Notre Dame Journal of Formal Logic
Volume 22, Number 1, January 1981

Probabilistic, Truth-Value, and Standard Semantics and the Primacy of Predicate Logic

JOHN A. PAULOS

Hartry Field maintained the thesis that reference and conceptual role together could account for all facts about the notion of meaning: "Truth-theoretic [Tarskian] semantics and conceptual-role semantics must supplement each other: truth-theoretic semantics cannot account for certain differences in sense unaccompanied by differences in reference; and conceptual-role semantics, though it deals nicely with questions of intra-speaker synonymy, cannot properly answer questions about inter-speaker synonymy or about relations between language and the world" ([2], p. 380). In the course of developing this thesis he provided a probabilistic semantics for the predicate calculus. This semantics, unlike the standard Tarski approach, dispenses with the notions of truth and reference and uses only the epistemic notion of subjective conditional probability. In this paper we give a proof of the equivalence of these two semantic approaches which also demonstrates their equivalence to another non-referential semantics, the truth-value (or substitution-theoretic) semantics of Leblanc [4], Dunn and Belnap [1], and others. Indeed Field's probabilistic semantics, it will be seen, is most naturally viewed as a generalization of truth-value semantics, the conditional probability of a sentence (pair) being determined in much the same nonreferential way as the truth value of a sentence. We close by discussing the possibility (and impossibility) of developing a probabilistic semantics for extensions of predicate logic and the implications this has for its primacy among logics.

Let L be a countable first-order language and let Pr be a (conditional probability) function from pairs of sentences of L to the interval $[0,1]$. "$Pr(A|B)$" is to be read "the probability of A given B". Following Field we say Pr is a *reasonable conditional probability function*[1] if there is some countable

Received January 3, 1979; revised February 5, 1980

extension L^+ of L, obtained by adding constants, and a function Pr^+ extending Pr such that the following axioms hold, A, B and C being sentences of L^+.

A1 $Pr^+(A|B) \geqslant 0$

A2 $Pr^+(A|A) = 1$

A3 $\exists A \exists B\, Pr^+(A|B) \neq 1$

A4 *If* $Pr^+(A|B) = 1$ *and* $Pr^+(B|A) = 1$, *then* $Pr^+(C|A) = Pr^+(C|B)$ *for all* C

A5 $Pr^+(A \wedge B|C) = Pr^+(A|B \wedge C) \cdot Pr^+(B|C)$

A6 $Pr^+(A \wedge B|C) \leqslant Pr^+(A|C)$

A7 $Pr^+(\neg B|A) = 1 - Pr^+(B|A)$, *unless* $Pr^+(C|A) = 1$ *for all* C

A8 $Pr^+(\exists x\, F(x)|C) = \lim_{n\to\infty} Pr^+(F(a_1) \vee F(a_2) \ldots \vee F(a_n)|C)$, *where* $a_1, a_2, \ldots$, *is an enumeration of all the terms of* L^+ *and* F *has one free variable.*

Note that it is not part of the definition (although it will follow from it) that if A is logically equivalent to B and C is logically equivalent to D, then $Pr^+(A|C) = Pr^+(B|D)$. This is not an axiom because we are giving an alternative account of semantics and thus cannot use standard semantic notions such as logical equivalence in our definition without question-begging. The notions of truth and reference (and hence that of models) are also dispensed with in favor of Pr, a (subjective) conditional probability function. How this is done follows.

A sentence B in L is said to be *certain* under a reasonable Pr if for all C in L, $Pr(B|C) = 1$. B is said to be *probabilistically valid* if B is certain for all reasonable Pr. If $\Gamma = \{A_1, A_2, \ldots, A_n, \ldots\}$, then we stipulate that $Pr(\Gamma|C) = \lim_{n\to\infty} Pr(A_1 \wedge \ldots \wedge A_n|C)$. The inference from Γ to B is probabilistically valid if for all reasonable Pr and all C in L, we have $Pr(\Gamma|C) \leqslant Pr(B|C)$.

Turning now to truth-value (substitution-theoretic) semantics as developed by Leblanc, Dunn and Belnap, and others we say that a function v from the sentences of L to $\{0,1\}$ is a *completable truth-valuation* if there is a countable extension L^+ of L and an extension v^+ of v such that the following axioms hold, A and B being sentences of L^+.

1. $v^+(\neg A) = 1 - v^+(A)$
2. $v^+(A \wedge B) = \min(v^+(A), v^+(B))$
3. $v^+(\exists x\, F(x)) = 1$ iff there is a term a_i in L^+ such that $v^+(F(a_i)) = 1$. (Equivalently $v^+(\exists x\, F(x)) = \lim_{n\to\infty} (F(a_i) \vee \ldots \vee F(a_n))$ where the a_i enumerate the terms of L^+.)

