a minimally acceptable *J*-valuation of $L_{\mathcal{D}}^{\mathfrak{f}}$. Let $\mathcal{D} \subseteq \mathcal{D}^*$. Then $\mathcal{V}$ is a 0^{th}*-level J-valuation of* $L_{\mathcal{D}}^{\mathfrak{f}}$ with respect to $\mathcal{D}^*$. (In dealing with the different levels below, it is assumed that $\mathcal{D}^*$ is held fixed.)

Let $\mathcal{V}$ be an n^{th}-level *J*-valuation of $L_{\mathcal{D}}^{\mathfrak{f}}$ with respect to $\mathcal{D}^*$. $\mathcal{V}$ is *quantificationally adequate with respect to* $\mathcal{D}^*$ *at the* n^{th}*-level* iff for every pseudo-sentence $(\forall\alpha)A$ or $(\exists\alpha)A$ of $L_{\mathcal{D}}^{\mathfrak{f}}$, we have: (1) if for every $\mathcal{D}'$, $\mathcal{V}'$ such that $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{D}^*$ and $\mathcal{V}'$ is an n^{th}-level *J*-valuation of $L_{\mathcal{D}'}^{\mathfrak{f}}$ with respect to $\mathcal{D}^*$ that includes $\mathcal{V}$, we have $\mathcal{V}'(A[\alpha; \rho]) = +$ for every $\rho \in \mathcal{D}'$, then $\mathcal{V}[(\forall\alpha)A] = +$; and (2) if for every such $\mathcal{D}'$, $\mathcal{V}'$ we have $\mathcal{V}(A[\alpha; \rho]) \neq +$ for every $\rho \in \mathcal{D}'$, then $\mathcal{V}[(\exists\alpha)A] = -$.

Let $\mathcal{V}$ be an n^{th}-level *J*-valuation of $L_{\mathcal{D}}^{\mathfrak{f}}$ with respect to $\mathcal{D}^*$. Then the n^{th}*-level family determined by* $\mathcal{V}$ *with respect to* $\mathcal{D}^*$ is the set $X = \{\mathcal{V}' \mid$ There is a $\mathcal{D}'$ such that $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{D}^*$, $\mathcal{V}'$ is an n^{th}-level valuation of $L_{\mathcal{D}'}^{\mathfrak{f}}$ with respect to $\mathcal{D}^*$, $\mathcal{V}'$ is quantificationally adequate with respect to $\mathcal{D}^*$ at the n^{th}-level, and $\mathcal{V}$ is included in $\mathcal{V}'\}$.

Let $\mathcal{V}$ be an n^{th}-level *J*-valuation of $L_{\mathcal{D}}^{\mathfrak{f}}$ with respect to $\mathcal{D}^*$ that is quantificationally adequate with respect to $\mathcal{D}^*$ at the n^{th}-level. Let X be the

n^{th}-level family determined by $\mathcal{V}$. Then $\mathcal{V}$ is an *$(n + 1)$st-level J-valuation of $L_{\mathcal{D}}^{\mathfrak{f}}$ with respect to $\mathcal{D}^*$* iff:

(1) If A is a pseudo-sentence of $L_{\mathcal{D}}^{\mathfrak{f}}$ that is assigned + by no $\mathcal{V}' \in X$, then $\mathcal{V}(A) = -$

(2) If $\mathcal{V}[A \vee B] = +$ and every $\mathcal{V}' \in X$ that assigns + to either A or B also assigns + to pseudo-sentence C of $L_{\mathcal{D}}^{\mathfrak{f}}$, then $\mathcal{V}(C) = +$

(3) If every $\mathcal{V}' \in X$ that assigns + to pseudo-sentence A of $L_{\mathcal{D}}^{\mathfrak{f}}$ also assigns + to pseudo-sentence B of $L_{\mathcal{D}}^{\mathfrak{f}}$, then $\mathcal{V}[A \supset B] = +$

