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N-POLAR LOGIC OF CLASSES

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1 *Class calculus* In the following we make reference to certain notions without explaining them, but they have the usual meaning. Among these notions are: set, element, belonging, nonbelonging, decomposition, validity, etc.*

1.1 *General definitions*

D1.1: A set M_1 is a *subset* of the set M, if all elements of the set M_1 are elements of M. If M_1 is a subset of the set M, then M is an extended set (extension) of M_1.

D1.2: We call *bidisjunctive subsets* of the set M two subsets M_1 and M_2, which have no common elements; that is to say, if an element belongs to the subset M_1 this element does not belong to the subset M_2, while if an element belongs to the subset M_2 that element does not belong to the subset M_1.

D1.3: We call *n-disjunctive subsets* of the set M the subsets M_i, $i = 1, 2, \ldots, n$, in a way that these subsets should be bidisjunctive in pairs.

D1.4: If a set can be decomposed following a certain criterion (intensional or extensional) into n n-disjunctive subsets, so that each element of the set M should be at the same time an element of one of these subsets, then these subsets are called *classes*. In order to differentiate between the classes and the other subsets, we shall indicate the classes with a_i, $i = 1, 2, \ldots, n$. The n classes will be listed arbitrarily and numbered $a_1, a_2, \ldots, a_n$. Although the listing has been made arbitrarily, it will be maintained throughout the calculation. The deduced formulas for sets and subsets will be, of course, valuable for classes as well.

D1.5: If a set can be decomposed into n classes, we say that this set has *variance n*, and the classes of this set are *n-variant*. If to each element of

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an n-variant class there is one sole element of each of the other n-1 n-variant classes corresponding, then these classes are called *n-polar.*

1.2 *Alphabet*

$a, b, c, \ldots, a_1, a_2, \ldots$	class variables (classes)
$M, P, Q, \ldots, M_1, M_2, \ldots$	sets (subsets)
ϵ	symbol of belonging (binary predicate)
$=$	symbol of equivalence (binary predicate)
$\mathsf{K}^i \ i = 1, 2, \ldots, n$	symbols of complementations (unary functors)
$\mathsf{U}_i^n \ i = 1, 2, \ldots, n$	n-ary operants
(,)	parentheses.

Observations:

1. Equivalence is a reflexive, symmetrical and transitive relationship.

2. In a set M of variance n each n-variant subset has n - 1 complementary subsets $\mathsf{K}_M^t M_i$, $t = 1, 2, \ldots, n$. The subset M_i and the other n - 1 complementary subsets of this subset are n-disjunctive subsets. $\mathsf{K}_M^t M_i$ is to be read "complement of order t of the subset M_i as compared to the set M".

3. An n-ary operant U_k^n will be followed by n sets (subsets, classes)—different or not—and in this case $\mathsf{U}_k^n P_1 P_2 \ldots P_n$ will be called an *operation.*

1.3 *Definitions and abbreviations*

D1.6: $\mathsf{K}_M^t a_k =_{df} a_{k+t}$, $a_k \epsilon M$, $k, t = 1, 2, \ldots, n$ and (a_k are classes).

D1.7: $\mathsf{U}_k^n\{M_s\}_{1,n} =_{df} \mathsf{U}_k^n M_1 M_2 \ldots M_n$ $\qquad k = 1, 2, \ldots, n.$

or

$\mathsf{U}_k^n\{M_s\}_{s(1,n)} =_{df} \mathsf{U}_k^n M_1 M_2 \ldots M_n.$ $\qquad k = 1, 2, \ldots, n.$

D1.8: $\mathrm{Perm}\,\{M_s\}_{1,n}$ represents the set of permutations of the n elements M_s, $s = 1, 2, \ldots, n$.

D1.9: $\widehat{\mathrm{Perm}}\,\{M_s\}_{1,n}$ is any permutation of the set $\mathrm{Perm}\,\{M_s\}_{1,n}$, that is to say $\widehat{\mathrm{Perm}}\{M_s\}_{1,n} \epsilon \ \mathrm{Perm}\,\{M_s\}_{1,n}$.

D1.10: $\tilde{\mathsf{U}}_k^n\{M_s\}_{1,p}$, $p \epsilon N$ (set of natural numbers), $p \neq n$, (*composed operation*) is an operation U_k^n of n operations U_k^n, which in their turn are operations of n operations U_k^n, etc., to the complete exhaustion of the p sets M_s. For example:

$$\tilde{\mathsf{U}}_k^n\{M_s\}_{1,n^2} = \mathsf{U}_k^n(\mathsf{U}_k^n\{M_s\}_{1,n})(\mathsf{U}_k^n\{M_s\}_{n+1,2n}) \ldots \ldots (\mathsf{U}_k^n\{M_s\}_{n^2-n+1,n^2}).$$

If $n^q < p < n^{q+1}$, $q = 0, 1, 2, \ldots$, then the p sets M_i $i = 1, 2, \ldots, p$, will be completed to n^{q+1} through the repetition of $n^{q+1} - p$ sets. Which set and how many times it is going to be repeated is established arbitrarily. Examples:

1. $\tilde{\mathsf{U}}_k^3\{M_s\}_{1,5} = \mathsf{U}_k^3(\mathsf{U}_k^3 M_1 M_2 M_3)(\mathsf{U}_k^3 M_4 M_5 M_5)(\mathsf{U}_3^3 M_5 M_5 M_5)$

or

$\tilde{\mathsf{U}}_k^3\{M_s\}_{1,5} = \mathsf{U}_k^3(\mathsf{U}_k^3 M_1 M_2 M_3)(\mathsf{U}_k^3 M_2 M_3 M_4)(\mathsf{U}_k^3 M_3 M_4 M_5)$, etc.

2. $\tilde{\mathsf{U}}_k^7\{M_s\}_{1,3} = \mathsf{U}_k^7 M_1 M_2 M_2 M_2 M_3 M_3 M_3$

or

$\tilde{\mathsf{U}}_k^7\{M_s\}_{1,3} = \mathsf{U}_k^7 M_1 M_1 M_1 M_1 M_1 M_2 M_3$, etc.

D1.11: $\mathsf{U}_{k+1}^n\{K_M^s M_t\}_{s(1,n)} =_{df} \mathfrak{J}_k^n,\ k,\ t = 1, 2, \ldots, n,\ M_t \in M.$

$\mathfrak{J}_k^n$ are called *n-variant class constants.*

D1.12: $\mathsf{U}_1^n\{K_M^s M_t\}_{s(1,n)} =_{df} M,\ M_t \in M,\ t = 1, 2, \ldots, n.$

D1.13: $\subseteq \{M_s\}_{1,n} =_{df} (\mathsf{U}_1^n\{K_M^{n-s} M_s\}_{s(1,n)} = M)$ $\qquad M_s \in M,\ s = 1, 2, \ldots, n$

D1.14: $\mathsf{U}_{nu+v}^n\{P_s\}_{1,n} =_{df} \mathsf{U}_v^n\{P_s\}_{1,n}$ $\qquad u \in N,\ v = 1, 2, \ldots, n.$

D1.15: $\mathsf{K}_M^p \mathsf{K}_M^q M_1 =_{df} \mathsf{K}_M^{p+q} M_1$ $\qquad M_1 \in M,\ p,\ q = 1, 2, \ldots, n.$

1.4 *Terms and formulas* Each n-variant class variable, n-variant class constant, or set, represents an n-variant (n-polar) term. If b is an n-variant term, then $\mathsf{K}_M^i b$, $b \in M$, $i = 1, 2, \ldots, n$, are n-variant terms as well.

If b_i, $i = 1, 2, \ldots, n$, are n n-variant terms, then $\mathsf{U}_k^n\{b_s\}_{1,n}$, $k = 1, 2, \ldots, n$, are n-variant terms as well. If a and b are n-variant terms, then $a \in b$ and $a = b$ are formulas of the n-variant class calculus (of the n-polar class logic).

In the following, particular cases are given for $n = 2$ and $n = 3$ for certain formulas. For $n = 2$ we note that $\mathsf{U}_1^2 M_1 M_2 = M_1 \cup M_2$, $\mathsf{U}_2^2 M_1 M_2 = M_1 \cap M_2$ and $\mathfrak{J}_1^2 = \emptyset$.

