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Career Induction for Quantifiers

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In [7], I showed that Belnap's method of "career induction" comes to an unexpected halt at the *second degree*. While the halt was called for relevant logics, the method is quite general. What it amounts to, roughly, is that for a large number of logical systems, one can find, for each formula A in the vocabulary, a formula A^* such that: (i) A^* is of the second degree and (ii) A is provable in the system iff A^* is provable therein. Thus, for example, if we could find a decision procedure for the second-degree formulas of R, we could solve the decision question for all of R. (This question is open.)

The purpose of the present note is to generalize the method, with the particular aim of showing that first-order relevant logics are also second-degree reducible. Again, the method remains quite general, so that the decision to apply it to the analysis of R^x and its kin may be written off to an idiosyncratic interest of the author. It is to be hoped that readers with other problems will not be put off by this interest.

It often happens, in logical analysis, that iteration of some particles is held to *increase* the complexity of a formula, whereas other particles lead to no such increase. Thus, for example, iterated occurrences of $\Box$ in modal logic are held to increase the "degree of modal involvement", whereas iterated occurrences of particles like & and $\sim$, being merely truth-functional, do not increase degree. It is very easy to tie numbers to this scheme. For illustrative purposes, consider a sentential logic formulated with just two n-ary connectives c and C, of which c is to be thought of as degree-nonincreasing and C as degree-increasing. Then a sensible and familiar specification of the degree of a formula is the following. The degree of every propositional variable p shall be 0. The degree of $c(A_1, \ldots, A_n)$ shall be the greatest among the degrees of $A_1, \ldots, A_n$. And the degree of $C(A_1, \ldots, A_n)$ shall be one greater than the maximal degree of the A_i. It is easy to see that what this scheme measures is the depth of nesting of the degree-raising connective C among the formulas A of our sample language.

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This scheme has been applied to a number of logics. In relevant logics, for example, Belnap [4] counted the relevant implication $\rightarrow$ as degree-increasing, and the truth-functional connectives &, v, $\sim$ and the quantifiers $\forall x$, $\exists x$ as degree-nonincreasing. (The scheme has obvious consequences for defined connectives; for example, the relevant coimplication $\leftrightarrow$ as defined in [1] is also degree-increasing, by one.) On the scheme, [4] provides a complete semantics and a decision procedure for the first-degree formulas of E and R, and a complete semantics as well for the first-degree formulas of the relevant predicate logics E^x and R^x. This limitation of relevant complexity is moreover important, since, although there is now a complete semantics for all of E and R, improvement of Belnap's results on the other two points (decidability and quantificational completeness) has been hard to come by. Even the second degree has proved recalcitrant. Reference 7 and this paper show why. If the technical problems of relevant logic could be solved at the second degree, they could be solved. Period. But, one hopes, they may be a little easier to solve there.

1 ***Propositional logics*** I now summarize the method of [7], in a slightly generalized way. (For more motivational details and discussion of application to relevant logics, see that paper.) This method is intended for sentential logics. As ingredients of a sentential logic, we expect at least the following (as primitive or defined notions), with the attached properties.

Notion	Property
Sentential variables	Admits substitution rule
Equivalence $\leftrightarrow$	Replacement rule is derivable, $\vdash A \leftrightarrow A$
Conjunction &	Binds premisses in deduction, admits $\&I$
Implication $\supset$	Satisfies deduction theorem, admits $\supset E$
Other particles	None required.

Other than the very ordinary specifications just listed (which, however, do require slightly more spelling out), no requirements were placed on a *sentential logic* in [7] but the ones just listed. In particular, it need not be assumed that the equivalence $\leftrightarrow$ is defined or definable in terms of the implication $\supset$.

