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SIGNIFICANCE AND ILLATIVE COMBINATORY LOGICS

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Introduction If first order propositional calculus is extended by allowing the replacement of propositional variables by terms formed by application using the combinators **K** and **S** as well as **P** (implication), the system becomes inconsistent (see [5]). However, as the presence of these combinators in a system allows the definition of all recursive functions, such an extension can be desirable. Many of the significance logics of Goddard and Routley ([7]) contain somewhat restricted versions of first order predicate calculus and it is, therefore, of interest to see whether terms involving combinators can be consistently substituted for the well formed formulas (wffs) of these systems. In addition some of these significance logics, when extended in this way, may be interesting systems of illative combinatory logic. To enhance the similarities the significance operator S can be replaced by **H**, an operator standing for "is a proposition", which is used in for example [1], [2], and [6].

The extension Our extension of any of the significance logics will involve allowing:

(a) The substitution of terms involving combinators and implication for propositional variables.
(b) The use of the equality axioms of pure combinatory logic (see [5]).
(c) The use of the following rule for combinatory equality (=):

Rule Eq *If* $x = y$, *then* $x \vdash y$.

The C_i *systems* The system C_0 is equivalent to first order propositional calculus and so becomes inconsistent when terms involving combinators are introduced. This is also the case for C_1 (and C_2, C_3, C_5, and C_6 which have C_1 as a subsystem), but C_1 has a restricted or modified modus ponens, so the inconsistency proof must be rewritten. Modified modus ponens is stated as follows:

R_2' *If* $\vdash A$ *and* $\vdash A \supset B$ *then* $\vdash B$, *provided no wff*[1] *is uncovered in* A *and covered in* B.

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A wff C is covered in a wff A if and only if C occurs in A and every occurrence of C in A is within the scope of some occurrence of S when A is written in primitive notation; a wff C is uncovered in A if and only if C occurs in A and not every occurrence of C is within the scope of some occurrence of S.

Using the combinators we can, for A arbitrary, define a term X such that

$$X = SX \supset . X \supset A$$

After appropriate substitution Axiom 1.2′ of S_1 leads to:

$$\vdash X \supset . SX \supset (X \supset A) :\supset: X \supset SX . \supset . X \supset (X \supset A)$$

A theorem of S_1 is

$$\vdash X \supset X$$

which is by Rule Eq,

$$\vdash X \supset (SX \supset . X \supset A).$$

Then by R_2' we have

$$\vdash X \supset SX \supset . X \supset (X \supset A).$$

Axiom 1.6′ of S_1 and substitution give

$$\vdash X \supset SX$$

and R_2' gives

$$\vdash X \supset (X \supset A).$$

From this it is easy to prove

$$\vdash X \supset A$$

and hence

$$\vdash SX \supset . X \supset A$$

which is

$$\vdash X.$$

Thus by R_2'

$$\vdash A.$$

The S_i *systems* Many of these systems resemble the one proved inconsistent in [4]. We have there the following axiom and rules which together with Rule Eq are inconsistent.

1. "wff" here, and below, replaces "variables" in the version of R_2' given in [7], because the presence of wffs involving the combinators must be allowed for.

P	$X, X \supset Y \vdash Y$
H	$X \vdash \mathbf{H}X$
DTP	*If* $X \vdash Y$, *then* $\mathbf{H}X, \mathbf{H}Y \vdash X \supset Y$.
HP	$\mathbf{H}X, \mathbf{H}Y \vdash \mathbf{H}(X \supset Y)$.
axiom **H**	$\vdash \mathbf{H}^n X$

When we compare the systems S_i to this we of course need to replace S by $\mathbf{H}$. The system S_0 has no explicit S and so does not fall under the inconsistency. All the other systems have S with Axiom H for $n = 2$ and Rule HP. S_4 and S_6 also have Rules P and H and, as was shown in [3], DTP and are therefore inconsistent upon the extension to combinatory logic.

S_1 has $\vdash X \supset SX$ instead of Rule H and R_2' instead of Rule P, as did C_1, but more importantly, it has, as was shown in [3], no deduction theorem. The inconsistency of [4], therefore, does not apply to it. In a similar way this inconsistency does not apply directly to the systems AS_1, IS_1, HS_1, L_3S_1, S_2, S_3, or S_5. All of these, except HS_1 and L_3S_1, however, involve new operators that have no counterpart in illative combinatory logic and are, therefore, of less interest from that point of view.

We will now look at combinatory logic versions of the remaining systems, S_1, HS_1, and L_3S_1. In combinatory logic there are no variables and we, therefore, cannot express axioms in terms of variables, let alone two types of variables. We can rewrite the axioms of S_1 (with $\mathbf{H}$ for S) as rules as follows:

1.1 $\mathbf{H}X, \mathbf{H}Y \vdash X \supset (Y \supset X)$.
1.2 $\mathbf{H}X, \mathbf{H}Y, \mathbf{H}Z \vdash X \supset (Y \supset Z) .\supset. (X \supset Y) \supset (X \supset Z)$.
1.3 $\mathbf{H}X, \mathbf{H}Y \vdash (\sim X \supset \sim Y) \supset (Y \supset X)$.
1.4 $\mathbf{H}X, \mathbf{H}Y \vdash \mathbf{H}(X \supset Y)$.
1.5 $(X \supset Y) \vdash \mathbf{H}X \supset \mathbf{H}Y$.
1.6 $\mathbf{H}(X \supset Y) \vdash \mathbf{H}X$.
1.7 $\mathbf{H}X \vdash \mathbf{H}(\sim X)$.
1.8 $\mathbf{H}(\sim X) \vdash \mathbf{H}X$.