(4) If $\mathcal{V}[(\exists\alpha)A] = +$ and every $\mathcal{V}' \in X$ that assigns + to $A[\alpha; \rho]$ for some ρ in its domain also assigns + to pseudo-sentence C of $L_{\mathcal{D}}^{\mathfrak{f}}$, then $\mathcal{V}(C) = +$

A minimally acceptable J-valuation $\mathcal{V}$ of $L_{\mathcal{D}}^{\mathfrak{f}}$ is an *acceptable J-valuation of $L_{\mathcal{D}}^{\mathfrak{f}}$ with respect to $\mathcal{D}^*$* iff $\mathcal{V}$ is an n^{th}-level J-valuation of $L_{\mathcal{D}}^{\mathfrak{f}}$ with respect to $\mathcal{D}^*$ for every $n \geqslant 0$.

In the definitions of quantificational adequacy and $(n + 1)$st-level J-valuations, some of the criteria are not features of the concept of justification. A quantificationally adequate J-valuation assigns + to a sentence $(\forall\alpha)A$ if all instances of A are "forced" to have value + in all possible futures. (These futures are *epistemically* possible. It might happen that a true mathematical sentence receives value − in an epistemically possible future—if that sentence is deductively independent of current mathematical knowledge.) Such compulsion can only be due to the sentences that are currently justified, but the compulsion might be rooted in a relation of logical consequence that outruns deducibility. Similarly, in defining $(n + 1)$st-level J-valuation, we insist that if A is not assigned + in any possible future, then A must be assigned − at present. Because the only thing that might keep A from being + in some possible future is A's incompatibility with the sentences that are already +. It is initially only a conjecture that any such incompatibility must be recognizable.

The following definitions are also found in [1].

A sentence A is *logically justified with respect to $\mathcal{D}$, $\mathcal{D}^*$* iff A is justified for every acceptable J-valuation of $L_{\mathcal{D}}^{\mathfrak{f}}$ with respect to $\mathcal{D}^*$ for every interpreting function $\mathfrak{f}$ of L with respect to $\mathcal{D}$. A sentence A is *logically justified* iff A is logically justified with respect to every nonempty $\mathcal{D}$, $\mathcal{D}^*$ for which $\mathcal{D} \subseteq \mathcal{D}^*$.

An *inference sequence* $A_1, \ldots, A_n/B$ $(n \geqslant 0)$ is *J-valid with respect to $\mathcal{D}$, $\mathcal{D}^*$* iff there is no acceptable J-valuation of $L_{\mathcal{D}}^{\mathfrak{f}}$ with respect to $\mathcal{D}^*$ that assigns + to each of $A_1, \ldots, A_n$ and assigns 0 or − to B. An inference sequence is *J-valid* iff it is J-valid with respect to all suitable domain pairs. And a set of sentences can *justify* a sentence A with respect to one domain pair or every pair.

In [1] it is shown that the deductive system establishes all logically justified sentences and J-valid inference sequences, and that it enables us to deduce all the sentences justified by a given set of sentences. The semantic definitions above incorporate conjectures about the behavior of +, 0, −. (Of course, we could retain these definitions even if the conjectures had proved false, but the defined expressions would not be useful ones.) These completeness results establish the conjectures, at least insofar as the connectives and quantifiers are concerned. (When we supplement the logical expressions with other expressions and their justification conditions, it might still happen that the consequences of justified sentences were not all deducible.) At the same time the results establish the completeness of intuitionist logic for justification. For the semantic account also incorporates our understanding of logical consequence and incompatibility. Intuitionist logic is exactly the right logic for the general concept of justification.

3 A simpler semantic account The semantic account in the preceding section made use of levels of J-valuations. These levels do not correspond in a natural way to a feature of the concept of justification. Instead levels are linked to the complexity of sentences of L. The more complex a sentence is, the higher is the level we must reach before we can be sure that all J-valuations treat it correctly. (E.g., a 0^{th}-level J-valuation might assign 0 to a sentence $[A \,\&\, {\sim}A]$. All first-level valuations assign this sentence $-$. But a first-level valuation might assign 0 to $[[A \,\&\, B] \supset C] \supset [A \supset [B \supset C]]$.)