1.5 *System of axioms*

Ax1 $\mathsf{K}_M^n M_1 = M_1$, where $M_1 \in M$

Ax2 $\mathsf{K}_M^j \mathsf{U}_q^n\{M_s\}_{1,n} = \mathsf{U}_{q+j}^n\{\mathsf{K}_M^j M_s\}_{s(1,n)}$, where $M_s \in M$, $q, j = 1, 2, \ldots, n$

For $n = 2$; $q = 1$: $\mathsf{K}_M^1(M_1 \cup M_2) = \mathsf{K}_M^1 M_1 \cap \mathsf{K}_M^1 M_2$
$q = 2$: $\mathsf{K}_M^1(M_1 \cap M_2) = \mathsf{K}_M^1 M_1 \cup \mathsf{K}_M^1 M_2$

For $n = 3$; $j = 1$; $q = 1$: $\mathsf{K}_M^1 \mathsf{U}_1^3 M_1 M_2 M_3 = \mathsf{U}_2^3 \mathsf{K}_M^1 M_1 \mathsf{K}_M^1 M_2 \mathsf{K}_M^1 M_3$
$j = 2$; $q = 1$: $\mathsf{K}_M^2 \mathsf{U}_1^3 M_1 M_2 M_3 = \mathsf{U}_3^3 \mathsf{K}_M^2 M_1 \mathsf{K}_M^2 M_2 \mathsf{K}_M^2 M_3$
$j = 1$; $q = 2$: $\mathsf{K}_M^1 \mathsf{U}_2^3 M_1 M_2 M_3 = \mathsf{U}_3^3 \mathsf{K}_M^1 M_1 \mathsf{K}_M^1 M_2 \mathsf{K}_M^1 M_3$
$j = 2$; $q = 2$: $\mathsf{K}_M^2 \mathsf{U}_2^3 M_1 M_2 M_3 = \mathsf{U}_1^3 \mathsf{K}_M^2 M_1 \mathsf{K}_M^2 M_2 \mathsf{K}_M^2 M_3$
$j = 1$; $q = 3$: $\mathsf{K}_M^1 \mathsf{U}_3^3 M_1 M_2 M_3 = \mathsf{U}_1^3 \mathsf{K}_M^1 M_1 \mathsf{K}_M^1 M_2 \mathsf{K}_M^1 M_3$
$j = 2$; $q = 3$: $\mathsf{K}_M^2 \mathsf{U}_3^3 M_1 M_2 M_3 = \mathsf{U}_2^3 \mathsf{K}_M^2 M_1 \mathsf{K}_M^2 M_2 \mathsf{K}_M^2 M_3$.

Ax3 $\mathsf{U}_k^n\{M_s\}_{1,n} = \mathsf{U}_k^n \widehat{\mathrm{Perm}}\{M_s\}_{1,n}$, where $k = 1, 2, \ldots, n$.

For $n = 2$; $k = 1$: $M_1 \cup M_2 = M_2 \cup M_1$
$k = 2$: $M_1 \cap M_2 = M_2 \cap M_1$.

For $n = 3$: $\cup_k^3 M_1 M_2 M_3 = \cup_k^3 M_2 M_1 M_3$ $\quad k = 1, 2, 3$

$\cup_k^3 M_1 M_2 M_3 = \cup_k^3 M_3 M_1 M_2$ $\quad k = 1, 2, 3,$

etc.

Ax4 $\cup_k^n(\cup_k^n\{M_s\}_{1,n})\{M_s\}_{n+1,2n-1}$
$= \cup_k^n(\cup_k^n\{M_{s+i}\}_{s(1,n)})\{M_{s+i}\}_{s(n+1,2n-1)}$
where $k = 1, 2, \ldots, n$, $i = 1, 2, \ldots, n - 1$, $M_{2n-1+p} = M_p$, $p = 1, 2, \ldots, n$.

For $n = 2$; $k = 1$: $(M_1 \cup M_2) \cup M_3 = (M_2 \cup M_3) \cup M_1$

$k = 2$: $(M_1 \cap M_2) \cap M_3 = (M_2 \cap M_3) \cap M_1$.

For $n = 3$; $i = 1$: $\cup_k^3(\cup_k^3 M_1 M_2 M_3) M_4 M_5 = \cup_k^3(\cup_k^3 M_2 M_3 M_4) M_5 M_1$

$i = 2$: $\cup_k^3(\cup_k^3 M_1 M_2 M_3) M_4 M_5 = \cup_k^3(\cup_k^3 M_3 M_4 M_5) M_1 M_2$,

$k = 1, 2, 3.$

Ax5 $\cup_{i+1}^n\{M_s\}_{1,n-1} \cup_i^n\{P_t\}_{1,n} = \cup_i^n\{\cup_{i+1}^n\{M_s\}_{1,n-1} P_t\}_{t(1,n)}$ where $i = 1, 2, \ldots, n$.

For $n = 2$; $i = 1$: $M \cap (P_1 \cup P_2) = (M \cap P_1) \cup (M \cap P_2)$

$i = 2$: $M \cup (P_1 \cap P_2) = (M \cup P_1) \cap (M \cup P_2)$.

For $n = 3$; $\cup_{i+1}^3 M_1 M_2(\cup_i^3 P_1 P_2 P_3)$
$= \cup_i^3(\cup_{i+1}^3 M_1 M_2 P_1)(\cup_{i+1}^3 M_1 M_2 P_2)(\cup_{i+1}^3 M_1 M_2 P_3)$ where $i = 1, 2, 3$.

1.6 *Preliminary notions—the axioms of the calculus* The sequence $\sigma(n, i)$, $n > i$, is determined as follows:

$$\sigma(n, i) = i - 1, i - 2, \ldots, 2, 1, n, n - 1, \ldots, i + 1, i$$

Example: $\sigma(7, 4) = 3, 2, 1, 7, 6, 5, 4$

Let a set of n natural numbers $\{\nu_i\}$, $i = 1, 2, \ldots, n$, be given, so that $\nu_i \geqslant 1$ and $\nu_i \leqslant n$. The numbers ν_i can be different from each other or not. Example: $\{\nu_i\} = 1, 2, 4, 4, 4, 5, 6$. Since the number of terms is 7, we shall consider $n = 7$. Placing the numbers of ν_i, according to the order of the sequence $\sigma(n, i)$, we obtain the sequence $\tau(n, i)$. In the above example we can state: $\tau(7, 4) = 2, 1, 6, 5, 4, 4, 4$.

We write τ_1 for the first term of the sequence $\tau(n, i)$. In our case $\tau_1 = 2$.

AxC1 $\cup_k^n\{\mathfrak{J}_{\nu_s}^n\}_{s(1,n)} = \mathfrak{J}_{\tau_1}^n$, $k = 1, 2, \ldots, n$

Example: We are to calculate the class constant equivalent to the expression: $\alpha = \cup_8^{13}\mathfrak{J}_1^{13}\mathfrak{J}_1^{13}\mathfrak{J}_1^{13}\mathfrak{J}_2^{13}\mathfrak{J}_2^{13}\mathfrak{J}_4^{13}\mathfrak{J}_4^{13}\mathfrak{J}_8^{13}\mathfrak{J}_9^{13}\mathfrak{J}_{10}^{13}\mathfrak{J}_{11}^{13}\mathfrak{J}_{11}^{13}\mathfrak{J}_{12}^{13}$.
$\sigma(13, 8) = 7, 6, 5, 4, 3, 2, 1, 13, 12, 11, 10, 9, 8,$
$\{\nu_i\} = 1, 1, 1, 2, 2, 4, 4, 8, 9, 10, 11, 11, 12.$
$\tau(13, 8) = 4, 4, 2, 2, 1, 1, 1, 12, 11, 11, 10, 9, 8.$
$\tau_1 = 4$. Consequently $\alpha = \mathfrak{J}_4^{13}$.

For $n = 2$, $k = 1$: $\emptyset \cup M = M$.

$k = 2$: $\emptyset \cap M = \emptyset$.

AxC2 *If* $\subseteq \{M_s\}_{1,n}$, *then* $\cup_k^n\{M_s\}_{1,n} = M_{k-1}$ $\quad k = 1, 2, \ldots, n$

For $n = 2$, $k = 1$: *If* $\subseteq M_1 M_2$, *then* $M_1 \cup M_2 = M_2$.
$k = 2$: *If* $\subseteq M_1 M_2$, *then* $M_1 \cap M_2 = M_1$.