A sentence B in L is *true* under a completable v if $v(B) = 1$. B is *truth-value valid* if for all completable v, $v(B) = 1$. The inference from Γ to B is truth-value valid if for every completable v whenever $v(\Gamma) = 1$, then $v(B) = 1$. If $\Gamma = \{A_1, \ldots, A_n, \ldots\}$, then $v(\Gamma) = 1$ iff $v(A_i) = 1$ for all i.

These somewhat similar sets of definitions can, of course, be extended in an obvious way to "probabilistic satisfiability", "truth-value satisfiability", etc. We now have the following important theorem.

Equivalence Theorem I *Inferences (and sentences) are probabilistically valid if and only if they are truth-value valid.*

Proof: (Although he did not mention truth-value semantics, the proof is due essentially to Field [2].) Suppose the inference from Γ to A is not truth-value valid. Then there is some completable truth-valuation v such that $v(\Gamma) = 1$ and

$v(A) = 0$. Associate with v a conditional probability function Pr_v such that for all pairs of L sentences B and C

$$Pr_v(B|C) = \begin{cases} 1 & \text{if } v(C \to B) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $v(C) = 1$ for some L sentence C, then $Pr_v(\Gamma|C) = 1$ and $Pr_v(A|C) = 0$. Thus if Pr_v can be shown to be reasonable, the inference from Γ to A is not probabilistically valid. An extension of Pr_v satisfies the limit requirement in the definition of reasonable conditional probability function because of the extension clause in the definition of completable truth-value assignment and the substitution interpretation of the existential quantifier. The other requirements for a reasonable conditional probability function are also easily seen to be satisfied by Pr_v. Consider for example $Pr^+(B \wedge C|D) \leqslant Pr^+(B|D)$. If $v(D \to B) = 1$, then $v(D \to B \wedge C) = 0$ or 1. If $v(D \to B) = 0$, then $v(D \to B \wedge C) = 0$. Pr_v thus satisfies the condition and extensions Pr_v^+ of Pr_v satisfy it by the extension clause in the definition of completable truth-value assignment.

Conversely, let Pr be a reasonable Pr which invalidates the inference from Γ to A and let $\Delta_0 = \Gamma \cup \{\neg A\}$. There is thus an extension Pr^+ of Pr defined on sentences of L^+ satisfying conditions 1-8 above. Order all the sentences of L^+ as $S_1, S_2, S_3, \ldots$. Since Pr^+ invalidates the inference from Γ to A, there is a C such that $Pr^+(\Gamma|C) > Pr^+(A|C)$ and hence by conditions 5, 6, and 7 on Pr^+, we have that $Pr^+(\Delta_0|C) > 0$. Choose such a sentence C. Then define Δ_{n+1} as follows.

$$\Delta_{n+1} = \begin{cases} \text{(i) } \Delta_n \cup \{s_n\} & \text{if } Pr^+(\Delta_n \cup \{s_n\}|C) > 0 \text{ and } s_n \neq \exists x\, F(x) \\ \text{(ii) } \Delta_n \cup \{s_n\} \cup \{F(a_k)\} & \text{if } Pr^+(\Delta_n \cup \{s_n\}|C) > 0 \text{ and } s_n = \exists x\, F(x).\ a_k \text{ is the first term in } L^+ \text{ for which } Pr^+(\Delta_n \cup \{F(a_1) \vee \ldots \vee F(a_k)\}|C) > 0 \\ \text{(iii) } \Delta_n \cup \{\neg s_n\} & \text{if } Pr^+(\Delta_n \cup \{s_n\}|C) = 0 \text{ and } s_n \neq \neg\exists x\, F(x) \\ \text{(iv) } \Delta_n \cup \{\neg s_n\} \cup F(a_k) & \text{if } Pr^+(\Delta_n \cup \{s_n\}|C) = 0 \text{ and } s_n = \neg\exists x\, F(x).\ a_k \text{ is as above.} \end{cases}$$

(The following argument shows that the a_k mentioned above in clauses (ii) and (iv) do exist when Δ_n is infinite: If Δ_n is finite, a_k exists by condition 8 of the definition of a reasonable probability function. We have in clause (ii) that $0 < Pr^+(\Delta_n \cup \{s_n\}|C) = \lim_{r\to\infty} Pr^+(\Delta_n^r \cup \{s_n\}|C)$ where Δ_n^r consists of the first r sentences in Δ_n. By condition 5 this equals

$$\lim_{r\to\infty} Pr^+(s_n|\Delta_n^r \cup \{C\})\cdot Pr^+(\Delta_n^r|C)$$

which by the limit requirement on Pr^+ equals

$$\lim_{r\to\infty} \lim_{k\to\infty} Pr^+(F(a_1) \vee \ldots \vee F(a_k)|\Delta_n^r \cup \{C\})\cdot Pr^+(\Delta_n^r|C).$$