In what follows I will provide a simpler semantic account that makes no use of levels of J-valuations. This account employs a tree structure to characterize acceptable J-valuations. Each node is occupied by a J-valuation. The descendants of a J-valuation are its possible futures (each node is one of its own descendants). The shortcomings of the matrices (and the generalized matrix conditions) are overcome by considering the relations between a J-valuation and the future J-valuations that are possible with respect to it.

Let $\mathcal{D}$ be a nonempty domain. Let $\mathfrak{f}$ be a function which assigns individuals in $\mathcal{D}$ to individual constants of L. Let $\mathcal{D} \subseteq \mathcal{D}^*$. Let $\mathcal{V}$ be a minimally acceptable J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}}$. And let X be a tree structure whose top node is $\mathcal{V}$, and whose remaining nodes are (occupied by) minimally acceptable J-valuations of $L^{\mathfrak{f}}_{\mathcal{D}'}$, for $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{D}^*$. In characterizing X, it will be understood that $\mathcal{V}'$, $\mathcal{V}''$ are nodes of X, that $\mathcal{V}'$ is a minimally acceptable J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}'}$, and $\mathcal{V}''$ is a minimally acceptable J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}''}$. Then X is an *acceptable J-structure with respect to* $\langle \mathfrak{f}, \mathcal{D}, \mathcal{D}^* \rangle$ iff

(1) Every node $\mathcal{V}'$ of X is included in its (immediate) successors (if any).
(2) Every node $\mathcal{V}'$ has a descendant which is a minimally acceptable J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}^*}$.
(3) If $(\forall\alpha)A$ is a pseudo-sentence of $L^{\mathfrak{f}}_{\mathcal{D}'}$, and every descendant $\mathcal{V}''$ of $\mathcal{V}'$ assigns + to $A[\alpha;\rho]$ for every ρ in $\mathcal{D}''$, then $\mathcal{V}'[(\forall\alpha)A] = +$.
(4) If $(\exists\alpha)A$ is a pseudo-sentence of $L^{\mathfrak{f}}_{\mathcal{D}'}$, and no descendant $\mathcal{V}''$ of $\mathcal{V}'$ assigns + to $A[\alpha;\rho]$ for some ρ in $\mathcal{D}''$, then $\mathcal{V}'[(\exists\alpha)A] = -$.
(5) If there is no descendant of $\mathcal{V}'$ which assigns + to pseudo-sentence A of $L^{\mathfrak{f}}_{\mathcal{D}'}$, then $\mathcal{V}'(A) = -$.
(6) If $[A \vee B]$, C are pseudo-sentences of $L^{\mathfrak{f}}_{\mathcal{D}'}$, $\mathcal{V}'[A \vee B] = +$, and every descendant of $\mathcal{V}'$ which assigns + to A or to B also assigns + to C, then $\mathcal{V}'(C) = +$.
(7) If A, B are pseudo-sentences of $L^{\mathfrak{f}}_{\mathcal{D}'}$, and every descendant of $\mathcal{V}'$ which assigns + to A also assigns + to B, then $\mathcal{V}'[A \supset B] = +$.
(8) If $(\exists\alpha)A$, C are pseudo-sentences of $L^{\mathfrak{f}}_{\mathcal{D}'}$, $\mathcal{V}'[(\exists\alpha)A] = +$, and every descendant $\mathcal{V}''$ of $\mathcal{V}'$ which assigns + to $A[\alpha;\rho]$ for some ρ in $\mathcal{D}''$ also assigns + to C, then $\mathcal{V}'(C) = +$.

The rationale for clause (2) is that nothing that is known should prevent us from obtaining knowledge about whatever is accessible to knowledge. (Knowing about some things should not keep us from knowing about others.) It should be noted that every node $\mathcal{V}'$ of X is the top node of an acceptable J-structure with respect to $\langle \mathfrak{f}, \mathcal{D}', \mathcal{D}^* \rangle$.