1.7 *Certain theorems of class calculus*

T1 $\quad \mathsf{K}_M^{un+v} M_1 = \mathsf{K}_M^v M_1 \qquad M_1 \in M,\ u \in N,\ v = 1, 2, \ldots, n$

Proof:
(D1.15) $\mathsf{K}_M^{un+v} M_1 = \mathsf{K}_M^{un} \mathsf{K}_M^v M_1$
(D1.15) $\mathsf{K}_M^{un+v} M_1 = \underbrace{\mathsf{K}_M^n \mathsf{K}_M^n \ldots \mathsf{K}_M^n}_{u\ times} \mathsf{K}_M^v M_1$
(Ax1) $\mathsf{K}_M^{un+v} M_1 = \mathsf{K}_M^v M_1$

T2 $\quad a_{nu+v} = a_v \qquad a_v \in M,\ u \in N,\ v = 1, 2, \ldots, n$

Proof:
(D1.6) $a_{nu+v} = \mathsf{K}_M^{nu} a_v$
(D1.15) $a_{nu+v} = \underbrace{\mathsf{K}_M^n \mathsf{K}_M^n \ldots \mathsf{K}_M^n}_{u\ times} \mathsf{K}_M^v a$
(Ax1) $a_{nu+v} = a_v$

T3 $\quad \mathsf{U}_k^n \{a_s\}_{1+p, n+p} = \mathsf{U}_k^n \{a_s\}_{1,n} \qquad p \in N,\ k = 1, 2, \ldots, n$

Proof: Follows from T2 and Ax3.

T4 $\quad \mathsf{U}_k^n \{a_{s+i}\}_{s(1,n)} = \mathsf{U}_k^n \{a_s\}_{1,n} \qquad k, i = 1, 2, \ldots, n$

Proof: Follows from T3.

T5 $\quad \mathsf{K}_M^q \mathfrak{I}_k^n = \mathfrak{I}_{k+q}^n \qquad q, k = 1, 2, \ldots, n$

Proof: By replacing the sets with classes in D1.11:

(D1.11) $\mathsf{K}_M^q \mathfrak{I}_k^n = \mathsf{K}_M^q \mathsf{U}_{k+1}^n \{\mathsf{K}_M^s a_1\}_{s(1,n)} \qquad a_1 \in M$
(Ax2) $\mathsf{K}_M^q \mathfrak{I}_k^n = \mathsf{U}_{k+q+1}^n \{\mathsf{K}_M^q \mathsf{K}_M^s a_1\}_{s(1,n)}$
(D1.15) $\mathsf{K}_M^q \mathfrak{I}_k^n = \mathsf{U}_{k+q+1} \{\mathsf{K}_M^{q+s} a_1\}_{s(1,n)}$
(D1.6) $\mathsf{K}_M^q \mathfrak{I}_k^n = \mathsf{U}_{k+q+1} \{a_{1+q+s}\}_{s(1,n)}$
(T4) $\mathsf{K}_M^q \mathfrak{I}_k^n = \mathsf{U}_{k+q+1}^n \{a_{1+s}\}_{1,n}$
(D1.6) $\mathsf{K}_M^q \mathfrak{I}_k^n = \mathsf{U}_{k+q+1} \{\mathsf{K}_M^s a_1\}_{s(1,n)}$
(D1.11) $\mathsf{K}_M^q \mathfrak{I}_k^n = \mathfrak{I}_{k+q}^n$

T6 $\quad \mathfrak{I}_{nu+v}^n = \mathfrak{I}_v^n \qquad u \in N,\ v = 1, 2, \ldots, n$

Proof:
(D1.11) $\mathfrak{I}_{nu+v}^n = \mathsf{U}_{nu+v+1}^n \{\mathsf{K}_M^s M_1\}_{s(1,n)} \qquad M_1 \in M$
(D1.14) $\mathfrak{I}_{nu+v}^n = \mathsf{U}_{v+1}^n \{\mathsf{K}_M^s M_1\}_{s(1,n)}$
(D1.11) $\mathfrak{I}_{nu+v}^n = \mathfrak{I}_v^n$

T7 $\quad \mathfrak{I}_n^n = M.$

Proof: Follows from D1.11 and D1.12.

T8 $\quad \tilde{\mathsf{U}}_k^n M \mathfrak{I}_{k-1}^n = \mathfrak{I}_{k-1}^n \qquad k = 1, 2, \ldots, n$

Proof:
(T7) $\tilde{\mathsf{U}}_k^n M \mathfrak{I}_{k-1}^n = \mathsf{U}_k^n \mathfrak{I}_n^n \mathfrak{I}_{k-1}^n$
(AxC1) $\tilde{\mathsf{U}}_k^n M \mathfrak{I}_{k-1}^n = \mathfrak{I}_{k-1}^n$

For $n = 2$, $k = 1$: $M \cup M = M$.
$k = 2$: $M \cap \emptyset = \emptyset$.

For $n = 3$, $\mathsf{U}_1^3 MMM = M$.

$$\mathsf{U}_2^3 MM\mathfrak{J}_1^3 = \mathfrak{J}_1^3 \text{ or } \mathsf{U}_2^3 M\mathfrak{J}_1^3\mathfrak{J}_1^3 = \mathfrak{J}_1^3$$
$$\mathsf{U}_3^3 MM\mathfrak{J}_2^3 = \mathfrak{J}_2^3 \text{ or } \mathsf{U}_3^3 M\mathfrak{J}_2^3\mathfrak{J}_2^3 = \mathfrak{J}_2^3$$

T9 $\tilde{\mathsf{U}}_k^n M\mathfrak{J}_k^n = M$ $\quad k = 1, 2, \ldots, n$

Proof: (T7) $\tilde{\mathsf{U}}_k^n M\mathfrak{J}_k^n = \mathsf{U}_k^n\mathfrak{J}_n^n\mathfrak{J}_k^n$
(AxC1) $\tilde{\mathsf{U}}_k^n M\mathfrak{J}_k^n = \mathfrak{J}_n^n$
(T7) $\tilde{\mathsf{U}}_k^n M\mathfrak{J}_k^n = M$

For $n = 2$, $k = 1$: $M \cup \emptyset = M$
$k = 2$: $M \cap M = M$
For $n = 3$: $\mathsf{U}_1^3 MM\mathfrak{J}_1^3 = M$ or $\mathsf{U}_1^3 M\mathfrak{J}_1^3\mathfrak{J}_1^3 = M$
$\mathsf{U}_2^3 MM\mathfrak{J}_2^3 = M$ or $\mathsf{U}_2^3 M\mathfrak{J}_2^3\mathfrak{J}_2^3 = M$
$\mathsf{U}_3^3 MMM = M$

T10 $\tilde{\mathsf{U}}_k^n\mathfrak{J}_p^n = \mathfrak{J}_p^n$ $\quad p, k = 1, 2, \ldots, n$

Proof: Follows from AxC1.

T11 $\mathsf{U}_1^n\{\mathfrak{J}_s^n\}_{1,n} = M$

Proof: Follows from AxC1, taking into account D1.12.

T12 $\tilde{\mathsf{U}}_k^n M = M$ $\quad k = 1, 2, \ldots, n$

Proof: Follows from T10, taking into account D1.12.