Suppose now that we are given a sentential logic S, formulated in a language L. We define a transformation * on L that will usually reduce degree, with the further property that A is a theorem of S iff A^* is a theorem of S. Let A be a given formula of L, and let $SUB(A)$ be the set of all subformulas of A, including A itself. With each subformula B of A, we associate a sentential variable p_B of L, as follows: if B is an atomic formula q, p_B shall be q. Otherwise p_B may be any sentential variable of L, subject to the requirement that if B and C are distinct subformulas of A, p_B and p_C shall be distinct sentential variables. (We assume but do not specify a function that will determine the p_B exactly.)

We now lay down some *defining axioms*[1] for the p_B: If B is atomic, the defining axiom for p_B shall be

(1) $p_B \leftrightarrow p_B$.

Otherwise, where B is of the form $cB_1 \ldots B_n$, where c is an n-ary connective, the defining axiom shall be

(2) $p_B \leftrightarrow cp_{B_1} \ldots p_{B_n}$.

Given the replacement property of $\leftrightarrow$, the effect of these defining axioms, taken together, is to make each subformula B in $SUB(A)$ equivalent to p_B.

Let now $AX(A)$ be the conjunction of all the defining axioms for the p_B. Consider the formula

(3) $AX(A) \supset p_A$,

which we shall henceforth take as the A^* we are seeking. On the elementary properties that we have been assuming for S, it is easy to see that this A^* will be provable in S iff A itself is provable in S. For, by iteration of replacement properties, p_A is derivable in S from A and the members of $AX(A)$. Applying the deduction theorem for $\supset$, and using the binding properties of &, the following is a theorem of S.

(4) $A \supset . AX(A) \supset p_A$ (i.e., $A \supset A^*$).

Accordingly, if A is a theorem of S, so by $\supset E$ is A^*. On the other hand, suppose that A^* is a theorem of S. Simply substitute B for p_B throughout (3), for each B in $SUB(A)$. It is easy to see that the antecedent of (3) is, on this substitution, a conjunction of explicit identities $B \leftrightarrow B$, which, by our assumption that $\leftrightarrow$ is reflexive and by $\&I$, is a theorem of S. By $\supset E$, so also is A a theorem of S, since on our specifications it is A itself which is substituted for p_A. So A is a theorem of S iff A^* is a theorem of S, as claimed.

Moreover, the transformation which takes A into (3) will usually reduce degree. How it will reduce degree is a function of the degrees to be assigned to sentential variables and logical particles. We make the assumption that the conjunction & itself is *degree-nonincreasing*, that sentential variables are assigned degree 0, and that other particles increase degree by at most one. We allow that sentential constants, to be consistent with the scheme, may be assigned either degree 0 or degree 1, the latter counting as the *degree-increasing* option for such constants, if present.

There are now four cases for how much we have reduced degree, in general. In the worst case, all particles (except &, but including $\leftrightarrow$ and $\supset$) are degree-increasing. In that case, A^* will be, at worst, a *third*-degree formula. For the p_B will be 0-degree, the $cp_{B_1 \ldots B_n}$ will be first-degree, the $p_B \leftrightarrow cp_{B_1 \ldots B_n}$ will be the second-degree (and hence $AX(A)$, which is the conjunction of such formulas, will be second-degree). So (3) itself, which has $AX(A)$ as antecedent, will be bumped up to the third degree by its main implication. If neither of $\leftrightarrow$ or $\supset$ are degree-increasing, then two of the bumping up stages are omitted and A^* is, at worst, a first-degree formula. The intermediate two cases occur when just one of $\leftrightarrow$, $\supset$ is degree-increasing, in which case A^* is, at worst, a second-degree formula. (This was the basis for the reduction for the relevant logics themselves in [7], where $\leftrightarrow$ was taken as relevant and degree-increasing, and $\supset$ was taken as material and degree-nonincreasing, producing a second-degree reduction.)