Let $\mathcal{D}$, $\mathcal{D}^*$ be nonempty domains such that $\mathcal{D} \subseteq \mathcal{D}^*$. Let $\mathfrak{f}$ be a function which assigns individuals in $\mathcal{D}$ to individual constants of L. It must be proved that a function $\mathcal{V}$ is an acceptable J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}}$ with respect to $\mathcal{D}^*$ iff $\mathcal{V}$ is the top node of an acceptable J-structure with respect to $\langle \mathfrak{f}, \mathcal{D}, \mathcal{D}^* \rangle$.

We prove the 'if' part first. Let X be an acceptable J-structure with respect to $\langle \mathfrak{f}, \mathcal{D}, \mathcal{D}^* \rangle$. And let $\mathcal{V}$ be the top node of X.

Proofs of the following results are omitted when they are straightforward.

Lemma 1 *$\mathcal{V}$ is a minimally acceptable J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}}$.*

Lemma 2 *Suppose $\mathcal{V}$ is an n^{th}-level J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}}$ with respect to $\mathcal{D}^*$. Then $\mathcal{V}$ is quantificationally adequate with respect to $\mathcal{D}^*$ at the n^{th}-level.*

Proof: (i) Suppose $(\forall\alpha)A$ is a pseudo-sentence of $L^{\mathfrak{f}}_{\mathcal{D}}$. And suppose that for every $\mathcal{D}'$, $\mathcal{V}'$ such that $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{D}^*$, $\mathcal{V}'$ is an n^{th}-level J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}'}$ with respect to $\mathcal{D}^*$, and $\mathcal{V}'$ includes $\mathcal{V}$, $\mathcal{V}'[\alpha; \rho] = +$ for every $\rho \in \mathcal{D}'$. Every descendant $\mathcal{V}''$ of $\mathcal{V}$ in X is an n^{th}-level J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}''}$ with respect to $\mathcal{D}^*$. So every descendant $\mathcal{V}''$ assigns + to $A[\alpha; \rho]$ for every $\rho \in \mathcal{D}''$. So $\mathcal{V}[(\forall\alpha)A] = +$.

Similarly, (ii) if $(\exists\alpha)A$ is a pseudo-sentence of $L^{\mathfrak{f}}_{\mathcal{D}}$ and $\mathcal{V}'(A[\alpha; \rho]) \neq +$ for every $\rho \in \mathcal{D}'$ for all $\mathcal{D}'$, $\mathcal{V}'$ such that $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{D}^*$, $\mathcal{V}'$ is an n^{th}-level J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}'}$ with respect to $\mathcal{D}^*$, and $\mathcal{V}'$ includes $\mathcal{V}$, then $\mathcal{V}[(\exists\alpha)A] = -$.

Lemma 3 *Suppose $\mathcal{V}$ is an n^{th}-level J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}}$ with respect to $\mathcal{D}^*$. Then $\mathcal{V}$ is an $(n+1)$st-level J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}}$ with respect to $\mathcal{D}^*$.*

Proof: By Lemma 2, $\mathcal{V}$ is quantificationally adequate with respect to $\mathcal{D}^*$ at at the n^{th} level. Let Y be the n^{th}-level family determined by $\mathcal{V}$. Every node of X is a member of Y. So the features common to members of Y are shared by members of X. This is sufficient to show that $\mathcal{V}$ is an $(n+1)$st-level J-valuation.

Theorem 1 *$\mathcal{V}$ is an acceptable J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}}$ with respect to $\mathcal{D}^*$.*

For the following lemmas and Theorem 2, let $\mathcal{D}$, $\mathcal{D}^*$ be nonempty domains such that $\mathcal{D} \subseteq \mathcal{D}^*$. Let $\mathfrak{f}$ be a function which assigns individuals in $\mathcal{D}$ to individual constants of L. Let $\mathcal{V}$ be an acceptable J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}}$ with respect to $\mathcal{D}^*$. Let X be a tree structure whose top node is $\mathcal{V}$, and which is constructed according to: If $\mathcal{V}'$ is a node of X and is an acceptable J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}'}$ with respect to $\mathcal{D}^*$, where $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{D}^*$, and $\mathcal{D}''$, $\mathcal{V}''$ are such that $\mathcal{D}' \subseteq \mathcal{D}'' \subseteq \mathcal{D}^*$ and $\mathcal{V}''$ is an acceptable J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}''}$ with respect to $\mathcal{D}^*$ which includes $\mathcal{V}'$, then $\mathcal{V}''$ is an (immediate) successor of $\mathcal{V}'$. F is the natural-deduction system of intuitionist logic that is described in [1].