T13 $\mathsf{U}_{k+1}^n\{\mathsf{K}_M^q P_s\}_{s(1,n)} = \mathsf{U}_{k+1}^n\{\mathsf{K}_M^{q+t} K_M^{n-t} P_s\}_{s(1,n)}$ $\quad P_s \in M,\ k,\ q,\ t = 1, 2, \ldots, n$

Proof: (T1) $\mathsf{U}_{k+1}^n\{\mathsf{K}_M^q P_s\}_{s(1,n)} = \mathsf{U}_{k+1}^n\{\mathsf{K}_M^{q+n} P_s\}_{s(1,n)}$
(T1) $\mathsf{U}_{k+1}^n\{\mathsf{K}_M^q P_s\}_{s(1,n)} = \mathsf{U}_{k+1}^n\{\mathsf{K}^{q+n-t+t} P_s\}_{s(1,n)}$
(D1.15) $\mathsf{U}_{k+1}^n\{\mathsf{K}_M^q P_s\}_{s(1,n)} = \mathsf{U}_{k+1}^n\{\mathsf{K}_M^{q+t}\, \mathsf{K}_M^{n-t} P_s\}_{s(1,n)}$

T14 $\subseteq \{M_s\}_{1,n} = \subseteq \{\mathsf{K}_M^{n-q} M_{q+s}\}_{s(1,n)}$ $\quad M_s \in M,\ M_{n+v} = M_v,\ q,\ v = 1, 2, \ldots, n$

Proof: (D1.13) $\subseteq \{M_s\}_{1,n} = \mathsf{U}_1^n\{\mathsf{K}_M^{n-s} M_s\}_{1,n}$
(D1.7) $\subseteq \{M_s\}_{1,n} = \mathsf{U}_1^n(\mathsf{K}_M^{n-1} M_1\, \mathsf{K}_M^{n-2} M_2 \ldots \mathsf{K}_M^{n-q+1} M_{q-1}$
$\mathsf{K}_M^{n-q} M_q\, \mathsf{K}_M^{n-q-1} M_{q+1} \ldots \mathsf{K}_M^2 M_{n-2}\, \mathsf{K}_M^1 M_{n-1} M_n)$
(D1.15; Ax1; T1) $\subseteq \{M_s\}_{1,n} = \mathsf{U}_1^n(\mathsf{K}_M^{n+q-1}(\mathsf{K}_M^{n-q} M_1)\, \mathsf{K}_M^{n+q-2}(\mathsf{K}_M^{n-q} M_2)$
$\ldots \mathsf{K}_M^1(\mathsf{K}_M^{n-q} M_{q-1})\, \mathsf{K}_M^{n-q} M_q\, \mathsf{K}_M^{n-1}(\mathsf{K}_M^{n-q} M_{q+1})$
$\ldots \mathsf{K}_M^{q+2}(\mathsf{K}_M^{n-q} M_{n-2})\, \mathsf{K}_M^{q+1}(\mathsf{K}_M^{n-q} M_{n-1})\, \mathsf{K}_M^q(\mathsf{K}_M^{n-q} M_n))$
(Ax3; Ax4) $\subseteq \{M_s\}_{1,n} = \mathsf{U}_1^n(\mathsf{K}_M^{n-1}(\mathsf{K}_M^{n-q} M_{q+1})\, \mathsf{K}_M^{n-2}(\mathsf{K}_M^{n-q} M_{q+2})$
$\ldots \mathsf{K}_M^2(\mathsf{K}_M^{n-q} M_{q-2})\, \mathsf{K}_M^1(\mathsf{K}_M^{n-q} M_{q-1})\, \mathsf{K}_M^{n-q} M_q)$
(D1.7) $\subseteq \{M_s\}_{1,n} = \mathsf{U}_1^n\{\mathsf{K}_M^{n-s}(\mathsf{K}_M^{n-q} M_{q+s})\}_{s(1,n)}$
(D1.13) $\subseteq \{M_s\}_{1,n} = \subseteq \{\mathsf{K}_M^{n-q} M_{q+s}\}_{s(1,n)}$

2 *N-polar judgements*

2.1 *Definitions*

D2.1: $a \neq \mathfrak{J}_k^n \;.=_{df}\; .\; a \in \mathsf{K}_M^{n-k}\mathfrak{J}_k^n,$ $\quad k = 1, 2, \ldots, n$

that is to say, in $a \neq \mathfrak{J}_k^n$ we understand that a belongs to the set M, but is not equivalent to the class constant $\mathfrak{J}_k^n$, which also belongs to the set M.

D2.2: $(k, t)\{b_s\}_{1,n} =_{df} \mathsf{U}^n_{k+1}\{\mathsf{K}_M^{k+st-t-1} b_s\}_{s(1,n)}$ $k = 1, 2, \ldots, n-1,\ t = 1, 2, \ldots, n$

$$\mathsf{U}^n_{k+1}\{\mathsf{K}_M^{k+st-t-1} b_s\}_{s(1,n)} \neq \mathfrak{I}^n_k.$$

D2.3: $(n, t)\{b_s\}_{1,n} =_{df} \mathsf{U}^n_1\{\mathsf{K}_M^{st-t-1} b_s\}_{s(1,n)}$ $i = 1, 2, \ldots, n$

D2.4: $\mathbf{J}_{k,t} =_{df} (k, t)\{b_s\}_{1,n}$ $k, t = 1, 2, \ldots, n$

$\mathbf{J}_{k,t}(k, t = 1, 2, \ldots, n)$ are called *n-polar judgements*, where k is an indicator of quality and t an indicator of quantity. After the different values taken by the indicators k and t, there will exist n^2 types (kinds) of n-polar judgements. Each n-polar judgement is going to have n b_s terms.

2.2 *Examples:* For $n = 2$ there will exist $2^2 = 4$ types of bipolar judgements. Let P and Q be subsets of the set M, of variance 2. We note: $\mathsf{U}^2_1 PQ = P \cup Q$; $\mathsf{U}^2_2 PQ = P \cap Q$; $\mathsf{K}^1_M P = \mathsf{K} P$; $\mathfrak{I}^2_1 = \emptyset$.

The four types of bipolar judgement are going to be:

1. $\mathbf{J}_{1,1} = (1, 1)PQ = P \cap \mathsf{K}Q.$ $(P \cap \mathsf{K}Q \neq \emptyset)$

Particular negative judgement: $P\,\mathrm{o}\,Q$.

2. $\mathbf{J}_{1,2} = (1, 2)PQ = P \cap Q.$ $(P \cap Q \neq \emptyset)$

Particular affirmative judgement: $P\,\mathrm{i}\,Q$.

3. $\mathbf{J}_{2,1} = (2, 1)PQ = \mathsf{K}P \cup Q.$

Universal affirmative judgement: $P\,\mathrm{a}\,Q$.

4. $\mathbf{J}_{2,2} = (2, 2)PQ = \mathsf{K}P \cup \mathsf{K}Q.$

Universal negative judgement: $P\,\mathrm{e}\,Q$.

For $n = 3$ there will exist $3^2 = 9$ types of judgements. Take P, Q, R as three subsets of the set M, of the variance 3. According to the definitions D2.2, D2.3, and D2.4 there are:

1. $\mathbf{J}_{1,1} = (1, 1)PQR = \mathsf{U}^3_2 P\mathsf{K}^1_M Q\mathsf{K}^2_M R$ $(\mathsf{U}^3_2 P\mathsf{K}^1_M Q\mathsf{K}^2_M R \neq \mathfrak{I}^3_1)$
2. $\mathbf{J}_{1,2} = (1, 2)PQR = \mathsf{U}^3_2 P\mathsf{K}^2_M Q\mathsf{K}^1_M R$ $(\mathsf{U}^3_2 P\mathsf{K}^2_M Q\mathsf{K}^1_M R \neq \mathfrak{I}^3_1)$
3. $\mathbf{J}_{1,3} = (1, 3)PQR = \mathsf{U}^3_2 PQR$ $(\mathsf{U}^3_2 PQR \neq \mathfrak{I}^3_1)$
4. $\mathbf{J}_{2,1} = (2, 1)PQR = \mathsf{U}^3_3 \mathsf{K}^1_M P\mathsf{K}^2_M QR$ $(\mathsf{U}^3_3 \mathsf{K}^1_M P\mathsf{K}^2_M QR \neq \mathfrak{I}^3_2)$
5. $\mathbf{J}_{2,2} = (2, 2)PQR = \mathsf{U}^3_3 \mathsf{K}^1_M PQ\mathsf{K}^2_M R$ $(\mathsf{U}^3_3 \mathsf{K}^1_M PQ\mathsf{K}^2_M R \neq \mathfrak{I}^3_2)$
6. $\mathbf{J}_{2,3} = (2, 3)PQR = \mathsf{U}^3_3 \mathsf{K}^1_M P\mathsf{K}^1_M Q\mathsf{K}^1_M R$ $(\mathsf{U}^3_3 \mathsf{K}^1_M P\mathsf{K}^1_M Q\mathsf{K}^1_M R \neq \mathfrak{I}^3_2)$
7. $\mathbf{J}_{3,1} = (3, 1)PQR = \mathsf{U}^3_1 \mathsf{K}^2_M PQ\mathsf{K}^1_M R$
8. $\mathbf{J}_{3,2} = (3, 2)PQR = \mathsf{U}^3_1 \mathsf{K}^2_M P\mathsf{K}^1_M QR$
9. $\mathbf{J}_{3,3} = (3, 3)PQR = \mathsf{U}^3_1 \mathsf{K}^2_M P\mathsf{K}^2_M Q\mathsf{K}^2_M R$

