

## On Fleissner's Diamond

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Fleissner [1], in the course of showing that  $V = L$  implies every normal topological space is collectionwise Hausdorff, used a strengthening of Jensen's  $\diamond$  principle, denoted  $\diamond_{SS}$ , and often called "diamond for stationary systems". Mathias [3] stated  $\diamond_{SS}$  explicitly and asked whether for  $\aleph_1$ , for example,  $\diamond_{SS}$  follows from the related principles  $\diamond_{\aleph_1}^*$  or  $\diamond_{\aleph_1}^+$ . The purpose of this paper\* is to show that these implications may fail even under relatively nice conditions. This result was announced in [4].

For the remainder of the paper  $\lambda$  denotes a regular uncountable cardinal and  $S$  a stationary subset of  $\lambda$ . The reader may, for simplicity, want to identify  $\lambda$  with  $\aleph_1$ .

We now introduce the various sorts of  $\diamond$ -sequences under consideration and mention some of the connections between them.

**Definition 1** A sequence  $\langle A_\alpha : \alpha \in S \rangle$  is a  $\diamond_S$ -sequence if for each  $\alpha \in S$ ,  $A_\alpha \subseteq \alpha$  and for every  $A \subseteq \lambda$ ,  $\{\alpha \in S : A \cap \alpha = A_\alpha\}$  is stationary (in  $\lambda$ ).

**Definition 2** A sequence  $\langle P_\alpha : \alpha \in S \rangle$  is a weak  $\diamond_S$ -sequence ( $w$ - $\diamond_S$ -sequence) if each  $P_\alpha$  is a set of subsets of  $\alpha$ , and for every  $A \subseteq \lambda$ ,  $\{\alpha \in S : A \cap \alpha \in P_\alpha\}$  is stationary. If, in addition,  $\overline{P_\alpha} \leq \overline{\alpha}$  for each  $\alpha \in S$ , we call  $\langle P_\alpha : \alpha \in S \rangle$  a  $\diamond_S$ -sequence.

The above definitions obviously involve an abuse of terminology. Notice however, that  $\langle A_\alpha : \alpha \in S \rangle$  is a  $\diamond_S$ -sequence in the sense of Definition 1 iff  $\{\langle A_\alpha \rangle : \alpha \in S\}$  is a  $\diamond_S$ -sequence in the sense of Definition 2.

Kunen has proved the following result relating the existence of the two types of  $\diamond_S$ -sequences.

**Theorem 1 (Kunen)** *If there is a  $\diamond_S$ -sequence  $\langle P_\alpha : \alpha \in S \rangle$ , with  $P_\alpha \subseteq P(\alpha)$ , then there is a  $\diamond_S$ -sequence  $\langle A_\alpha : \alpha \in S \rangle$ , with  $A_\alpha \subseteq \alpha$ .*

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An obvious question is whether the  $A_\alpha$ 's in Kunen's Theorem may be chosen so that  $A_\alpha \in P_\alpha$ . This cannot always be done, as witnessed by Theorem 5 below.

We next briefly describe Fleissner's results [1].

**Definition 3** A *stationary system* is a sequence  $\langle S_A : A \subseteq \lambda \rangle$  such that each  $S_A$  is stationary in  $\lambda$  and if  $A \cap \alpha = B \cap \alpha$ , then  $S_A \cap (\alpha + 1) = S_B \cap (\alpha + 1)$ , for each  $\alpha < \lambda$ .

**Definition 4**  $\diamond_{SS}$  holds if for any stationary system  $\langle S_A : A \subseteq \lambda \rangle$  there is a sequence  $\langle T_\alpha : \alpha < \lambda \rangle$  such that  $T_\alpha \subseteq \alpha$  and for each  $A \subseteq \lambda$ ,  $\{\alpha \in S_A : T_\alpha = A \cap \alpha\}$  is stationary, Fleissner [1] showed.

**Theorem 2 (Fleissner)**  $V = L$  implies  $\diamond_{SS}$ .

The proof is similar to Jensen's "minimal counterexample proof" of  $\diamond$  in  $L$  (cf. [2]).

**Theorem 3 (i) (Fleissner)** Assume  $\diamond_{SS}$ . If  $\langle P_\alpha : \alpha \in S \rangle$  is a  $w$ - $\diamond_S$ -sequence, then there are  $A_\alpha \in P_\alpha$ , for  $\alpha \in S$ , such that  $\langle A_\alpha : \alpha \in S \rangle$  is a  $\diamond_S$ -sequence.