Lemma 4 *The structure X satisfies conditions (1) and (2) of the definition of 'acceptable J-structure with respect to $\langle \mathfrak{f}, \mathcal{D}, \mathcal{D}^* \rangle$'.*

For the following lemmas, let $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{D}^*$, let $\mathcal{V}'$ be an acceptable J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}'}$ with respect to $\mathcal{D}^*$, and let $\mathcal{V}'$ include $\mathcal{V}$.

Lemma 5 *Let $(\forall\alpha)A$ be a pseudo-sentence of $L^{f}_{\mathcal{D}'}$. For every $n > 0$, let there be a $\mathcal{D}''$ such that $\mathcal{D}' \subseteq \mathcal{D}'' \subseteq \mathcal{D}^*$ and an n^{th}-level J-valuation $\mathcal{V}''$ of $L^{f}_{\mathcal{D}''}$ with respect to $\mathcal{D}^*$ such that $\mathcal{V}''$ includes $\mathcal{V}'$ and, for some $\rho \in \mathcal{D}''$, $\mathcal{V}''(A[\alpha;\rho]) \neq +$. Then there is a $\mathcal{D}''$ such that $\mathcal{D}' \subseteq \mathcal{D}'' \subseteq \mathcal{D}^*$ and an acceptable J-valuation $\mathcal{V}''$ of $L^{f}_{\mathcal{D}''}$ with respect to $\mathcal{D}^*$ such that $\mathcal{V}''$ includes $\mathcal{V}'$ and, for some $\rho \in \mathcal{D}''$, $\mathcal{V}''(A[\alpha;\rho]) \neq +$.*

Proof: Suppose $\mathcal{D}' = \mathcal{D}^*$. Then there is a $\rho \in \mathcal{D}'$ such that $\mathcal{V}'(A[\alpha;\rho]) \neq +$. But $\mathcal{V}'$ includes $\mathcal{V}'$.

Suppose $\mathcal{D}' \neq \mathcal{D}^*$. Let σ be an individual in $\mathcal{D}^*$ such that $\sigma \notin \mathcal{D}'$. Let $\mathcal{D}'' = \mathcal{D}' \cup \{\sigma\}$. Let Y be the set of pseudo-sentences B of $L^{f}_{\mathcal{D}'}$ such that $\mathcal{V}'(B) = +$. Let F'' be the deductive system F of [1] extended to pseudo-sentences of $L^{f}_{\mathcal{D}''}$, and let Y' be the closure of Y under deducibility in F''.

By Theorem 2 of [1], there is an acceptable J-valuation $\mathcal{V}'$ of $L^{f}_{\mathcal{D}''}$ with respect to $\mathcal{D}^*$ such that for every pseudo-sentence B of $L^{f}_{\mathcal{D}''}$, $\mathcal{V}''(B) = +$ iff $B \in Y'$, and $\mathcal{V}''(B) = -$ iff $\sim B \in Y'$.

Suppose $A[\alpha;\sigma] \in Y'$. Then $A[\alpha;\sigma]$ is deducible from Y. But then $(\forall\alpha)A$ is deducible from Y. This deduction can be carried out in the system F' obtained by extending F to pseudo-sentences of $L^{f}_{\mathcal{D}'}$. Since F' is sound for acceptable J-valuations of $L^{f}_{\mathcal{D}'}$, $\mathcal{V}'[(\forall\alpha)A] = +$. This *contradicts* the assumption of the lemma. So $A[\alpha;\sigma] \notin Y'$. Hence $\mathcal{V}''(A[\alpha;\sigma]) \neq +$.