If we interchange limits and use condition 5 once more we get

$$\lim_{k\to\infty} \lim_{r\to\infty} Pr^+(\Delta_n^r \cup (F(a_1) \vee \ldots \vee F(a_k))|C)$$

or, equivalently,

$$\lim_{k\to\infty} Pr^+(\Delta_n \cup (F(a_1) \vee \ldots \vee F(a_k)) | C).$$

Thus there is an a_k such that

$$Pr^+(\Delta_n \cup (F(a_1) \vee \ldots \vee F(a_k)) | C) > 0.$$

The demonstration for clause (iv) is similar.)

An easy induction shows $Pr^+(\Delta_n | C) > 0$ for all n. Let $\Delta = \bigcup_{n=1} \Delta_n$. By a standard argument (using $Pr^+(\Delta_n | C) > 0$ and the reasonableness of Pr), we see that

$$v^+(A) = \begin{cases} 1 & \text{if } A \in \Delta \\ 0 & \text{if } A \notin \Delta \end{cases}$$

is a truth-valuation on L^+ and thus that its restriction v to sentences in L is a completable truth-valuation on L such that $v(\Gamma) = 1$ and $v(A) = 0$.

Equivalence Theorem II *Inferences (and sentences) are truth-value valid if and only if they are (Tarski) valid.*

Proof: By the work of Leblanc, Dunn and Belnap, and others, truth-value (substitution-theoretic) semantics is equivalent to standard Tarski semantics. Roughly, countable models are "Henkinized" (every element is named) and thus shown to correspond to completable truth-valuations. Since validity in countable structures is equivalent to validity in general by the (downward) Löwenheim-Skolem theorem, the equivalence of truth-value and standard semantics is obtained. Combining these results we have the following.

Extended Equivalence and Completeness Theorem *Probabilistic validity (satisfiability) = truth-value validity (satisfiability) = standard validity (satisfiability).*

We thus see that truth-value semantics is nonreferential and appeals only to the notion of truth. It dispenses with models, utilizing instead the notion of a completable truth-valuation. Probabilistic semantics is also nonreferential and hence also dispenses with models. It, however, substitutes for completable truth-valuation the epistemic notion of a reasonable (subjective) conditional probability function. Our approach above demonstrates how the latter is a natural generalization of the former.[2]

Finally something should be mentioned about the possibility of proving an Extended Equivalence Theorem (or merely an Equivalence Theorem II) for a logic stronger than the predicate calculus. Consider the logic $L(Q)$ which is the predicate calculus with an added quantifier "$Qx\phi(x)$" which is taken to mean "for uncountably many x, $\phi(x)$." This logic (which has a complete axiomatization) satisfies a *weak* downward Löwenheim-Skolem theorem: If Γ, a set of $L(Q)$ sentences, has a model of uncountable cardinality, then it has an elementarily equivalent one of cardinality $\aleph_1$. Thus to get a truth-value semantics for $L(Q)$, call a function v from sentences of $L(Q)$ to $\{0,1\}$, a completable truth-valuation if there is an uncountable ($\aleph_1$) extension L^+ of L and an extension v^+ of v such that the following conditions hold, A and B being sentences of L^+.

1. $v^+(\neg A) = 1 - v^+(A)$
2. $v^+(A \wedge B) = \min(v^+(A), v^+(B))$
3. $v^+(\exists x\, F(x)) = 1$ iff there is a term a_α in L^+ such that $v^+(F(a_\alpha)) = 1$
4. $v^+(Qx\, F(x)) = 1$ iff $v^+(F(a_\alpha)) = 1$ for uncountably many a_α.

It is easy to see that Equivalence Theorem II goes through in essentially the same way (Henkinization and weak downward Löwenheim-Skolem theorem). Equivalence Theorem I goes through but with a little more difficulty if we add the following condition to the definition of a reasonable probability function:

A9 *Pr has an extension Pr^+ defined on L^+, an extension of L obtained by adding $\aleph_1$ constants, such that (in addition to satisfying conditions 1-8) $Pr^+(Qx\, F(x))|C) = \sup Pr^+\left(\bigwedge F(a_\alpha)|C\right)$, the sup being taken over all uncountable conjunctions of L^+.*

(Note that Pr^+ is thus conceived as a function from possibly uncountable sets of formulas to [0,1] and thus is philosophically problematic.) Obvious clauses governing $s_n = Qx\, F(x)$ and $s_n = \neg Qx\, F(x)$ must be added to the definition of Δ_{n+1}. Thus the Extended Equivalence Theorem can be shown to hold for $L(Q)$ although it is much less natural than in the case of the predicate logic.