While we have required of what is here called a *sentential logic* that it shall have all the properties that we have listed, and while they are quite ordinary properties that most sentential logics of general interest can be shown to have without further fuss, there are a few points on which the reader needs to be cautioned. The first is that the replacement rule for $\leftrightarrow$ be *derivable*. This means, on usage freely borrowed from Curry [6], that the rule must hold *under hypothesis*. For, in assuming A and $AX(A)$ *on hypothesis* and then applying the replacement rule (repeatedly), we are able to derive p_A, which is the basis for the theorem scheme (4). But while most systems of logic will support a replacement rule for a suitably chosen $\leftrightarrow$, they will not necessarily support it as a derivable rule, but merely as an *admissible* rule (again, in the Lorenzen-Curry sense), which, if added to the primitive rules of a given system, does not produce any new theorems. So, while the replacement rule for $\leftrightarrow$ is the common situation even in its stronger, derivable form (as is witnessed by its presence in relevant logics, which differ from the norm on most points on which it is possible reasonably to differ), there are many interesting situations (in weak modal logics, for example), in which this rule will be underivable. If such systems are denied here the honorific title "sentential logic", it is not that they are uninteresting, but merely that we are uninterested in them for the purposes of this paper.

Of the properties presupposed for &, I have not much to say. The model that I have in mind for & is truth-functional conjunction, though all that I use of that model is the thought that conjunctions can be asserted iff their conjuncts can be asserted. Since & tends to be the particle with the most stable properties over a large variety of logical systems, most systems will treat these assumptions in their stride. Interesting, however, are systems in which the truth-functional & is absent, like some fragments of R, but in which its place is taken by an *intensional* conjunction $\circ$. Suffice it to say here that, by being careful, we can adapt the above argument to such situations. But note that the *point* of the argument remains only if $\circ$ itself can be counted as degree-nonincreasing (for $AX(A)$ is typically a long conjunction). While such reckoning may be useful for some purposes, it is less natural than our standard assumption that & is degree-nonincreasing, when we think of degree itself as a measure of the departure of a formula from a truth-functional norm.

The properties of $\supset$ are more interesting. For one thing, while a system usually provides little choice as to which connective shall count as its $\leftrightarrow$ (if replacement is to be a derivable rule, only the strongest candidate is likely to do—for example, strict equivalence in modal logics, not mere material equivalence), we may have some choice of $\supset$. Since the crucial property is the deduction theorem, we need only satisfy ourselves that the following is true: If C is derivable in S from A,B, then $A \supset . B \supset C$ is a theorem of S. (This is derivability in the ordinary sense, which [1] calls "Official".) For example, there are a couple of connectives definable in R with the necessary properties: the material implication $\supset$, and an intuitionist sort of implication as well. Caution: if we choose $\supset$ to be a strict implication, in the modal sense, or a relevant one, it does not necessarily satisfy such a deduction theorem. But such connectives will ordinarily satisfy *some* form of the deduction theorem, whence again we can often reinstate the argument so that it goes through with a strong $\rightarrow$ as the principal connective of the A^* defined by (3). If we do this for R or E in particular, for example, our reduction still goes through, though in this case it is a *third-degree* reduction rather than a second-degree one, since $\rightarrow$ (unlike the material $\supset$ used to define A^* in [7]) is naturally taken as degree-increasing. Finally, we have assumed that the rule $\supset E$ of modus ponens for our chosen implication is admissible in S. While, ordinarily, this is the most uncontroversial of principles, it can become controversial if the chosen $\supset$ is not the preferred implication of the system. Thus, for example, the reduction of [7] depended for the relevant logics on the nontrivial result that $\supset E$ is admissible in these systems for *material* implication.

*2 **Quantification theory*** Having summarized and supplemented [7] with these remarks, we now turn to the real business of this paper, which is to extend the above techniques to quantification theory. The key step in reducing degree above lay in letting a sentential variable p_B stand in for a formula B. We did this, using the equivalence $\leftrightarrow$ of an arbitrary system S, by assuming defining axioms that, essentially, recapitulated the formation rules that produced B in the first place. The key properties of $\leftrightarrow$ that went into this recapitulation were *congruence properties*.