We consider in $\mathbf{J}_{3,2}$ $R = M$, consequently $\mathbf{J}_{3,2} = \mathsf{U}^3_1 \mathsf{K}^2_M P\mathsf{K}^1_M QM$. According to AxC2 $\mathsf{U}^3_1 \mathsf{K}^2_M P\mathsf{K}^1_M QM = M$. Consequently, according to definition D1.13: $\mathbf{J}_{3,2} = (3, 2)PQR = \subseteq PQR$. According to the same deduction we get:

$$\mathbf{J}_{n,n-1} = (n, n-1)\{Q_s\}_{1,n} \;.=.\; \subseteq \{Q_s\}_{1,n}$$

2.3 *Relationships among the different types of judgements* The rules of immediate inference are as follows:

R1 Contradiction rule:

$$\mathsf{K}_M^q \mathbf{J}_{k,t} = \mathbf{J}_{k+q,t} \qquad k, t, q = 1, 2, \ldots, n$$

Proof: Follows from Ax2.

R2 Obversion rule:

$$(k,t)\{Q_s\}_{1,n} = (k, t+q)\{\mathsf{K}_M^{q(s-1)} Q_s\}_{s(1,n)} \qquad k, t, q = 1, 2, \ldots n$$

Proof: Follows from T13.

R3 (Simple) conversion rule:

$$(k,n)\{Q_s\}_{1,n} = (k,n)\, \widehat{\mathsf{Perm}}\{Q_s\}_{1,n} \qquad k = 1, 2, \ldots, n$$

Proof: Follows from Ax3.

R4 Subalternation rule:

$$\subseteq \{\mathbf{J}_{q,t+q-1}\}_{q(1,n)} \qquad n \text{ - prime number}, t = 1, 2, \ldots, n.$$

Proof:

(R4.1) (D1.13; R1) $\mathsf{U}_1^n\{\mathsf{K}_M^{n-q}\mathbf{J}_{q,t+q-1}\}_{q(1,n)} = \mathsf{U}_1^n\{\mathbf{J}_{n,t+q-1}\}_{q(1,n)}$
(R4.2) (D2.3) $\mathbf{J}_{n,t+q-1} = \mathsf{U}_1^n\{\mathsf{K}_M^{s(t+q-1)-t-q} b_s\}_{s(1,n)}$
(R4.3) (D1.15) $\mathbf{J}_{n,t+q-1} = \mathsf{U}_1^n\{\mathsf{K}_M^{s(t+q-1)}\, \mathsf{K}_M^{n-t-q} b_s\}_{s(1,n)}$
(R4.4) We note $\mathsf{K}_M^{n-t-q} b_s = c_s$, hence R4.3 becomes:
(R4.5) $\mathbf{J}_{n,t+q-1} = \mathsf{U}_1^n\{\mathsf{K}_M^{s(t+q-1)} c_s\}_{s(1,n)}$

Introducing this value into R4.1, we get:

(R4.6) $\mathsf{U}_1^n\{\mathsf{K}_M^{n-q}\mathbf{J}_{q,t+q-1}\}_{q(1,n)} = \mathsf{U}_1^n\{\mathsf{U}_1^n\{\mathsf{K}_M^{s(t+q-1)} c_s\}_{s(1,n)}\}_{q(1,n)}$
(R4.7) (Ax3; Ax4) $\mathsf{U}_1^n\{\mathsf{K}_M^{n-q}\mathbf{J}_{q,t+q-1}\}_{q(1,n)}$
$= \mathsf{U}_1^n\{\mathsf{U}_1^n\{\mathsf{K}_M^{s(t+q-1)} c_s\}_{q(1,n)}\}_{s(1,n)}$

According to an arithmetical theorem we get:

(R4.8) (D1.12) $\mathsf{U}_1^n\{\mathsf{K}_M^{s(t+q-1)} c_s\}_{q(1,n)} = \mathsf{U}_1^n\{\mathsf{K}_M^{p} c_s\}_{p(1,n)} = M$

Consequently:

(R4.9) $\mathsf{U}_1^n\{\mathsf{K}_M^{n-q}\mathbf{J}_{q,t+q-1}\}_{q(1,n)} = M$

Now consider $s \neq n$, (n being a prime number). According to D1.13 R4 is proved for $s \neq n$.

For $s = n$ we get $c_n = \mathsf{K}_M^{n-t-q} b_n$ (R4.4). In this case through the substitution of this value in R4.8, we get:

(R4.10) $\mathsf{U}_1^n\{\mathsf{K}_M^{n-t-q} b_n\}_{q(1,n)} = M$

Rule R4 is thus demonstrated, but only for the case when the polarity (variance) of the sets is expressed by a prime number. We can deduce that a subalternation relationship exists only among judgements of prime polarity.

R5 (Total) contraposition rule:

$$(k,1)\{Q_s\}_{1,n} = (k,1)\{\mathsf{K}_M^q Q_{s+n-q}\}_{s(1,n)} \qquad q = 1, 2, \ldots, n$$

Proof: Follows from T14, by the substitution of q with $n - q$.

R6 Technical rule 1

$$\mathbf{J}_{un+v,t} = \mathbf{J}_{v,t}. \qquad u \in N,\ v = 1, 2, \ldots, n$$

Proof: Deducible from D1.15 and Ax1.

R7 Technical rule 2

$$\mathbf{J}_{k,un+v} = \mathbf{J}_{k,v} \qquad u \in N,\ v = 1, 2, \ldots, n$$

Based on Rules R1-R5 we get for $n = 2$ (a prime) the following immediate inferences. We shall use the notation introduced in **2.2.** The formulas in parentheses are the basis of these inferences.

a. Immediate direct inferences:

a.01	$\mathsf{K}(1,1)PQ = (2,1)PQ$	i.e., $\mathsf{K}(P\,\mathrm{o}\,Q) = P\,\mathrm{a}\,Q$	(R1)
a.02	$\mathsf{K}(1,2)PQ = (2,2)PQ$	i.e., $\mathsf{K}(P\,\mathrm{i}\,Q) = P\,\mathrm{e}\,Q$	(R1)
a.03	$\mathsf{K}(2,1)PQ = (1,1)PQ$	i.e., $\mathsf{K}(P\,\mathrm{a}\,Q) = P\,\mathrm{o}\,Q$	(R1)
a.04	$\mathsf{K}(2,2)PQ = (1,2)PQ$	i.e., $\mathsf{K}(P\,\mathrm{e}\,Q) = P\,\mathrm{i}\,Q$	(R1)
a.05	$(1,1)PQ = (1,2)P\mathsf{K}Q$	i.e., $P\,\mathrm{o}\,Q = P\,\mathrm{i}\,\mathsf{K}Q$	(R2)
a.06	$(1,2)PQ = (1,1)P\mathsf{K}Q$	i.e., $P\,\mathrm{i}\,Q = P\,\mathrm{o}\,\mathsf{K}Q$	(R2)
a.07	$(2,1)PQ = (2,2)P\mathsf{K}Q$	i.e., $P\,\mathrm{a}\,Q = P\,\mathrm{e}\,\mathsf{K}Q$	(R2)
a.08	$(2,2)PQ = (2,1)P\mathsf{K}Q$	i.e., $P\,\mathrm{e}\,Q = P\,\mathrm{a}\,\mathsf{K}Q$	(R2)
a.09	$(1,2)PQ = (1,2)QP$	i.e., $P\,\mathrm{i}\,Q = Q\,\mathrm{i}\,P$	(R3)
a.10	$(2,2)PQ = (2,2)QP$	i.e., $P\,\mathrm{e}\,Q = Q\,\mathrm{e}\,P$	(R3)
a.11	$(1,1)PQ \subseteq (2,2)PQ$	i.e., $P\,\mathrm{o}\,Q \subseteq P\,\mathrm{e}\,Q$	(R4)
a.12	$(1,2)PQ \subseteq (2,1)PQ$	i.e., $P\,\mathrm{i}\,Q \subseteq P\,\mathrm{a}\,Q$	(R4)
a.13	$(2,1)PQ = (2,1)\mathsf{K}Q\mathsf{K}P$	i.e., $P\,\mathrm{a}\,Q = \mathsf{K}Q\,\mathrm{a}\,\mathsf{K}P$	(R5)
a.14	$(1,1)PQ = (1,1)\mathsf{K}Q\mathsf{K}P$	i.e., $P\,\mathrm{o}\,Q = \mathsf{K}Q\,\mathrm{o}\,\mathsf{K}P$	(R5)