(ii) The converse also holds.

*Proof:* (i) For each  $A \subseteq \lambda$  let  $S_A = \{\alpha \in S : A \cap \alpha \in P_\alpha\}$ . Then  $\langle S_A : A \subseteq \lambda \rangle$  is easily seen to be a stationary system. Let  $\langle T_\alpha : \alpha < \lambda \rangle$  be a sequence as given by  $\diamond_{SS}$ . Define  $A_\alpha$  so that if  $\alpha \in S$  then  $A_\alpha = T_\alpha$ , and  $A_\alpha$  is an arbitrary element of  $P_\alpha$  otherwise. Then  $\langle A_\alpha : \alpha \in S \rangle$  is a  $\diamond_S$ -sequence.

(ii) Let  $\langle S_A : A \subseteq \omega_1 \rangle$  form a stationary system. Let  $P_\alpha = \{X \subseteq \alpha : \text{for some (or, equivalently, for every) } A \subseteq \omega_1 \text{ such that } A \cap \alpha = X, \alpha \in S_A\}$ . Now, for every  $A \subseteq \omega_1$ ,  $\{\alpha : A \cap \alpha \in P_\alpha\} = S_A$ , and hence is stationary. Thus,  $\langle P_\alpha : \alpha < \omega_1 \rangle$  is a  $w$ - $\diamond$ -sequence. Now, by hypothesis, there are  $A_\alpha \in P_\alpha$ ,  $\alpha < \omega_1$ , such that  $\langle A_\alpha : \alpha < \omega_1 \rangle$  is a  $\diamond$ -sequence. Then, for a stationary set of  $\alpha < \omega_1$ ,  $A \cap \alpha = A_\alpha$ , whence  $A \cap \alpha \in P_\alpha$ , and so  $\alpha \in S_A$ . Thus  $\diamond_{SS}$  holds.

We will require two additional definitions before proceeding to our results.

**Definition 5**  $\langle P_\alpha : \alpha \in S \rangle$  is a  $\diamond_S^*$ -sequence iff  $\overline{P_\alpha} \leq \overline{\alpha}$  for each  $\alpha \in S$  and for every  $A \subseteq \lambda$  there is a closed, unbounded (club) set  $C$  such that  $\forall \alpha \in S \cap C$ ,  $A \cap \alpha \in P_\alpha$ . A  $w$ - $\diamond_S^*$ -sequence is defined similarly except there is no restriction that  $\overline{P_\alpha} \leq \overline{\alpha}$ .  $\diamond_S^*$  means that a  $\diamond_S^*$ -sequence exists.  $w$ - $\diamond_S^*$  is defined analogously.

**Definition 6**  $\langle P_\alpha : \alpha \in S \rangle$  is a  $\diamond_S^+$ -sequence iff  $\overline{P_\alpha} \leq \overline{\alpha}$  and for every  $A \subseteq \lambda$  there is a club  $C \subseteq \lambda$  such that for all  $\alpha \in S \cap C$ , both  $A \cap \alpha \in P_\alpha$  and  $C \cap \alpha \in P_\alpha$ . A  $w$ - $\diamond_S^+$ -sequence,  $\diamond_S^+$  and  $w$ - $\diamond_S^+$  are defined analogously.

Both  $\diamond_S^*$  and  $\diamond_S^+$  hold in  $L$ . The existence of a single  $\diamond_S^*$ -sequence has ramifications for all  $w$ - $\diamond_S$  sequences as we next show.

**Theorem 4** Assume  $\diamond_S^*$ . If  $\langle P_\alpha : \alpha \in S \rangle$  is a  $w$ - $\diamond$ -sequence then there is  $P_\alpha^* \subseteq P_\alpha$ ,  $\overline{P_\alpha^*} \leq \overline{\alpha}$  such that  $\langle P_\alpha^* : \alpha \in S \rangle$  is a  $\diamond_S$ -sequence.

*Proof:* Let  $\langle Q_\alpha : \alpha \in S \rangle$  be a  $\diamond_S^*$ -sequence. Let  $P_\alpha^* = P_\alpha \cap Q_\alpha$ . Then  $\overline{P_\alpha^*} \leq \overline{\alpha}$ . Fix  $A \subseteq \lambda$ . Let  $B$  be the stationary set provided for  $A$  by the definition of  $w$ - $\diamond_S$ -sequence. Let  $C$  be the club set provided for  $A$  by the definition of  $\diamond_S^*$ -sequence.