We ask ourselves, what are the appropriate congruence properties in predicate logic (which we assume formulated with predicate letters, but without function symbols or identity)? What the sentential analogy suggests is that our defining axioms should now set every subformula B of A equivalent to a *predicate letter*. While that thought is not exactly well-formed, it is near enough to being well-formed that we can make use of it. For each open formula $B(x_1, \ldots, x_n)$ of a first-order language stands for, in Russell's terminology, a *propositional function*. In particular, where exactly $x_1, \ldots, x_n$ are free in $B(x_1, \ldots, x_n)$, this formula may be viewed as an n-ary propositional function, whose arguments are n-tuples of objects of an intended domain and whose values are propositions. In order to reduce complexity, we should identify this propositional function with one determined by an n-ary *atomic* predicate F_B. If we had the machinery we could introduce something like Church's λ notation (to distinguish expressions denoting functions from open formulas, by writing the former as $\lambda x_1 \ldots \lambda x_n B(x_1, \ldots, x_n)$). Thus, using ordinary '=' for equality between propositional functions, we would want defining axioms, on analogy to those that we had before, which would identify each such λ expression (corresponding to a subformula of a given A) with an F_B, in the sense that $F_B = \lambda x_1 \ldots \lambda x_n B(x_1, \ldots, x_n)$ would be derivable in an appropriate sense from defining axioms. For we could then use F_B to do the work of the propositional function determined by a complicated B.

Essentially, this will be our plan, although, since we do not have the λ notation in a first-order language, we shall have to simulate it. Ordinarily, we can simulate it quite easily, by causing to be derivable from our defining axioms sentences of the form

(5) $\forall x_1 \ldots \forall x_n(F_B x_1 \ldots x_n \leftrightarrow B(x_1, \ldots, x_n))$,

where F_B is the new predicate letter that we have picked to go with a given complex formula B, in exactly the variables $x_1, \ldots, x_n$ (occurring free).

So our plan will be as before, except that it is formulas of the form (5) that we wish to be derivable in S from the defining axioms determined by a given A. Once again, let $SUB(A)$ be the class of all subformulas of a given formula A. With each B in $SUB(A)$, we now associate a predicate letter F_B, as follows. If B is an atomic formula $Gx_1 \ldots x_n$, F_B shall be G. Otherwise, as before, F_B may be any predicate letter, provided that it is distinct from all the F_C, $C \in SUB(A)$ and C distinct from B. Again, since the number of subformulas of any formula is finite, there is no problem about specifying a definite plan to pick the F_B. Note that the plan has different effects at the atomic level and at the molecular level. For example, if B is Fxy and C is Fyx and D is Fyz and E is Fyy, all of F_B, F_C, F_D, and F_E are just F. But, where P is any formula, all of $F_{B\&P}$, $F_{C\&P}$, etc., are different.

Where B is an atomic formula $Gx_1 \ldots x_n$, F_B is an n-ary predicate letter, since it is just G. This is the case whether the x_i are distinct or not. But where B is nonatomic, we shall determine the n-adicity of F_B strictly by the *number* of free variables which occur in B (and *not* by the number of *occurrences* of free variables in case any are repeated). We do not count bound variables. That is, where exactly n variables occur free in B, we call B an *n-ary formula*. And F_B, in this case, shall be an *n-ary predicate letter*. In particular, if B is a *sentence* (*no* free variables), F_B shall be a 0-ary predicate letter (i.e., a sentential variable).

As before, we now wish to chase up the formation tree of our given formula A in order to lay down defining axioms. They are essentially as before, except that we have to be careful to assure that the n-ariness comes out right. This time all our defining axioms shall be the *universal closures* of the formulas actually displayed (gotten, let us say, by prefacing universal quantifiers in the order in which free variables, if any, occur in the formula in question). But we may abuse language to the extent of sometimes referring to the formulas themselves as axioms. As before, where B is atomic the defining axiom for B shall be just

(6) $B \leftrightarrow B$.