I'd like to close with a question and a comment.

For what other logics can we prove an Extended Equivalence Theorem? It seems impossible, for example, that such a theorem could hold for (full) second-order logic or for $L(Q^c)$, where Q^c is Chang's equicardinality quantifier. In the case of second-order logic there is no obvious way of handling the $Pr^+(\forall X\, \mathcal{F}(X))$ or $v^+(\forall X\, \mathcal{F}(X))$ clause since the Löwenheim-Skolem theorem fails. In the case of $L(Q^c)$, ($Q^c_X \mathcal{F}(X)$ is true in a model $\mathfrak{M}$ iff the cardinality of the set of elements satisfying $\mathcal{F}$ equals the cardinality of $\mathfrak{M}$), although there is a weak ($\aleph_1$) downward Löwenheim-Skolem theorem, a natural probabilistic semantics seems unlikely since the quantifier explicitly refers to the domain of the model. Indeed the referential powers of both these logics seem to be too great for a conceptual-role (probabilistic semantic) analysis.

Per Lindstrom [5] proved that predicate logic is the largest logic[3] satisfying completeness (or compactness) and the (strong) downward Löwenheim-Skolem theorem. Completeness is a desirable property to have, but it has always been more difficult to justify the desirability of the Löwenheim-Skolem theorem. Why should a logic satisfy this theorem? An answer to this is that some version of the theorem is probably necessary to ensure that a logic's referential (standard semantic) and conceptual-role (probabilistic semantic) powers match, an important desideratum. Note the critical use of the (downward) Löwenheim-Skolem theorem in the proof of the Extended Equivalence Theorem for both the predicate logic and $L(Q)$. More cogent is the case of second-order logic where the failure of the Löwenheim-Skolem theorem seems to be the reason a probabilistic semantics cannot be provided; there can never be enough substitution instances for the $Pr^+(\forall X \mathcal{F}(X))$ clause. This problem is a general one. It is very hard to see how a probabilistic (or any nonreferential) semantics can be provided for a logic which doesn't satisfy some version of the Löwenheim-Skolem theorem.

Thus Lindstrom's theorem, the argument above for the necessity of at least some downward Löwenheim-Skolem theorem, and philosophical difficulties with probability functions from uncountable sets of formulas to [0,1] make a strong philosophical case for the primacy of the predicate calculus among logics. One might hope for a more precise argument, but it is doubtful that there is one. Tharp, for example, who provides a definitive characterization of the monadic predicate logic, comments that he is "not confident there is a completely watertight argument for the (primacy of the) full predicate logic" ([6], p. 16.

NOTES

1. Field [2] has a philosophically more palatable definition which he proves equivalent to the one used here. The latter is however easier to work with mathematically.
2. A related but less interesting generalization of truth-value semantics is what might be called (nonreferential) Boolean-value semantics. The definitions are analogous to those in the truth-value case except that {0,1} is replaced by a complete Boolean algebra.
3. Although a probabilistic semantics probably can be developed for intuitionistic and modal logics, we are limiting ourselves here to model-theoretic logics as defined in Lindstrom [5], logics with cardinality quantifiers, game quantifiers, infinite connectives, and other nonstandard extensions, but in which nevertheless the standard satisfaction (truth) relation between formulas and models still holds. Barwise and others have recently developed "soft model theory", the abstract study of these logics and their properties.

REFERENCES

[1] Dunn, M. and N. D. Belnap, "Substitution interpretation of the quantifiers," *Noûs,* vol. 2 (1968), pp. 177-185.

[2] Field, H., "Logic, meaning, and conceptual role," *The Journal of Philosophy,* vol. 74 (1977), pp. 379-409.

[3] Keisler, H. J., "Logic with the quantifier 'there exist uncountably many'," *Annals of Mathematical Logic,* vol. 1 (1970), pp. 1-93.

[4] Leblanc, H., *Truth-Value Semantics,* North-Holland, Amsterdam, 1976.

[5] Lindstrom, P., "On extensions of elementary logic," *Theoria,* vol. 35 (1969), pp. 1-11.

[6] Tharp, L. H., "Which logic is the right logic?" *Synthese,* vol. 31 (1975), pp. 1-21.

Department of Mathematics
Temple University
Philadelphia, Pennsylvania 19122