Then  $S \cap C \cap B$  is stationary, and for every  $\alpha \in S \cap C \cap B$ , both  $A \cap \alpha \in P_\alpha$  and  $A \cap \alpha \in Q_\alpha$ . Thus  $A \cap \alpha \in P_\alpha^*$ .

For  $\lambda$  regular and stationary  $S_1 \subseteq S_2 \subseteq \lambda$  the following implications hold.

$$\begin{array}{c} \diamond_{S_2}^* \Rightarrow \diamond_{S_1}^* \Rightarrow \diamond_{S_1} \Rightarrow \diamond_{S_2} \\ \uparrow \qquad \qquad \uparrow \\ \diamond_{S_2}^+ \Rightarrow \diamond_{S_1}^+ \end{array}$$

On the other hand,  $\diamond_{S_2}$  does not imply  $\diamond_{S_1}$  as shown in [5].

We are now ready for our main result.

**Theorem 5** *Assume  $1 < k \leq \aleph_0$ . The following are consistent:*

(1)  *$GCH + \diamond_{\omega_1}^+ + \langle P_\alpha: \alpha < \omega_1 \rangle$  is a  $\diamond_{\omega_1}$ -sequence,  $\overline{P}_\alpha = k$  and no refinement  $\langle P'_\alpha: \alpha < \omega_1 \rangle$ ,  $P'_\alpha \subsetneq P_\alpha$ ,  $\alpha < \omega_1$ , is a  $\diamond_{\omega_1}$ -sequence.*

(2)  *$GCH + \diamond_S$  for every stationary  $S \subseteq \omega_1 + \langle P_\alpha: \alpha < \omega_1 \rangle$  is a  $w\text{-}\diamond_{\aleph_1}$ -sequence,  $\overline{P}_\alpha = \aleph_1$  for every  $\alpha < \omega_1$ , and no refinement  $\langle P'_\alpha: \alpha < \omega_1 \rangle$ ,  $P'_\alpha \subsetneq P_\alpha$ ,  $\alpha < \omega_1$  is a  $\diamond_{\omega_1}$ -sequence.*

*Proof:* (1). The idea is to start with a model  $V$  of  $GCH$  and add sequences  $\langle R_\alpha: \alpha < \omega_1 \rangle$  and  $\langle S_\alpha: \alpha < \omega_1 \rangle$  such that in the generic extension  $\langle R_\alpha: \alpha < \omega_1 \rangle$  will be a  $\diamond_{\omega_1}^+$ -sequence and  $\langle S_\alpha: \alpha < \omega_1 \rangle$  will be a  $\diamond_{\omega_1}$ -sequence with  $|S_\alpha| = k$ , and such that  $\langle S_\alpha: \alpha < \omega_1 \rangle$  cannot be refined, i.e., there is no  $\langle T_\alpha: \alpha < \omega_1 \rangle$  with  $T_\alpha \subsetneq S_\alpha$  and  $\langle T_\alpha: \alpha < \omega_1 \rangle$  a  $\diamond_{\omega_1}$ -sequence. We will make sure that  $\langle R_\alpha: \alpha < \omega_1 \rangle$  is a  $\diamond_{\omega_1}^+$ -sequence by considering each subset of  $\omega_1$  in the extension and forcing a club set as in the definition of  $\diamond_{\omega_1}^+$ . We will make sure that  $\langle S_\alpha: \alpha < \omega_1 \rangle$  cannot be refined by considering each possible refinement of it in the extension and forcing a counterexample to its being a  $\diamond_{\omega_1}$ -sequence. Each of these ideas involves iterations of length  $\omega_2$ , and by dovetailing we can accomplish both simultaneously.

Rather than describing the forcing as a genuine iteration we will define notions of forcing  $X_\alpha$ ,  $\alpha \leq \omega_2$ , by induction on  $\alpha$ , and eventually use  $X_{\omega_2}$  for the extension. Assume now that  $X_\beta$  has been defined for all  $\beta < \alpha$ . Assume also that for each  $\beta > 1$  a term  $\tau_\beta$  of the language of forcing with  $X_\beta$  has been selected such that

- (i) if  $\beta$  is even, then  $0 \Vdash_{X_\beta} \text{“}\tau_\beta: \omega_1 \rightarrow 2\text{”}$
- (ii) if  $\beta$  is odd, then  $0 \Vdash_{X_\beta} \text{“}\tau_\beta: \omega_1 \rightarrow k\text{”}$ .

