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On the Number of Generators of an Ideal

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A countably complete ideal I over a set S is *κ-saturated* if the Boolean algebra $P(S)/I$ does not have a subset of size κ of pairwise disjoint elements.* I is *λ-generated* if it has a subset X of size λ such that I is the smallest σ-ideal containing X. We denote by *sat*(I), *gen*(I) the least cardinal number κ (the least cardinal number λ) such that I is κ-saturated (λ-generated).

In [2], Baumgartner and Taylor prove that if every σ-ideal over ω_1 is $\aleph_2$-generated then every σ-ideal over ω_1 is $\aleph_3$-saturated, and ask the following question: *Can one prove that every $\aleph_2$-generated σ-ideal over ω_1 is $\aleph_3$-saturated?*

We answer this question in the negative:

Theorem 1 *It is consistent that the closed unbounded filter over ω_1 is $\aleph_2$-generated but not $\aleph_3$-saturated.*[1]

In fact, a σ-ideal can have $\aleph_2$ generators and not be κ-saturated for arbitrarily large κ:

Theorem 2 *Let M be a model of $V = L$ and let κ and λ be (in M) cardinals such that $\kappa \leqslant \lambda$ and $cf\,\kappa \geqslant \omega_2$, $cf\,\lambda \geqslant \omega_2$. Then there is a generic extension $M[G]$ in which*

$$gen(F) = \aleph_2,\ sat(F) = \kappa^+,\ 2^{\aleph_1} = \lambda$$

(where F is the closed unbounded filter over ω_1).[2]

Proof of Theorem 1: Let M be a model of ZFC in which $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} \geqslant \aleph_3$. We extend M generically by adjoining $\aleph_2$ closed unbounded subsets

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of ω_1 which will generate the closed unbounded filter in the extension. We adjoin the $\aleph_2$ closed unbounded subsets successively, using iterated forcing. The extension will still satisfy $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} \geqslant \aleph_3$ and by a theorem of Jech and Prikry this implies that no σ-ideal over ω_1 is $\aleph_3$-saturated (cf. [4]).

Let us consider the following notion of forcing $(Q,<)$: A forcing condition $q \in Q$ is a pair

$$q = (s,C)$$

where

(1) s is a closed countable subset of ω_1
(2) C is a closed unbounded subset of ω_1
(3) $max(s) < min(C)$.

The partial ordering on Q is defined as follows: $q = (s,C)$ is stronger than $q' = (s',C')$ iff

(4) $C \subseteq C'$
(5) s extends s'
(6) $s - s' \subseteq C'$.

The notion of forcing $(Q,<)$ is countably closed: if $\{(s_n,C_n):n \in \omega\}$ is a descending sequence of condition, then the condition (s,C) where $s = \bigcup_n s_n \cup \{sup_n max(s_n)\}$ and $C = \bigcap_n C_n$ is stronger than all of them. If $s = s'$ then the conditions (s,C) and (s',C') are compatible. Hence every incompatible set of conditions has size at most $2^{\aleph_0}$ and since $2^{\aleph_0} = \aleph_1$, $(Q,<)$ satisfies the $\aleph_2$-chain condition.

Also, we have $|Q| = 2^{\aleph_1}$.

Let G be a generic set of conditions. Since Q is σ-closed and has the $\aleph_2$-chain condition, all cardinals and cofinalities are the same in $M[G]$ as in M. Moreover, $M[G]$ has no new countable sets of ordinals, and $(2^{\aleph_0})^{M[G]} = (2^{\aleph_0})^M = \aleph_1$ and $(2^{\aleph_1})^{M[G]} = (2^{\aleph_1})^M$. Let

(7) $C_G = \bigcup \{s\colon \text{for some } C, (s,C) \in G\}$.

The set C_G is a closed unbounded subset of ω_1, and since G is Q-generic, we can easily see that

(8) if $C \in M$ is a closed unbounded subset of ω_1, then there is $\alpha < \omega_1$ such that $C_G - \alpha \subseteq C$.

In other words, every closed unbounded subset of ω_1 in the ground model contains an end segment of the set C_G.

The notion of forcing $(P,<)$ is obtained by iterating the above construction $\aleph_2$ times. We assume that the reader is familiar with the basic facts on iterated forcing; these can be found, among others, in [3] p. 457, or in [1].

We consider an iterated forcing of length ω_2, where at successor stages we use the notion $(Q,<)$ described above, and at limit stages take either direct or inverse limits; namely, we take inverse limits at limit ordinals of cofinality ω and direct limits at limit ordinals of cofinality $> \omega$.