Rule 1.1 of S_1 becomes Rule H and Rule 1.4 of S_1, Rule P.

Rule 1.2 *If* $\vdash A$ *and* $\vdash SB$, *then* $\vdash \mathbf{S}_B^R A|$, *where R is an S-restricted variable;*

merely becomes a case of replacing the indeterminate R in $\mathbf{H}R \vdash A$, *by a term B for which* $\vdash \mathbf{H}B$ *holds*. Similarly,

Rule 1.3 *If* $\vdash A$, *then* $\vdash \mathbf{S}_B^P A|$, *where P is an* S-*unrestricted variable and B is a wff, or both P and B are S-restricted variables;*

is a case of replacing the indeterminate X in $\vdash A$ by a term B or the indeterminate X in $\mathbf{H}X \vdash A$ by another indeterminate.

The system we have now, however, is not quite as strong as S_1 as we can no longer use 1.6, 1.7, Rule H, and Modus Ponens to prove $\vdash \mathbf{H}(\mathbf{H}X)$ for arbitrary X.

If we introduce a universal category $\mathbf{E}$ (such that $\vdash \mathbf{E}X$ for all X) we can rewrite 1.4-1.8 as

1.4 $\mathbf{E}X,\ \mathbf{E}Y \vdash \mathbf{H}X \supset \mathbf{H}Y \supset \mathbf{H}(X \supset Y)$
1.5 $\mathbf{E}X,\ \mathbf{E}Y \vdash \mathbf{H}(X \supset Y) \supset (\mathbf{H}X \supset \mathbf{H}Y)$

etc.,

so that at least $\mathbf{E}X \vdash \mathbf{H}(\mathbf{H}X)$ becomes provable.

Another alternative might be to introduce restricted generality (Ξ) with the rule:

Rule Ξ $\Xi XY, XU \vdash YU$

(ΞXY will often be written as $Xu \supset_u Yu$)

Implication can then be defined by:

$$X \supset Y = \Xi(\mathbf{K}X)(\mathbf{K}Y)$$

and we can write the axioms of S_1 as:

1.4 $\vdash \mathbf{H}x \supset_x : \mathbf{H}y \supset_y . x \supset (y \supset x)$
$\vdots$
1.8 $\vdash \mathbf{H}(\sim x) \supset_x \mathbf{H}x$

A deduction theorem is still not provable, but if we also replaced Rule H by

$$\vdash x \supset_x \mathbf{H}x,$$

as it is in [1] and we have a very similar system to that of [1] and [2]. The inconsistency does not arise as $\vdash \mathbf{H}(\mathbf{H}X)$ is no longer provable. This change is therefore one, as was the first one suggested above, that eliminates a theorem that is basic to all S_i systems.

The three types of transformation, of S_1, to a system of illative combinatory logic can also be applied to the system L_3S_1 of [7] which is very similar.

The system HS_1, however, has a complicated rule which changes some theorems of S_1 (all of these are also theorems of HS_1) into special theorems called $\mathbf{H}$-theorems. We can prove

$$\vdash_{\mathrm{H}} p \supset p$$

and

$$\vdash_{\mathrm{H}} p \supset (p \supset q) :\supset (p \supset q),$$

and these together with Rule Eq for $\vdash_{\mathrm{H}}$ are sufficient to prove a contradiction. Rule Eq restricted to $\vdash$ only, may allow HS_1 to avoid inconsistency.

REFERENCES

[1] Bunder, M. W., "A deduction theorem for restricted generality," *Notre Dame Journal of Formal Logic*, vol. XIV (1973), pp. 341-346.

[2] Bunder, M. W., "Propositional and predicate calculuses based on combinatory logic," *Notre Dame Journal of Formal Logic*, vol. XV (1974), pp. 25-34.

[3] Bunder, M. W., "Deduction theorems in significance logics," *Notre Dame Journal of Formal Logic*, vol. XX (1979), pp. 695-700.

[4] Bunder, M. W., and R. K. Meyer, "On the inconsistency of systems similar to $\mathcal{J}_{21}^*$," *The Journal of Symbolic Logic*, vol. 43 (1978), pp. 247-249.

[5] Curry, H. B. and R. Feys, *Combinatory Logic*, Vol. I, North Holland Co., Amsterdam (1958).

[6] Curry, H. B., J. R. Hindley, and J. P. Seldin, *Combinatory Logic*, Vol. II, North Holland Co., Amsterdam (1972).

[7] Goddard, L. and R. Routley, *The Logic of Significance and Context*, Scottish Academic Press, Edinburgh (1973).

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