Now let  $X_\alpha$  consist of all functions  $p$  with domain a countable subset of  $\alpha$  containing 0 and 1, such that

- (iii)  $p(0)$  is a sequence  $\langle R_\xi: \xi < \delta \rangle$ , where  $\delta < \omega_1$ ,  $\overline{R}_\xi < \omega_1$  and  $R_\xi \subseteq P(\xi)$
- (iv)  $p(1)$  is a sequence  $\langle S_\xi: \xi < \gamma \rangle$ , where  $\gamma < \omega_1$ ,  $S_\xi = \langle S_{\xi,n}: n < k \rangle$ ,  $S_{\xi,n} \subseteq \xi$
- (v) if  $\beta \in \text{dom } p$ ,  $\beta > 1$  and  $\beta$  even, then  $p \upharpoonright \beta \in X_\beta$  and  $p(\beta)$  is a function mapping some countable  $\delta$  onto 2 such that  $\{\xi: p(\beta)(\xi) = 1\}$  is closed, and

$$p \upharpoonright \beta \Vdash_{X_\beta} \text{“}\forall \xi \text{ a limit } < \omega_1 [p(\beta)(\xi) = 1 \rightarrow (\{\eta < \xi: \tau_\beta(\eta) = 1\}, \{\eta < \xi: p(\beta)(\eta) = 1\} \in R_\xi)]\text{”}$$

- (vi) if  $\beta \in \text{dom } p$ ,  $\beta > 1$  and  $\beta$  odd, then  $p \upharpoonright \beta \in X_\beta$  and  $p(\beta)$  is a function mapping some countable  $\gamma$  into 2 such that

$$p \upharpoonright \beta \Vdash_{\overline{X_\beta}} \text{“}\forall \xi \text{ a limit } \leq \gamma \forall n < k [n \neq \tau_\beta(\xi) \rightarrow (S_{\xi,n} \neq \{\eta < \xi : p(\beta)(\eta) = 1\})]\text{”}.$$

The ordering is defined so that  $p \leq q$  iff  $\text{dom } q \subseteq \text{dom } p$  and  $\forall \beta \in \text{dom } q$ ,  $q(\beta) \subseteq p(\beta)$ , (i.e., the smaller element gives more information).

Now assume  $G$  is  $X_{\omega_2}$ -generic. For each  $\alpha < \omega_2$  define

$$G_\alpha = \{\cup p(\alpha) : p \in G\}.$$

Then it is straightforward to show that

1.  $G_0$  is a sequence  $\langle R_\alpha : \alpha < \omega_1 \rangle$  such that  $R_\alpha \subseteq P(\alpha)$  and  $\overline{R_\alpha} < \aleph_1$  for each  $\alpha < \omega_1$ .
2.  $G_1$  is a sequence  $\langle S_\alpha : \alpha < \omega_1 \rangle$  such that  $S_\alpha = \langle S_{\alpha,n} : n < k \rangle$ , and  $S_{\alpha,n} \subseteq \alpha$ , for each  $\alpha < \omega_1$ ,  $n < k$ .
3. For  $\alpha > 1$ ,  $G_\alpha$  is unbounded in  $\omega_1$ .
4. For  $\alpha > 1$ ,  $\alpha$  even  $G_\alpha$  is closed in  $\omega_1$ .

It is convenient to work with a certain dense subset  $Y_\alpha$  of  $X_\alpha$ . We call a condition  $p \in X_\alpha$  a *normal  $\delta$ -sequence* iff

- (i)  $\text{dom } p(0) = \text{dom } p(1) = \delta + 1$
- (ii)  $\forall \beta \in \text{dom } p$ ,  $\beta > 1 \Rightarrow \text{dom } p(\beta) = \delta + 1$
- (iii)  $\forall \beta \in \text{dom } p$ ,  $\beta > 1 \Rightarrow$  there is some function  $f_\beta \in V$  such that  $p \upharpoonright \beta \Vdash_{\overline{X_\beta}} \tau_\beta \upharpoonright \delta = f_\beta$ .