(To the reader who was curious before, and who remains so, as to why these explicit identities have been chosen as defining axioms, the reason is that, for some versions of the deduction theorem for relevant implications, they are needed. While we have not dwelt on the point, which doesn't touch on our main concerns, we include it to smooth a few more applications of our methods to the One True Logic.)

Suppose now that B is of the form $cB_1 \ldots B_n$, where B is an n-ary connective. We may assume that we have already laid down defining axioms for each of the B_i, of the form

(7) $B_i' \leftrightarrow B_i''$,

for each i, and that B_i' is of the form $F_{B_i}x_1 \ldots x_n$, where $x_1, \ldots, x_n$ are all the variables that occur free in B_i itself. Let $y_1, \ldots, y_m$ be exactly the variables that occur free in B, without repetitions. Then our defining axiom shall be

(8) $F_B y_1 \ldots y_m \leftrightarrow cB_1' \ldots B_n'$.

Note that exactly the same variables occur free on the left and right sides of (8). (On the right, however, a y_i may occur more than once, which is not permitted on the left.)

Finally we come to the quantifiers. (Their accommodation is, after all, the point of the extension of the method.) So let B be of the form QxC, where Q is a quantifier (or any other 1-place variable-binding operator that works syntactically like a quantifier). As in the last case, we assume that $y_1, \ldots, y_m$ are exactly the variables that occur free in B, and that we have added a defining axiom for C of the form

(9) $C' \leftrightarrow C''$,

where the free variables of C' are $y_1, \ldots, y_m$, and possibly x (in the interesting case where the Qx of QxC is not vacuous). Then our defining axiom for B shall be

(10) $F_B y_1 \ldots y_m \leftrightarrow QxC'$.

We can now form A^* as before, given A. Let $AX(A)$ be the conjunction of all defining axioms for subformulas of A. (We cease abusing language to recall that $AX(A)$ is itself a *closed* formula.) Let A' be the atomic formula which occurs on the left-hand side of the defining axiom for A itself. Then let A^* be the analogue of (3), namely

(11) $AX(A) \supset A'$.

We now wish to show that, in every predicate logic S, A and A^* are again deductively equivalent, in the sense that one is a theorem of S iff the other is. To begin with, the biconditional $\leftrightarrow$ of a sentential logic is now in general too weak to justify a *derivable* replacement rule (unless one wants to go through a lot of explanation as to what " derivable" is now to mean). For such a rule now goes naturally not with formulas $B \leftrightarrow C$ but with their *closures*. Let us waste a definition, calling the wanted notion *exact equivalence* and symbolizing it by '='.

(12) $B = C =_{df} \forall x_1 \ldots \forall x_n(B \leftrightarrow C)$,

where $x_1, \ldots, x_n$ are, in the order of occurrence, the variables that occur free in $B \leftrightarrow C$. Note that, disabusing language, our actual defining axioms are all exact equivalences.

Our specification of a predicate logic now mirrors our specification of a sentential logic.

Notion	Property
Predicate variables	Admits substitution rule
Exact equivalence =	Derivable replacement rule, $\vdash A = A$
Conjunction &	As before
Implication $\supset$	As before
Other particles	None required.

The argument for the deductive equivalence of A and A^* is now as before. If A is a theorem of the predicate logic S, and one assumes the defining axioms for A and its subformulas on hypothesis, a succession of replacements of exact equivalents, looking up the construction tree for A, will yield A' as before, whence the deduction theorem will yield A^*. Going the other way, suppose A^* is a theorem. For each atomic expression $F_B y_1 \ldots y_m$ in A^*, substitute $B(y_1, \ldots, y_m)$ (according to the specifications of, say, [5]). This again turns the antecedent of A^* into a conjunction of exact equivalences and its consequent into A (modulo some well-known nuisances, which are greatly eased by thinking of the substitution as *of* $\lambda y_1 \ldots \lambda y_m B(y_1, \ldots, y_m)$ *for* F_B). Then again apply $\supset E$ to get A.