Lemma 6 *Let $(\exists\alpha)A$ be a pseudo-sentence of $L^{f}_{\mathcal{D}'}$. For every $n > 0$, let there be a $\mathcal{D}''$ such that $\mathcal{D}' \subseteq \mathcal{D}'' \subseteq \mathcal{D}^*$ and an n^{th}-level J-valuation $\mathcal{V}''$ of $L^{f}_{\mathcal{D}''}$ with respect to $\mathcal{D}^*$ such that $\mathcal{V}''$ includes $\mathcal{V}'$ and, for some $\rho \in \mathcal{D}''$, $\mathcal{V}''(A[\alpha;\rho]) = +$. Then there are a similar $\mathcal{D}''$, $\mathcal{V}''$ where $\mathcal{V}''$ is an acceptable J-valuation of $L^{f}_{\mathcal{D}''}$ with respect to $\mathcal{D}^*$.*

Proof: Let Y be the set of pseudo-sentences B of $L^{f}_{\mathcal{D}'}$ such that $\mathcal{V}'(B) = +$. And let F' be the deductive system F of [1] extended to pseudo-sentences of $L^{f}_{\mathcal{D}'}$.

Suppose $\mathcal{D}' = \mathcal{D}^*$. And suppose that for every $\rho \in \mathcal{D}'$, the set $Y \cup \{A[\alpha;\rho]\}$ is not consistent. But then, for every $\rho \in \mathcal{D}'$, the pseudo-sentence $\sim A[\alpha;\rho]$ is deducible in F' from Y. So, for every $\rho \in \mathcal{D}'$, $\mathcal{V}'(A[\alpha;\rho]) = -$. This *contradicts* the assumption of the lemma. Select $\sigma \in \mathcal{D}'$ such that $Y \cup \{A[\alpha;\sigma]\}$ is consistent. Let Y' be the closure of $Y \cup \{A[\alpha;\sigma]\}$ under deducibility in F'. By Theorem 2 of [1], there is an acceptable J-valuation $\mathcal{V}''$ of $L^{f}_{\mathcal{D}'}$ with respect to $\mathcal{D}^*$ such that for every pseudo-sentence B of $L^{f}_{\mathcal{D}'}$, $\mathcal{V}''(B) = +$ iff $B \in Y'$, and $\mathcal{V}''(B) = -$ iff $\sim B \in Y'$. So $\mathcal{V}''$ includes $\mathcal{V}'$ and $\mathcal{V}''(A[\alpha;\sigma]) = +$.

If $\mathcal{D}' \neq \mathcal{D}^*$ we can argue similarly to the proof of Lemma 5 to show that this lemma holds.

Lemma 7 *The structure X satisfies conditions (3) and (4) of the definition of 'acceptable J-structure with respect to $\langle f, \mathcal{D}, \mathcal{D}^*\rangle$'.*

Proof: (3) Let $(\forall\alpha)A$ be a pseudo-sentence of $L^{f}_{\mathcal{D}'}$. Let every descendant $\mathcal{V}''$ of $\mathcal{V}'$ (in X) assign $+$ to $A[\alpha;\rho]$ for every ρ in $\mathcal{D}''$. Suppose $\mathcal{V}'[(\forall\alpha)A] \neq +$. Then for every $n > 0$, there is a $\mathcal{D}''$ such that $\mathcal{D}' \subseteq \mathcal{D}'' \subseteq \mathcal{D}^*$ and an n^{th}-level J-valuation $\mathcal{V}''$ of $L^{f}_{\mathcal{D}''}$ with respect to $\mathcal{D}^*$ that includes $\mathcal{V}'$ and such that, for some $\rho \in \mathcal{D}''$, $\mathcal{V}''(A[\alpha;\rho]) \neq +$. By Lemma 5, there are a similar $\mathcal{D}''$, $\mathcal{V}''$

where $\mathcal{V}''$ is an acceptable J-valuation of $L^{\mathfrak{f}}_{\mathcal{D}''}$ with respect to $\mathcal{D}^*$. But then $\mathcal{V}''$ is a successor of $\mathcal{V}'$ in X. This is *impossible.* So $\mathcal{V}'[(\forall\alpha)A] = +$.