More precisely, we define, by induction, for each $\alpha \leqslant \omega_2$ an α-stage iteration $(P_\alpha, <_\alpha)$, the corresponding Boolean-valued model M^{P_α} and the notion of forcing $\Vdash_\alpha$, and a notion of forcing $Q_\alpha \in M^{P_\alpha}$:

(9) $P_0 = \{1\}, M^{P_0} = M, Q_0 = Q$

(10) P_α is the set of all α-sequences $p = \langle p_\xi : \xi < \alpha\rangle$ such that

(i) for every $\gamma < \alpha$, $p \restriction \gamma \in P_\gamma$ and $p \restriction \gamma \Vdash_\gamma p_\gamma \in Q_\gamma$

(ii) $\{\xi < \alpha : p(\xi) \neq 1\}$ is at most countable

(11) if $p, q \in P_\alpha$ then $p \leqslant_\alpha q$ iff for every $\gamma < \alpha$, $p \restriction \gamma \Vdash_\gamma p_\gamma \leqslant_\gamma q_\gamma$

(12) $Q_\alpha \in M^{P_\alpha}$ is the notion of forcing defined in M^{P_α} by (1)-(6).

Finally, we let $(P,<) = (P_{\omega_2}, <_{\omega_2})$ be the ω_2-stage iteration.

Since for each α, $\Vdash_\alpha Q_\alpha$ is countably closed and has the $\aleph_2$-chain condition, and because we iterate with countable support, it follows from basic facts on iterated forcing that $(P,<)$ is countably closed and has the $\aleph_2$-chain condition. And also, $|P| = 2^{\aleph_1}$.

Let G be an M-generic filter on $(P,<)$. Since P is σ-closed and has the $\aleph_2$-chain condition, all cardinals and cofinalities are preserved. Also, $M[G]$ has no new countable sets of ordinals, satisfies $2^{\aleph_0} = \aleph_1$, and $2^{\aleph_1}$ is the same in $M[G]$ as in M.

We shall show that in $M[G]$, the closed unbounded filter is $\aleph_2$-generated.

For each $\alpha < \omega_2$ let $G \restriction \alpha = \{p \restriction \alpha : p \in G\}$, and let $G_\alpha = \{p_\alpha : p \in G\}$. Clearly, G_α is (isomorphic to) an $M[G \restriction \alpha]$-generic filter on $(Q,<)$ (where Q is defined by (1)-(6) in $M[G \restriction \alpha]$). Thus for each $\alpha < \omega_2$, we can define a closed unbounded set $C_\alpha = C_{G_\alpha}$ as in (7) and we have

(13) every closed unbounded subset of ω_1 in $M[G \restriction \alpha]$ contains an end segment of the set C_α.

The proof will be completed when we show that every closed unbounded subset of ω_1 in $M[G]$ belongs to some $M[G \restriction \alpha]$, $\alpha < \omega_2$. This however is a well-known consequence of the fact that $(P,<)$ has the $\aleph_2$-chain condition and that P is the direct limit of P_α, $\alpha < \omega_2$.

Proof of Theorem 2: We start with a model M of $V = L$.[3] Let $\kappa \leqslant \lambda$ be cardinals of cofinality $\geqslant \omega_2$. First we extend M generically to a model M_1 in which $2^{\aleph_1} = \kappa$ by adjoining (using countable conditions) κ subsets of ω_1. Next we extend M_1 to a model M_2 by the notion of forcing P described in the proof of Theorem 1. And finally, we extend M_2 to M_3 by adjoining (via finite condition) λ subsets of ω.

The passage from M to M_2 is via a countably closed notion of forcing. As M is a model of $V = L$, M satisfies the $\diamond$ principle. It is easy to see that an extension via a countably closed notion of forcing preserves the $\diamond$ principle (every $\diamond$-sequence in the ground model is a $\diamond$-sequence in the extension). It follows from $\diamond$ that there are $2^{\aleph_1}$ almost disjoint stationary sets; hence M_2 satisfies that the closed unbounded filter is not κ-saturated. As M_2 is an extension of M_1 via $(P,<)$, M_2 also satisfies that the closed unbounded filter is $\aleph_2$-generated.

The passage from M_2 to M_3 uses a ccc notion of forcing. It is well-known that when forcing with a ccc set of conditions, every closed unbounded set in the extension contains a closed unbounded set in the ground model and every stationary set in the ground model remains stationary in the extension. Thus in M_3, the closed unbounded filter is still $\aleph_2$-generated, and still not κ-saturated. Also, it is generated by the closed unbounded filter in M_2. The closed unbounded filter in M_2 is κ^+-saturated (because $(2^{\aleph_1})^{M_2} = \kappa$) and by a theorem of Baumgartner and Taylor [2] it generates a κ^+-saturated filter in any ccc extension. Thus the closed unbounded filter in M_3 is κ^+-saturated.

NOTES

1. Added in proof: A similar result was obtained independently by A. Kanamori (see [5]). His construction required a large cardinal in the ground model.

2. In a letter to the author, J. Baumgartner states: "... you can raise the generation number of the club filter by iterating as far as you like. Thus you could get, for example $gen(F) = \aleph_3$, $sat(F) = \aleph_4$, $2^{\aleph} = \aleph_5$."

3. As the referee points out, it is not necessary to start with a model of $V = L$, since $\diamond$ is automatically obtained when forcing with a countably closed partial ordering that adds a subset of ω_1.

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