We let  $Y_\alpha$  be the set of all normal  $\delta$ -sequences for  $\delta \in Y_\alpha$ . We let  $\delta(p)$  be the  $\delta$  as described above.

It is quite easy to show that for each  $\alpha < \omega_2$ ,

5.  $Y_\alpha$  is  $\aleph_1$ -complete.

To see this, consider a sequence of elements of  $Y_\alpha$ ,  $q_0 > q_1 > \dots$ . We define  $q$  so that

- (i)  $\text{dom } q = \bigcup_{i \in \omega} \text{dom } q_i$ ,
- (ii)  $\forall \beta \in \text{dom } q$ , if  $\beta > 1$  then  $q(\beta) = \bigcup_{i \in \omega} q_i(\beta) \cup \{(\delta, 1)\}$
- (iii)  $q(0) = \bigcup_{i \in \omega} q_i(0) \cup \{(\delta, A)\}$  where  $\delta = \sup\{\delta(q_i) : i < \omega\}$  and  $A = \{\{\xi < \delta : q(\beta)(\xi) = 1\} : \beta \in \text{dom } q, \beta > 1, \beta \text{ even}\} \cup \{\{\xi < \delta : f_\beta(\xi) = 1\} : \beta \in \text{dom } q, \beta > 1, \beta \text{ even}\}$
- (iv)  $q(1) = \bigcup_{i \in \omega} q_i(1) \cup \{(\delta, D)\}$  where  $D$  is a sequence of length  $k$  of subsets of  $\delta$  containing none of the sets  $\{\{\xi < \delta : q(\beta)(\xi) = 1\} : \beta \in \text{dom } q, \beta > 1, \beta \text{ odd}\}$ .

Then  $q$  extends all the  $q_i$  and  $q$  is a normal  $\delta$ -sequence, whence in  $Y_\alpha$ . Of course, in verifying condition (iii) in the definition of normal  $\delta$ -sequence, we choose  $f_\beta = \bigcup_{n < \omega} f_{\beta,n}$ , where  $q_n \upharpoonright \beta \Vdash_{\overline{X_\beta}} \tau_\beta \upharpoonright \delta = f_{\beta,n}$ . It is now not difficult to check that

6.  $\forall p \in X_\alpha \forall \delta \in \omega_1 \exists q \in Y_\alpha [q < p \text{ and } \delta(q) \geq \delta]$ .

Next, using a standard  $\Delta$ -systems argument it follows that

7.  $Y_\alpha$  satisfies the  $\aleph_2$ -chain condition.

From the preceding we may conclude that if  $G$  is  $X_{\omega_2}$ -generic, then  $V[G]$  has the same cardinals as  $V$ , contains no new reals, and satisfies the *GCH*. As a consequence of this last fact, we can see that we could have chosen the  $\tau_\beta$ 's so that for  $\beta$  even  $\tau_\beta$  runs through all functions from  $\omega_1$  to 2 in  $V[G]$ , while for  $\beta$  odd  $\tau_\beta$  runs through all functions from  $\omega_1$  to  $k$  in  $V[G]$ .

Now, with the  $\tau_\beta$ 's chosen in this way it is easy to see that  $\langle R_\alpha: \alpha < \omega_1 \rangle$  will be a  $\diamond_{\omega_1}^+$ -sequence and  $\langle S_\alpha: \alpha < \omega_1 \rangle$  cannot be refined, where  $\langle R_\alpha: \alpha < \omega_1 \rangle$  is, of course,  $\cup \{p(0): p \in G\}$  and  $\langle S_\alpha: \alpha < \omega_1 \rangle$  is  $\cup \{p(1): p \in G\}$ . This follows since the sets  $\cup \{p(\beta): p \in G\}$ ,  $\beta > 1$ ,  $\beta$  even, are characteristic functions of the necessary club sets, while  $\cup \{p(\beta): p \in G\}$ ,  $\beta > 1$ ,  $\beta$  odd, add the necessary counterexamples. We will be finished if we can show that  $\langle S_\alpha: \alpha < \omega_1 \rangle$  is a  $\diamond$ -sequence in the sense of  $V[G]$ . This is the heart of the proof.