b. Immediate inferences deducible from direct inferences:

b.01	$\mathsf{K}(2,2)PQ \subseteq (2,1)PQ$	(a.04; a.12)
b.02	$\mathsf{K}(2,1)PQ \subseteq (2,2)PQ$	(a.03; a.11)
b.03	$(1,1)PQ \subseteq \mathsf{K}(1,2)PQ$	(a.11; a.02)
b.04	$(1,2)PQ \subseteq \mathsf{K}(1,1)PQ$	(a.12; a.01)
b.05	$\mathsf{K}(2,1)PQ \subseteq \mathsf{K}(1,2)PQ$	(a.03; a.02; a.11)
b.06	$\mathsf{K}(2,2)PQ \subseteq \mathsf{K}(1,1)PQ$	(a.04; a.01; a.12)
b.07	$(1,2)PQ \subseteq (2,1)QP$	(a.09; a.12)
b.08	$(1,1)PQ \subseteq (2,2)QP$	(a.11; a.10)
b.09	$(1,1)P\mathsf{K}Q \subseteq (2,1)PQ$	(a.06; a.12)
b.10	$(1,2)P\mathsf{K}Q \subseteq (2,2)PQ$	(a.05; a.11)
b.11	$(2,2)PQ = (2,1)Q\mathsf{K}P$	(a.10; a.08)
b.12	$(1,2)PQ = (1,1)Q\mathsf{K}P$	(a.09; a.06)
b.13	$(1,1)PQ \subseteq (2,1)\mathsf{K}QP$	(a.06; a.09; a.12)
b.14	$(1,2)PQ \subseteq (2,2)\mathsf{K}QP$	(a.05; a.11; a.10)
b.15	$(1,2)PQ \subseteq (2,1)\mathsf{K}Q\mathsf{K}P$	(a.12; a.13)
b.16	$(1,1)PQ \subseteq (2,2)\mathsf{K}Q\mathsf{K}P$	(a.14; a.11)
b.17	$(2,1)PQ = (2,2)\mathsf{K}QP$	(a.07; a.10)

b.18 $(1,1)PQ = (1,2)\mathrm{K}QP$ (a.05; a.09)
b.19 $(1,1)PQ \subseteq (2,1)Q\mathrm{K}P$ (a.11; a.10; a.08)
b.20 $(1,2)PQ \subseteq (2,2)Q\mathrm{K}P$ (a.09; a.12; a.07)
b.21 $(1,2)PQ \subseteq (2,1)\mathrm{K}P\mathrm{K}Q$ (a.09; a.12; a.13)
b.22 $(1,1)PQ \subseteq (2,2)\mathrm{K}P\mathrm{K}Q$ (a.14; a.11; a.10)
b.23 $(1,1)PQ \subseteq (2,1)\mathrm{K}PQ$ (a.05; a.09; a.12; a.13)
b.24 $(1,2)PQ \subseteq (2,2)\mathrm{K}PQ$ (a.09; a.06; a.11; a.10)

The notation $A \subseteq B$ was used instead of $\subseteq AB$.

c. The immediate inferences for $n = 3$ are given below:

c.01 $\mathrm{K}_M^1(1,1)PQR = (2,1)PQR$
c.02 $\mathrm{K}_M^2(1,1)PQR = (3,1)PQR$
c.03 $\mathrm{K}_M^1(1,2)PQR = (2,2)PQR$
c.04 $\mathrm{K}_M^2(1,2)PQR = (3,2)PQR$
c.05 $\mathrm{K}_M^1(1,3)PQR = (2,3)PQR$
c.06 $\mathrm{K}_M^2(1,3)PQR = (3,3)PQR$
c.07 $\mathrm{K}_M^1(2,1)PQR = (3,1)PQR$
c.08 $\mathrm{K}_M^2(2,1)PQR = (1,1)PQR$
c.09 $\mathrm{K}_M^1(2,2)PQR = (3,2)PQR$
c.10 $\mathrm{K}_M^2(2,2)PQR = (1,2)PQR$
c.11 $\mathrm{K}_M^1(2,3)PQR = (3,3)PQR$
c.12 $\mathrm{K}_M^2(2,3)PQR = (1,3)PQR$
c.13 $\mathrm{K}_M^1(3,1)PQR = (1,1)PQR$
c.14 $\mathrm{K}_M^2(3,1)PQR = (2,1)PQR$
c.15 $\mathrm{K}_M^1(3,2)PQR = (1,2)PQR$
c.16 $\mathrm{K}_M^2(3,2)PQR = (2,2)PQR$
c.17 $\mathrm{K}_M^1(3,3)PQR = (1,3)PQR$
c.18 $\mathrm{K}_M^2(3,3)PQR = (2,3)PQR$

c.19 $(1,1)PQR = (1,2)P\mathrm{K}_M^1Q\mathrm{K}_M^2R$
c.20 $(1,1)PQR = (1,3)P\mathrm{K}_M^2Q\mathrm{K}_M^1R$
c.21 $(1,2)PQR = (1,3)P\mathrm{K}_M^1Q\mathrm{K}_M^2R$
c.22 $(1,2)PQR = (1,1)P\mathrm{K}_M^2Q\mathrm{K}_M^1R$
c.23 $(1,3)PQR = (1,1)P\mathrm{K}_M^1Q\mathrm{K}_M^2R$
c.24 $(1,3)PQR = (1,2)P\mathrm{K}_M^2Q\mathrm{K}_M^1R$
c.25 $(2,1)PQR = (2,2)P\mathrm{K}_M^1Q\mathrm{K}_M^2R$
c.26 $(2,1)PQR = (2,3)P\mathrm{K}_M^2Q\mathrm{K}_M^1R$
c.27 $(2,2)PQR = (2,3)P\mathrm{K}_M^1Q\mathrm{K}_M^2R$
c.28 $(2,2)PQR = (2,1)P\mathrm{K}_M^2Q\mathrm{K}_M^1R$
c.29 $(2,3)PQR = (2,1)P\mathrm{K}_M^1Q\mathrm{K}_M^2R$
c.30 $(2,3)PQR = (2,2)P\mathrm{K}_M^2Q\mathrm{K}_M^1R$
c.31 $(3,1)PQR = (3,2)P\mathrm{K}_M^1Q\mathrm{K}_M^2R$
c.32 $(3,1)PQR = (3,3)P\mathrm{K}_M^2Q\mathrm{K}_M^1R$
c.33 $(3,2)PQR = (3,3)P\mathrm{K}_M^1Q\mathrm{K}_M^2R$
c.34 $(3,2)PQR = (3,1)P\mathrm{K}_M^2Q\mathrm{K}_M^1R$
c.35 $(3,3)PQR = (3,1)P\mathrm{K}_M^1Q\mathrm{K}_M^2R$
c.36 $(3,3)PQR = (3,2)P\mathrm{K}_M^2Q\mathrm{K}_M^1R$

c.37 $(1,3)PQR = (1,3)PRQ$
c.38 $(2,3)PQR = (2,3)PRQ$
c.39 $(1,3)PQR = (1,3)QPR$
c.40 $(2,3)PQR = (2,3)QPR$
c.41 $(1,3)PQR = (1,3)QRP$
c.42 $(2,3)PQR = (2,3)QRP$
c.43 $(1,3)PQR = (1,3)RPQ$
c.44 $(2,3)PQR = (2,3)RPQ$
c.45 $(1,3)PQR = (1,3)RQP$
c.46 $(2,3)PQR = (2,3)RQP$

c.47 $\subseteq \{(1,1)PQR\ (2,2)PQR\ (3,3)PQR\}$
c.48 $\subseteq \{(1,2)PQR\ (2,3)PQR\ (3,1)PQR\}$
c.49 $\subseteq \{(1,3)PQR\ (2,1)PQR\ (3,2)PQR\}$

c.50 $(1, 1)PQR = (1, 1)\,\mathsf{K}_M^1R\mathsf{K}_M^1P\mathsf{K}_M^1Q$
c.51 $(1, 1)PQR = (1, 1)\,\mathsf{K}_M^2Q\mathsf{K}_M^2R\mathsf{K}_M^2P$
c.52 $(2, 1)PQR = (2, 1)\,\mathsf{K}_M^1R\mathsf{K}_M^1P\mathsf{K}_M^1Q$
c.53 $(2, 1)PQR = (2, 1)\,\mathsf{K}_M^2Q\mathsf{K}_M^2R\mathsf{K}_M^2P$
c.54 $(3, 1)PQR = (3, 1)\,\mathsf{K}_M^1R\mathsf{K}_M^1P\mathsf{K}_M^1Q$
c.55 $(3, 1)PQR = (3, 1)\,\mathsf{K}_M^2Q\mathsf{K}_M^2R\mathsf{K}_M^2P$

With the aid of the above direct immediate inferences the composed immediate inferences of the tripolar judgements can be formed.