Lemma 6 is used in a similar fashion to show that X satisfies condition (4).

Lemma 8 *Let A be a pseudo-sentence of $L^{\mathfrak{f}}_{\mathcal{D}'}$, and let no descendant of $\mathcal{V}'$ in X assign + to A. Then $\mathcal{V}'(A) = -$.*

Proof: Suppose $\mathcal{V}'(A) \neq -$. Let Y be the set of pseudo-sentences of $L^{\mathfrak{f}}_{\mathcal{D}'}$ that are assigned + by $\mathcal{V}'$, and let F' be the deductive system of [1] extended to include pseudo-sentences of $L^{\mathfrak{f}}_{\mathcal{D}'}$. $Y \cup \{A\}$ is consistent, because otherwise $\sim A$ is deducible from Y in F', and $\mathcal{V}'(A) = -$. But then, by Theorem 2 of [1], there is an acceptable J-valuation $\mathcal{V}''$ of $L^{\mathfrak{f}}_{\mathcal{D}'}$ with respect to $\mathcal{D}^*$ such that $\mathcal{V}''$ includes $\mathcal{V}'$ and $\mathcal{V}''(A) = +$. This is *impossible.* So $\mathcal{V}'(A) = -$.

Lemma 9 *Let $\mathcal{V}'[A \vee B] = +$. Let C be a pseudo-sentence of $L^{\mathfrak{f}}_{\mathcal{D}'}$. And let every descendant of $\mathcal{V}'$ which assigns + to either A or B also assign + to C. Then $\mathcal{V}'(C) = +$.*

This is proved analogously to Lemma 8.

Lemma 10 *Let A, B be pseudo-sentences of $L^{\mathfrak{f}}_{\mathcal{D}'}$. Let every descendant of $\mathcal{V}'$ which assigns + to A also assign + to B. Then $\mathcal{V}'[A \supset B] = +$.*

Lemma 11 *Let $\mathcal{V}'[(\exists\alpha)A] = +$. Let C be a pseudo-sentence of $L^{\mathfrak{f}}_{\mathcal{D}'}$. And let every descendant $\mathcal{V}''$ of $\mathcal{V}'$ which assigns + to $A[\alpha; \rho]$ for some $\rho \in \mathcal{D}''$ also assign + to C. Then $\mathcal{V}'(C) = +$.*

Theorem 2 *The structure X is an acceptable J-structure with respect to $\langle \mathfrak{f}, \mathcal{D}, \mathcal{D}^* \rangle$.*

4 Some comparisons It is now possible to define an acceptable J-valuation as a node of an acceptable J-structure. Such a definition dispenses with the levels of the earlier definition. It might seem that I have merely traded the complication of levels for the complication of tree structures, but this is not correct. Tree structures were implicit in the earlier account. The first semantic account already made use of possible future valuations to determine the values of sentences in the present (in a present). The proof of Theorem 2 shows that an acceptable J-valuation (by the original definition) determines a tree structure of such valuations.

The tree structures of the present paper make explicit the structure of epistemic possibility. As we ordinarily conceive of possibility, the future contains a variety of alternative possibilities, only some of which will be realized. Whichever possibilities become actual, we will be faced with new possibilities. Our natural conception is of a tree structure. (It is important to emphasize here that I am talking about epistemic possibility, and not what Kripke has called metaphysical possibility. What is epistemically possible is what is allowed by current knowledge.) The present semantic account is more natural than the account of [1] both in eliminating levels and in rendering explicit the tree structure which is characteristic of possibility. The completeness proof for intuitionist logic shows that the concept of justification exactly "fits" our conception of epistemic possibility.