To this end, suppose  $A \subseteq \omega_1$  in  $V[G]$  and  $C$  a closed unbounded subset of  $\omega_1$  in  $V[G]$ . For  $\beta > 1$ ,  $\beta$  odd, we let

$$A_\beta = \{\xi: p(\beta)(\xi) = 1, p \in G\}.$$

Let  $\dot{A}$ ,  $\dot{C}$ , and  $\dot{A}_\beta$  be forcing names for  $A$ ,  $C$ , and  $A_\beta$  respectively. The argument divides into two cases.

*Case 1:* For all odd  $\beta < \omega_2$ ,  $A \neq A_\beta$ . It is sufficient to show that if  $p \Vdash \text{“}(\forall \beta < \omega_1)[\dot{A} \neq \dot{A}_\beta \text{ and } \dot{C} \text{ is club}]$ ,” then  $\exists q \leq p$  such that  $q \Vdash \text{“}\exists \alpha \in \dot{C} \exists n < k[\dot{A} \cap \alpha = S_{\alpha,n}]$ .”

First, using the current hypothesis, and the fact that  $V$  and  $V[G]$  have the same reals, there is a sequence  $q_0 \geq q_1 \geq \dots$ , in  $Y_{\omega_2}$  such that

( $\alpha$ )  $q_0 \leq p$  and for each  $n \in \omega$  there is some  $B_n \in V$  such that  $q_{n+1} \Vdash \text{“}B_n = \dot{A} \cap \delta(q_n) \ \& \ \{\alpha \in \dot{C}: \delta(q_n) \leq \alpha < \delta(q_{n+1})\} \neq \emptyset$ , and so that

( $\beta$ ) if  $n > 0$ , then for  $\beta \in \text{dom}(q_n)$ ,  $\beta > 1$ ,  $\beta$  odd,  $B_{n+1} \neq \{\eta < \delta(q_{n+1}): q_n(\beta)(\eta) = 1\}$ . Then if  $B = \bigcup_{n \in \omega} B_n$ ,  $B$  is not of the form  $\{\eta < \delta: \exists n q_n(\beta)(\eta) = 1\}$  for any  $n \in \omega$ ,  $\beta > 1$ ,  $\beta$  odd,  $\beta \in \bigcup_{n \in \omega} \text{dom } q_n$ .

We now select, as in the proof of Theorem 5, a lower bound  $q$ , making certain that if  $\delta = \sup\{\delta(q_n): n \in \omega\}$ , we define  $q(1)(\delta)(n) = B$ , for some  $n < k$  as in (iv) of the proof of Theorem 5. This is allowed by ( $\beta$ ). To see that  $q$  works we must use the fact that  $C$  is closed, so that  $\delta \in C$ , by ( $\alpha$ ).

*Case 2:* For some odd  $\beta < \omega_1$ ,  $A = A_\beta$ . This time it is sufficient to show that if  $p \Vdash \text{“}\dot{A} = \dot{A}_\beta \ \& \ \dot{C} \text{ is club”}$ ,  $\beta$  odd, then there is some  $q \leq p$  such that  $q \Vdash \text{“}\exists \alpha \in \dot{C} \exists n < k[\dot{A} \cap \alpha = S_{\alpha,n}]$ .”

We note first that if  $p \in X_{\gamma+1}$ , then there are  $q_0, q_1 \in Y_{\gamma+1}$  extending  $p$ , such that  $q_0 \upharpoonright \beta = q_1 \upharpoonright \beta$ , but  $q_0(\beta)$  and  $q_1(\beta)$  are incompatible. We combine this observation with the approach in Case 1 to obtain a binary tree of conditions  $q_s \in Y_{\omega_2}$ , where  $s$  ranges over all finite sequences of 0's and 1's such that

(i) if  $s \subseteq t$ , then  $q_t \leq q_s \leq p$ .

- (ii) if  $s$  and  $t$  are on the same level then  $q_s|\beta = q_t|\beta \leq p|\beta$ .
- (iii) if  $s$  and  $t$  are incomparable, then  $q_s(\beta)$  and  $q_t(\beta)$  are incompatible.
- (iv) if  $t$  is an immediate successor of  $s$ , then there is some sequence  $B_{q_t} \in V$  such that  $q_t \Vdash "B_{q_t} = \dot{A} \cap \delta(q_s) \ \& \ \{\alpha \in \dot{C} : \delta(q_s) \leq \alpha < \delta(q_t)\} \neq 0"$ .
- (v) if  $s, t$  are on level  $n + 1$ ,  $\beta \in \text{dom } q_{s|n}$ ,  $\gamma \in \text{dom } q_{t|n}$ ,  $\beta, \gamma$  odd, then  $q_s(\beta) \neq q_t(\gamma)$  except when  $\beta = \gamma, s = t$ .