3 *N-polar syllogistics*

3.1 *Preliminary notions* An n-polar syllogism is formed out of n n-polar judgements—presumptions—from which, according to certain laws, a new n-polar judgement—the conclusion—is deducible. We remind the reader that each n-polar judgement has n terms. Each term of the conclusion will be a term of at least one of the suppositions. The terms of the suppositions, which are also terms of the conclusion, are called *principal terms* of the presumptions. The terms of the presumptions, which are not principal terms, are called *middle terms*. Each presumption will contain for $n = 2$ a single middle term, while for $n > 2$, there will exist $(n - 1)/2$ or $(n + 1)/2$ different middle terms. Each middle term will be a term of at least two succesive presumptions. Between each two principal terms of a presumption we can interpose a maximum of $(n - 1)/2$ different middle terms. Between two middle terms of a presumption there can be a maximum of $(n - 1)/2$ principal terms. The total number of principal terms of the presumptions of a syllogism will be n, while the total number of presumptions of a syllogism of n-polar kind will be $n - 1$. Each principal term, except the first and the last (of the conclusion), will be at the same time a term of at least two succesive presumptions. The first principal term (of the conclusion) will be the first principal term of the last presumption; while the last principal term (of the conclusion) will be the last principal term of the first presumption. If we number the principal terms of a syllogism in the order in which they are to be found in the conclusion, then in each presumption the principal terms will appear in ascending order of their indices.

According to the placement of the middle terms of the n presumptions of an n-polar syllogism, the syllogisms can be classified in syllogistical figures. Out of the set of syllogistical figures we shall detach one—the fundamental syllogistical figure—in which, in each presumption, between two principal terms following one another there will be a middle term, while between two middle terms following one another there will be a principal term. The first presumption has as its last term the last principal term, while the last presumption has as its first term the first principal term.

Examples of fundamental syllogistical figures:

$n = 2$	$n = 3$	$n = 5$
$(k_1, t_1)\, mP_2$	$(k_1, t_1)\, P_2 m_2 P_3$	$(k_1, t_1)\, P_3 m_3 P_4 m_4 P_5$
$\underline{(k_2, t_2)\, P_1 m}$	$(k_2, t_2)\, m_1 P_2 m_2$	$(k_2, t_2)\, m_2 P_3 m_3 P_4 m_4$
$(k_0, t_0)\, P_1 P_2$	$\underline{(k_3, t_3)\, P_1 m_1 P_2}$	$(k_3, t_3)\, P_2 m_2 P_3 m_3 P_4$
	$(k_0, t_0)\, P_1 P_2 P_3$	$(k_4, t_4)\, m_1 P_2 m_2 P_3 m_3$
		$\underline{(k_5, t_5)\, P_1 m_1 P_2 m_2 P_3}$
		$(k_0, t_0)\, P_1 P_2 P_3 P_4 P_5$

In these examples the principal terms are P_i, $i = 1, 2, \ldots, n$, while the middle terms are m_j, $j = 1, 2, \ldots, n - 1$.

A more concentrated notation for syllogisms, which will be used in this paper, is the following:

$$\cup_n^n \{\mathbf{J}_{k_s, t_s}\}_{s(1,n)} \Rightarrow \mathbf{J}_{k_0, t_0},$$

where $\mathbf{J}_{k_s, t_s}$, $s = 1, 2, \ldots, n$, are the presumptions, and $\mathbf{J}_{k_0, t_0}$ is the conclusion. The symbol $\Rightarrow$ indicates "from . . . results . . .". Thus, $a \Rightarrow b$ is read: *from a results b.*

3.2 *Formation rules for valid n-polar syllogisms*

Rg1 Each syllogism (n-polar syllogism) will contain as presumptions $n - 1$ judgements $\mathbf{J}_{n,t}$, of which a maximum of one will have $t \neq n - 1$.

Rg2 One of the judgements (not exhausted in Rg1, there being n judgements) can be:

a. $\mathbf{J}_{n-p,t-p}$, $t \neq n - 1$, $p = 1, 2, \ldots, n - 1$, if the other $n - 1$ judgements are of type $\mathbf{J}_{n,n-1}$.

b. $\mathbf{J}_{n-p,n-p-1}$, $p = 1, 2, \ldots, n - 1$, if one of the other $n - 1$ judgements is of type $\mathbf{J}_{n,t}$ and $t \neq n - 1$.

3.3 *About the type of the conclusion judgement* The axioms of n-polar syllogistic are:

AxS1 $\cup_n^n \{\mathbf{J}_{k_s, t_s}\}_{s(1,n)} \Rightarrow \mathbf{J}_{k_1+k_2+\ldots+k_n,\, t_1+t_2+\ldots+t_n-1}$

AxS2 $\cup_n^n \{\mathbf{J}_{n, t_s}\}_{s(1,n)} \Rightarrow \mathbf{J}_{n-1,\, t_1+t_2+\ldots+t_n}$

Remark: If from the same presumptions different conclusions are deduced according to these two axioms, both syllogisms are valid.

From the above statement we can conclude: *The type of the conclusion judgement does not depend on the syllogistical figure; it depends only on the types of the presumption judgements.*

In the case of the judgements of type $\mathbf{J}_{n,n}$, according to rule R3 of (simple) conversion on judgements, there will exist $n!$ cases when the type of the conclusion judgement will be deducible according to AxS1, and $n!$ cases when the type of the conclusion judgement will be deducible according to AxS2. Consequently we can state that the minimal number of n-polar syllogistical figures is $2n!$.

3.4 *Valid bipolar syllogisms*

A. Deducible according to AxS1

$U_2^2(\mathbf{J}_{2,1}\mathbf{J}_{2,1}) \Rightarrow \mathbf{J}_{2,1}$	Barbara
$U_2^2(\mathbf{J}_{2,2}\mathbf{J}_{2,1}) \Rightarrow \mathbf{J}_{2,2}$	Celarent, Cesare
$U_2^2(\mathbf{J}_{2,1}\mathbf{J}_{1,2}) \Rightarrow \mathbf{J}_{1,2}$	Darii, Datisi
$U_2^2(\mathbf{J}_{2,2}\mathbf{J}_{1,2}) \Rightarrow \mathbf{J}_{1,1}$	Ferio, Ferison, Fresison, Festino
$U_2^2(\mathbf{J}_{2,1}\mathbf{J}_{2,2}) \Rightarrow \mathbf{J}_{2,2}$	Camestres, Camenes
$U_2^2(\mathbf{J}_{2,1}\mathbf{J}_{1,1}) \Rightarrow \mathbf{J}_{1,1}$	Baroco
$U_2^2(\mathbf{J}_{1,2}\mathbf{J}_{2,1}) \Rightarrow \mathbf{J}_{1,2}$	Disamis, Dimaris
$U_2^2(\mathbf{J}_{1,1}\mathbf{J}_{2,1}) \Rightarrow \mathbf{J}_{1,1}$	Bocardo

B. Deducible according to AxS2

$U_2^2(\mathbf{J}_{2,1}\mathbf{J}_{2,1}) \Rightarrow \mathbf{J}_{1,2}$	Barbari, Darapti, Bramantip
$U_2^2(\mathbf{J}_{2,2}\mathbf{J}_{2,1}) \Rightarrow \mathbf{J}_{1,1}$	Celaront, Cesaro, Felapton, Fesapo
$U_2^2(\mathbf{J}_{2,1}\mathbf{J}_{2,2}) \Rightarrow \mathbf{J}_{1,1}$	Camestrop, Camenop

3.5 *Sketches of valid bipolar syllogisms* As mentioned above, there will be at least 2.3! = 12 tripolar syllogistical figures. We shall analyze neither the forms of the syllogistical figures nor how many and which of the tripolar syllogisms belong to each of these syllogistical figures.