To determine the value of a sentence in the present, the semantic account of [1] made use of those epistemically possible futures that are (merely) logically possible. It made use of all logically possible futures—where logical possibility is determined by the meanings of the connectives and quantifiers. In the tree-structure semantics, the descendants of a given node need not include all logically possible futures. The tree structure is a heuristic model of a situation where the language L has been applied to a specific subject matter, and the logical justification conditions are supplemented by additional conditions. The tree structure need not contain all logically and epistemically possible futures of a given node, because features of a subject matter can be sufficient to rule out some logically possible futures. (In addition to having conditions derived from a specific subject matter, we might also insist on a stricter kind of justification than the general concept investigated here.)

The new characterization of acceptable J-valuations makes it easier to compare the present semantics with the semantic accounts of Kripke (found in [2]) and Beth (as explained in [2] and [3]). From the present standpoint, both of those accounts are less general than the account in this paper. Both of them are special cases of the present account, obtained by imposing additional justification conditions.

The present account allows a sentence $[A \vee B]$ to have value + when neither A nor B has this value. Similarly, a sentence $(\exists\alpha)A$ is allowed to have value + when no individual in the domain of the present justifies A. One way to impose additional justification conditions is to have $[A \vee \sim A]$ justified for every sentence A. This is to impose the "weakest" conditions on A. (The general conditions of the semantic account in this paper are not weak conditions. The general conditions can be supplemented by weak or strong conditions.) The "strongest" condition on '$\vee$' has $[A \vee B]$ justified just in case one of A, B is justified. One condition of intermediate strength is imposed if we do not accept excluded middle, but do take $[\sim A \vee \sim\sim A]$ to be justified for every sentence A. For $\exists$, the weakest condition amounts to accepting the following principle as valid: $\sim(\forall\alpha)\sim A/(\exists\alpha)A$. (Adopting the weakest condition for either $\vee$ or $\exists$ "collapses" the logic of justification into classical logic. So adopting the weakest condition for one forces us to adopt the weakest condition for the other.) The strongest condition on $\exists$ has $(\exists\alpha)A$ justified iff A is justified for some individual in the domain associated with the present. An intermediate condition on $\exists$ is obtained if we reject the weakest condition, but accept the following inference principle: $\sim(\forall\alpha)A/(\exists\alpha)\sim A$.

To characterize the semantic accounts of Kripke and Beth, I will consider tree structures whose nodes are valuations. Let X be an acceptable J-structure with respect to $\langle \mathfrak{f}, \mathscr{D}, \mathscr{D}^* \rangle$. X is a *Kripke J-structure with respect to* $\langle \mathfrak{f}, \mathscr{D}, \mathscr{D}^* \rangle$ iff every member of X imposes the strongest conditions on '$\vee$', '$\exists$'. A J-valuation is a *Kripke valuation* just in case it is the top node of a Kripke structure. (In Kripke's own presentation of his semantics, he recognizes only two values. It is the behavior of + which represents Kripke's account of intuitionist logic. Sentences with value 0 or − would be false on his account.)

X is a *Beth J-structure with respect to* $\langle \mathfrak{f}, \mathscr{D}, \mathscr{D}^* \rangle$ iff for every node $\mathscr{V}'$ of X, if $\mathscr{V}'$ is a J-valuation of $L^{\mathfrak{f}}_{\mathscr{D}'}$ and A is a pseudo-sentence which has value

+ on *every* branch of descendants of A, then $\mathcal{V}'(A) = +$. A J-valuation is a *Beth valuation* iff it is the top node of a Beth J-structure.

The "idea" behind the Beth semantics is related to the present treatment of the value $-$. For our account has it that a sentence A which is justified in no possible future is assigned the value $-$ in the present (so that $\sim A$ receives the value $+$). In the Beth semantics, a sentence which eventually becomes justified in every possible sequence of futures must already be justified. There may be situations for which the Beth semantics is plausible, but this semantics cannot provide a general account of justification. Adopting the Beth semantics would prevent us from assigning the strongest conditions to $\vee$ and $\exists$. (For a sentence $[A \vee B]$ might be justified when neither disjunct is.)

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