Notice that for any branch  $f \in 2^\omega$ , by (ii),  $\delta = \sup\{\delta(q_s) : s \subseteq f\}$  comes out the same. If we define  $B_f = \bigcup_{s \subseteq f} B_{q_s}$ , then  $B_f$  is a subset of  $\delta$  and, for different  $f$ ,  $B_f$  is different. We wish to select some  $g \in 2^\omega$  and find a lower bound  $q$  for the sequence  $q_s$ ,  $s$  an initial segment of  $g$ , as in the proof of 5. The only restriction we place on  $q$  is that  $q(1)(\delta)$  consists of sets of the form  $B_f$ .

Let us do this last part exactly. First let  $q_n = q_s|\beta$  for some (equivalently every)  $s$  of length  $n$ . Next, choose  $k$  distinct  $g_i \in 2^\omega$  ( $i < k$ ). Now, define  $q^\circ \in Y_\beta$  with  $\delta(q^\circ) = \delta$ , as in 5, but let  $q^\circ(1)(\delta) = \langle B_{g_i} : i < k \rangle$  and  $q^\circ(0)(\delta) = \{\{\xi < \delta : \text{for some } n < \omega, q_{g_i|n}(\gamma)(\xi) = 1\} : i < k \text{ and } \gamma \in \text{dom } q_{g_i|n} \text{ for some } n < \omega\} \cup \{\{\xi < \delta : q_{g_i|n} \Vdash_{X_\gamma} " \tau_\gamma(\xi) = 1" \text{ for some } \gamma : i < k \text{ and } \gamma \in \text{dom } q_{g_i|n} \text{ for some } n \in \omega\}\}$ . We want to extend  $q^\circ$  to a bound of  $\{q_{g_i|n} : n < \omega\}$  for some  $i < k$ . Next, since  $Y_\beta$  is dense in  $X_\beta$ , there is some  $q^1 \in Y_\beta$ ,  $q^1 \leq q^\circ$  such that  $q^1 \Vdash_{X_\beta} " \tau_\beta(\delta) = i"$ , for some fixed  $i < k$ . Now define  $q$  so that  $q(\alpha) = q^1(\alpha)$  for  $\alpha < \beta$  while  $q(\alpha) = \bigcup_{n < \omega} q_{g_i|n}(\alpha)$  for  $\alpha \geq \beta$ ,  $\alpha \in \bigcup_{n < \omega} \text{dom } q_{g_i|n}$ . Then,  $q(0)(\delta)$  is as required for  $q(\gamma)$ ,  $\gamma$  even (see 5 above), by the way we defined it;  $q(0)(\delta)$  works for  $\beta$  since  $q^1 \Vdash " \tau_\beta(\delta) = i"$ , and for any other  $\gamma$  by (v) above.

(2) The proof of Theorem 5(2) is similar to that of Theorem 5(1) and related to [5], so we only give a sketch. We will use forcing conditions  $X_\alpha$  similar to those used in Theorem 5(1), but with certain modifications. We will use  $P_\beta$  to indicate the set of  $p(\beta)$ 's for  $p \in X_\alpha$ .

First, we will now use  $P_1$  to add a  $w$ - $\diamond$ -sequence. Elements of  $P_1$  are of the form  $\langle S_\alpha : \alpha < \beta \rangle$ , where  $S_\alpha = \langle S_{\alpha,\xi} : \xi < \omega_1 \rangle$  and  $S_{\alpha,\xi} \subseteq \alpha$ . In order to ensure that later on the  $\aleph_2$ -chain condition holds and that conditions can be arbitrarily extended, we also make the requirement that if  $\alpha$  is a limit, then there are  $\alpha_n < \alpha$ ,  $\alpha_n < \alpha_{n+1}$ ,  $\alpha = \bigcup_{n < \omega} \alpha_n$ , and for each  $n < \omega$  and  $t \in 2^n$  a function  $g_t : \alpha_n \rightarrow 2$ , such that for  $s \subseteq t$ ,  $g_s \subseteq g_t$  and

$$\{S_{\alpha,\xi} : \xi < \omega_1\} = \{g : \text{for some } t \in 2^\omega, g = \bigcup_{n < \omega} g_{t|n}\},$$

where  $S_{\alpha,\xi}$  is identified with its characteristic function.