We present below sketches of the tripolar syllogisms, out of which the tripolar syllogisms can be obtained by permutation of the types of the presumptions. The sketches of the syllogisms below have been made according to the rules of formation for the valid syllogisms; the type of the conclusion judgement has been deduced according to the axioms of the syllogism; and each of the syllogisms belongs to at least one of the syllogistical figures. The sketches of the tripolar syllogisms are the following;

A. Deducible according to AxS1:

Sk1 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,2}\mathbf{J}_{3,2}) \Rightarrow \mathbf{J}_{3,2}$
Sk2 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,2}\mathbf{J}_{2,1}) \Rightarrow \mathbf{J}_{2,1}$
Sk3 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,2}\mathbf{J}_{1,3}) \Rightarrow \mathbf{J}_{1,3}$
Sk4 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,2}\mathbf{J}_{3,3}) \Rightarrow \mathbf{J}_{3,3}$
Sk5 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,2}\mathbf{J}_{2,2}) \Rightarrow \mathbf{J}_{2,2}$
Sk6 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,2}\mathbf{J}_{1,1}) \Rightarrow \mathbf{J}_{1,1}$
Sk7 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,2}\mathbf{J}_{3,1}) \Rightarrow \mathbf{J}_{3,1}$
Sk8 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,2}\mathbf{J}_{2,3}) \Rightarrow \mathbf{J}_{2,3}$
Sk9 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,2}\mathbf{J}_{1,2}) \Rightarrow \mathbf{J}_{1,2}$
Sk10 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,3}\mathbf{J}_{2,1}) \Rightarrow \mathbf{J}_{2,2}$
Sk11 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,3}\mathbf{J}_{1,3}) \Rightarrow \mathbf{J}_{1,1}$
Sk12 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,1}\mathbf{J}_{2,1}) \Rightarrow \mathbf{J}_{2,3}$
Sk13 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,1}\mathbf{J}_{1,3}) \Rightarrow \mathbf{J}_{1,2}$

B. Deducible according to AxS2:

Sk14 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,2}\mathbf{J}_{3,2}) \Rightarrow \mathbf{J}_{2,3}$
Sk15 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,2}\mathbf{J}_{3,3}) \Rightarrow \mathbf{J}_{2,1}$
Sk16 $U_3^3(\mathbf{J}_{3,2}\mathbf{J}_{3,2}\mathbf{J}_{3,1}) \Rightarrow \mathbf{J}_{2,2}$

4 *Variance of sets*

4.1 *About the subvariances of the sets* As defined in D1.5, the characteristic of a set to permit its discomposition into several classes is called *variance* of this set. The classes of an n-variance set are n-variant classes. A set of variance n can have any number of subvariances. For example, let a set of variance 3, be denoted M^3, its classes are a_1^3, a_2^3, and a_3^3. The superior index shows the variance of the set, the classes of which are a_1^3, a_2^3, and a_3^3. The set M^3 has subvariance 2 with its classes $a_1^2 = \cup_1^2 a_1^3 a_2^3$ and $a_2^2 = a_3^3$; or $a_1^2 = \cup_1^2 a_1^3 a_3^3$ and $a_2^2 = a_2^3$; or $a_1^2 = \cup_1^2 a_2^3 a_3^3$ and $a_2^2 = a_1^3$. The set M^3 can be discomposed into two bivariant classes in three ways.

A set of the variance 7, M^7, which has classes a_i^7, $i = 1, 2, \ldots, 7$, has:

1. subvariance 2 (in 105 ways) with bivariant classes of the following form:

$$a_1^2 = \cup_1^2(\cup_1^2 a_1^7 a_2^7)(\cup_1^2 a_3^7 a_4^7) \qquad a_2^2 = \cup_1^2 a_5^7(\cup_1^2 a_6^7 a_7^7).$$

2. subvariance 3 (in 70 ways) with trivariant classes of the following form:

$$a_1^3 = \cup_1^3 a_1^7 a_2^7 a_3^7 \qquad a_2^3 = \cup_1^3 a_4^7 a_5^7 a_6^7 \qquad a_3^3 = a_7^7.$$

3. subvariance 4 (in 35 ways) with 4-variant classes of the following form:

$$a_1^4 = \cup_1^4 a_1^7 a_2^7 a_3^7 a_4^7 \qquad a_2^4 = a_5^7 \qquad a_3^4 = a_6^7 \qquad a_4^4 = a_7^7.$$

The number of subvariances of a set with variance n is equal to the number of the divisors of $n - 1$ smaller than $n - 1$. If $p_1, p_2, \ldots, p_k$ are these divisors, the subvariances will be $n_j = 1 + p_j$, where $j = 1, 2, \ldots, k$. A set M of variance 13 will have 5 subvariances, since the divisors of $13 - 1 = 12$, smaller than 12, are $p_1 = 1$; $p_2 = 2$; $p_3 = 3$; $p_4 = 4$; $p_5 = 6$, and the set M will have subvariances 2, 3, 4, 5, and 7. Since each whole number has the divisor 1, each set of any variance will have the subvariance 2.

4.2 *About the complements of a subset* Given the sequence of inclusions $\subseteq \{M_s\}_{1,p}$, we shall consider each M_j as a class of M_{j+1}. If M_k has variance q, then M_{k-1} will have $q - 1$ complements with respect to M_k. If M_{k+1} is a set of variance m, then M_k will have $m - 1$ complements with respect to M_{k+1}. The number of the complements of the set M_{k-1} with respect to M_{k+1} is established according to the following sentence:

"*The variance and subvariance of M_{k+1} with respect to M_{k-1} will be equal with the common subvariances (or common variances) of M_{k+1} and M_k.*"

E.g., taking M_k as a set of variance 5, and M_{k+1} as a set of variance 7, the subvariances and variance of M_k are $r = 2, 3, 5$, and the subvariances and variance of M_{k+1} are $r = 2, 3, 4, 7$. The common subvariances are $r = 2$ and $r = 3$. The set M_{k-1} will have one (when $r = 2$) or two (when $r = 3$) complements with respect to M_{k+1}. If M_{k+1} has classes b_i^7, $i = 1, 2, \ldots, 7$, while one of these classes, b_1^7, has classes a_j^5, $j = 1, 2, \ldots, 5$, then considering $M_{k-1} = a_1^5$, the complement of M_{k-1} with respect to M_{k+1} will be in the case of the common variance $r = 2$:

$$K^1_{M_{k+1}} M_{k-1} = U_1^2(U_1^2(U_1^2 a_2^5 a_3^5)(U_1^2 a_4^5 a_5^5)\,U_1^2((U_1^2 b_2^7 b_3^7)\,U_1^2(U_1^2 b_4^7 b_5^7)(U_1^2 b_6^7 b_7^7))).$$

The complements of M_{k-1} with respect to M_{k+1} in the case of the subvariance $r = 3$ will be of the form:

$$K^1_{M_{k+1}} M_{k-1} = U_1^3(U_1^3 a_2^5 a_3^5 a_4^5)(U_1^3 b_2^7 b_3^7 b_4^7).$$
$$K^2_{M_{k+1}} M_{k-1} = U_1^3 a_5^5(U_1^3 b_5^7 b_6^7 b_7^7).$$

There exist a great number of ways of expressing these complements.

4.3 *About the univariant class* The univariant classes of a set are those classes which have no complements with respect to the extended set of this class nor with respect to the extended sets of this extended set. In other terms:

All the extended sets of this class are equivalent to each other, and each of these sets is equivalent with the given class.

An example of a univariant class is the set of all sets. On a set of variance 1 are defined a single operant U_1^1 and a single symbol of complementarity K^1. Let M be a univariant class, that is to say $n = 1$. According to AxC2 (**1.5**) we have:

$$U_1^1 M = M.$$

According to Ax1 we have:

$$K_M^1 M = M.$$

That is to say: The complement of a univariant set (particular case: the set of all sets) as compared to itself is just this univariant set (the set of all sets).

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