While we have *defined* exact equivalence = in a way agreeable to the predicate logics that one might meet in practice, nothing in our actual argument for the deductive equivalence of A and A^* in predicate logics S prevents that notion from being taken as primitive. For what counts are the reflexivity of = and its replacement properties under hypothetical extension. We also assume that =, whether primitive or defined, shall increase degree by one at most. On the definition (12), it suffices for this end to add to our previous specifications that & shall be degree-nonincreasing and that $\leftrightarrow$ shall increase degree by one at most, which specifications remain in force, that the universal quantifier $\forall$ shall also be degree-nonincreasing. Again, if we use degree to measure the non-truth-functional involvement of a formula, these specifications remain natural.

Finally, we can prove a theorem.

Reducibility theorem *Every predicate logic is third-degree reducible, i.e., every formula A of a predicate logic S, where S is subject to the conditions above, is deductively equivalent to a formula A*, which is of at most the third degree. If, in addition, at least one of =, $\supset$ is degree-nonincreasing, then S is second-degree reducible. If both of these particles are degree-nonincreasing, then S is first-degree-reducible.*

Observation: All previous assumptions about degree remain in force. Since we assume $\forall$ degree-nonincreasing, it suffices for = to be degree-nonincreasing that $\leftrightarrow$ should be, where the definition (12) is assumed. While we assume that no primitive particle increases degree by more than one, note that it does not matter for our results if *all* particles except & and $\forall$ are counted as degree-increasing, and that reductions in degree are got whenever one or both of $\leftrightarrow$, $\supset$ is degree-nonincreasing (as before), leaving all particles not specifically mentioned to be degree-increasing or not, as we choose.

Proof: As indicated, with the remarks limiting the degree of A^* being justified as in the sentential case.

As in the sentential case, the most familiar logics turn out to be predicate logics, and are hence third-degree reducible at worst, however degree is measured. Of course classical and intuitionist first-order logics are predicate logics in our sense. So are some first-order extensions of standard modal logics—e.g., the first-order extension of *S5* originally proposed by Barcan in [3]. Illustratively, however, I shall again consider the case of relevant logics, with an eye on the uncompleted task of furnishing the first-order versions of these logics with a complete model-theoretic semantics. (A proposal was made in [9], but, at least so far, there is no way of knowing whether it, or some variant thereof, is the correct proposal.) Again, we count degree as it was counted in [4]. The truth-functional connectives, including & and the *material* $\supset$, are degree-nonincreasing; so are the quantifiers. Relevant implication $\rightarrow$ (and accordingly the relevant coimplication $\leftrightarrow$ and the exact equivalence = defined therefrom by (12)) is degree-increasing. Under these conditions

Corollary *Let S be any of the relevant logics E, R, T, RM, and EM, and let S^x be the corresponding first-order logic.* (Formulations of all of these first-order systems will be found in [2], building on the sentential base offered in [1]. Axioms for quantifiers are as in [4], essentially.) *Then S^x is second-degree reducible.*

Proof: We need only verify that the logics in question meet our conditions on a predicate logic. The formulation of the systems with unrestricted axiom schemes is the key ingredient in guaranteeing the substitution rule for predicate letters. The replacement rule for exact equivalence can be shown derivable in the usual way. Conjunction has its ordinary properties. Again we prefer for this purpose to choose material implication as our $\supset$, since it is degree-nonincreasing. The rule $\supset E$ may be shown admissible for S^x as it was shown admissible for R^x in particular in [8]. The deduction theorem for $\supset$ may also be established in the ordinary way and = is of course reflexive. This is all that is required that S^x should be a predicate logic in our sense, falling accordingly under the case of the theorem where = is degree-increasing and $\supset$ is degree-nonincreasing. Thus relevant predicate logics are second-degree reducible as well, and once again the degree of relevant involvement need go no further than 2.

NOTE

1. Defining axioms in a sense are not to be taken as logical axioms, although in some cases as here they will be theorems of logic.

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