Next, as before, we add counterexamples in the odd places to make sure that the sequence we have added cannot be refined.

We will concentrate on  $\diamond_S$  for every stationary  $S$ . First, for  $\beta < \omega_2$  even let  $\tau_\beta$  be a forcing name for a subset of  $\omega_1$  such that every such  $X_{\omega_2}$ -name appears. Then, in defining  $X_{\beta+1}$  for  $\beta$  odd, we require that  $p(\beta)$  is of the form  $\langle T_\gamma : \gamma \in x \rangle$ , where  $T_\gamma \subseteq \gamma$  for each  $\gamma \in x$  and  $x \subseteq \omega_1$ .

The idea is that if the set  $S$  denoted by  $\tau_\beta$  turns out to be stationary, then the sequence of  $T_\gamma$ 's will form the required  $\diamond_S$ -sequence.

We begin by defining  $Y_\alpha$  and  $\delta(p)$  for  $p \in Y_\alpha$  as in (i). Now, suppose  $p \Vdash \text{“}\dot{A} \subseteq \omega_1 \ \& \ \dot{C} \text{ is a club set in } \omega_1\text{,”}$  and  $G$  is a  $X_{\omega_2}$ -generic set containing  $p$ . We work in  $V[G]$  to define  $\bar{C} = \{\delta: \exists q \in G \cap Y_{\omega_2} [\delta(q) = \delta \text{ and } q \Vdash \text{“}\sup \dot{C} \cap \delta = \delta\text{,”} \text{ and for some } A_\delta \in V, q \Vdash \text{“}\dot{A} \cap \delta = A_\delta\text{,”} \text{ and } q \Vdash \text{“}\delta \notin \tau_\beta\text{”}]\}$ .

We notice next that if  $\bar{C}$  is not stationary, then neither is the set denoted by  $\tau_\beta$ , in which case we need not concern ourselves with it. We thus assume that  $\bar{C}$  is stationary. Now, we can show that  $\{p \in X_{\omega_2}: p \leq q \text{ and } p \Vdash \text{“}\exists \alpha \in \dot{C} [A \cap \alpha = T_\alpha]\text{”}\}$ , where  $T_\alpha$  comes from the prospective  $\diamond_{\tau_\beta}$ -sequence, is dense below some condition in  $G$ , and so in  $V[G]$ ,  $\langle T_\alpha \rangle_{\alpha < \omega_1}$  really is a  $\diamond_{\tau_\beta}$ -sequence.

As a consequence of Theorem 5 (2) and Theorem 3 we obtain

**Corollary 1**      $\diamond_{\omega_1}^+$  does not imply  $\diamond_{SS}$ .

**Corollary 2**      $\diamond_S$  for every stationary  $S \subseteq \omega_1$  does not imply  $\diamond_{SS}$ .

As a consequence of Theorem 5 (1) we see that Kunen's result, Theorem 1, cannot be strengthened so as to select  $A_\alpha$  from  $P_\alpha$ .

We conclude with some brief remarks on generalizing Theorem 5 to other regular cardinals. If we wish to replace  $\aleph_1$  by  $\lambda^+$  we must replace the tree  $\bigcup_{n < \omega} 2^n$  by a downward closed  $T \subseteq \bigcup_{\alpha < \lambda} 2^\alpha$ , with  $\sum_{\alpha < \lambda} |T \cap 2^\alpha| < \lambda$  for  $\beta < \lambda$ , which (for some  $\lambda$ ) has to have  $\lambda^+$  branches (i.e., a Kurepa tree). However, there is no problem since we may start with  $V = L$ . We can also, for convenience, ignore the  $\alpha < \lambda$  in the definition of the conditions.

If we wish to replace  $\aleph_1$  by a strongly inaccessible cardinal  $\kappa$ , then in the definition of a condition we require that the domains of  $p(0)$  and  $p(1)$  include the cardinal of  $\text{dom } p$ , and in (v) and (vi) of the definition of condition, we replace “ $\forall \xi$  a limit” by “ $\forall \xi$  a strong limit cardinal”. We can then have either  $k < \kappa$  or  $\bar{P}_\alpha \leq \bar{\alpha}$  or  $\bar{P}_\alpha \leq \bar{\alpha}^+$ .

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