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GENERALIZED EQUIVALENCE AND THE PHRASEOLOGY OF CONFIGURATION THEOREMS

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1 *Introduction* We define the *generalized equivalence of* a finite set $\mathscr{J} = \{E_1, \ldots, E_n\}$ of statements, denoted $[\mathscr{J}]$ or $[E_1, \ldots, E_n]$, to be the conjunction for all $i \leqslant n$ of the implications $\wedge(\mathscr{J} - \{E_i\}) \rightarrow E_i$, where $\wedge\mathscr{T}$ denotes the conjunction of all the elements of set $\mathscr{T}$. The elements of $\mathscr{J}$ are *generalized equivalent* exactly when $[\mathscr{J}]$ holds. This notion was introduced in [3] and was applied there to quasigroup theory. While examples of generalized equivalence are common throughout mathematics, they are rampant in projective geometry—especially in configuration theorems such as those of Pappus and Desargues (see [2]).

For instance, we shall see that one formulation of Pappus' theorem is a (universally-quantified) generalized equivalence of 27 things. Another formulation is a generalized equivalence of nine things, each of which is in turn a generalized equivalence of three things. And yet another is a generalized equivalence of three things, each of which is a generalized equivalence of nine things. We intend to show how this profusion of generalized equivalences results from the distinctive behavior of quantifiers in the framework of configuration theorems, using methods which roughly parallel those of [3].

2 *Productive subsets of configurations* By a *regular configuration* we mean sets of "points" and "lines" with each point (or line) "incident" with exactly three lines (respectively, points) and such that the appropriate automorphism group is transitive (see Chapter 3 of [2]). We may also view a configuration as a set of incidences; that is, as the set of incident point-line pairs.

We shall use $A_1, X_1, A_2, X_2, \ldots$ and $a_1, x_1, a_2, x_2, \ldots$ as variables for points and lines respectively. The statement of the incidence of a point with a line will be denoted by juxtaposing their symbols; thus $A_i a_j$ denotes the incidence of point A_i with line a_j. We shall call a set $\mathscr{J}$ of three or more such incidence statements a *productive set* (with *producing variables* $A_1, A_2, \ldots, a_1, a_2, \ldots$ and *produced variables* $X_1, X_2, \ldots, x_1, x_2, \ldots$) whenever it satisfies all the following:

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P1: Each produced variable occurring in $\mathscr{S}$ occurs exactly three times.

P2 Every two variables occurring in $\mathscr{S}$ are linked by a sequence of variables with each pair of consecutive terms of the sequence appearing in a common element of $\mathscr{S}$.

P3 The cardinality of $\mathscr{S}$ is one more than twice the number of produced variables occurring in $\mathscr{S}$.

As examples of productive sets, consider $\mathscr{S}_1 = \{A_1x_1, A_6x_1, A_7x_1\}$, $\mathscr{S}_2 = \{A_6x_1, A_7x_1, X_1x_1, X_1a_4, X_1a_5\}$, and $\mathscr{S}_3 = \{A_{10}x_{10}, A_{10}x_2, A_2x_2, A_2x_7, A_7x_7, A_7x_{10}, X_3x_7, X_8x_{10}, X_9x_2, X_3a_6, X_8a_8, X_9a_9, X_3x_3, X_8x_3, X_9x_3\}$, where the subscripts used suggest an identification of the incidences with the nonzero entries of the incidence matrix of the Desargues configuration (Figure 1) with rows as points and columns as lines. Instead of actually listing the elements of a productive set, we can merely indicate the rows and columns of an incidence matrix which correspond to the produced variables. For instance, $\mathscr{S}_3$ above can be identified in Figure 1 with rows 3, 8, 9 and columns 2, 3, 7, 10.

1	0	0	1	1	0	0	0	0	0
0	1	0	1	0	0	1	0	0	0
0	0	1	0	0	1	1	0	0	0
0	0	0	1	0	1	0	0	1	0
0	0	0	0	1	0	0	1	1	0
1	0	0	0	0	1	0	1	0	0
1	0	0	0	0	0	1	0	0	1
0	0	1	0	0	0	0	1	0	1
0	1	1	0	0	0	0	0	1	0
0	1	0	0	1	0	0	0	0	1

Figure 1

Suppose $\mathscr{C}$ is a regular configuration and Π is a projective plane. An interpretation of a set of variables as points and lines of Π is called a *$\mathscr{C}$-general interpretation* if and only if, whenever the variables are identified with rows and columns of $\mathscr{C}$'s incidence matrix, the points and lines constructable using meet and join from the interpretations in Π of these variables are distinct whenever corresponding rows or columns of the matrix exist and are different. For instance, suppose $\mathscr{C}$ is the Desargues configuration and the variables A_1, A_2, a_1 are identified (without loss of generality since $\mathscr{C}$ is regular) with the first two rows and the first column of Figure 1. Then $\mathscr{C}$-general implies the distinctness of $A_1 \vee A_2$ and a_1 (since they correspond to different columns) and similarly that $A_1 \vee A_2 \wedge a_1 \neq A_2$ and that $A_1 \neq A_2$, where $\wedge$ and $\vee$ denote meet and join of the corresponding lines and points.

Let $\forall[\mathscr{S}]$ and $\exists\bigwedge\mathscr{S}$ be, respectively, the universal closure of $[\mathscr{S}]$ and the existential closure of $\wedge\mathcal{J}$, with the quantifications being over all the variables which have not been interpreted as fixed elements of a projective plane.

Lemma *Consider a productive set $\mathcal{J}$ as a subset of the incidences of a regular configuration $\mathcal{C}$ and suppose $\mathcal{J}$'s producing variables have been given a $\mathcal{C}$-general interpretation in a projective plane* Π. *Then*:

L1 *For each maximal proper subset $\mathcal{J}^*$ of $\mathcal{J}$, the produced variables of $\mathcal{J}$ can be assigned uniquely in* Π *so as to satisfy* $\wedge\mathcal{J}^*$.

L2 Π *satisfies* $\forall[\mathcal{J}]$ *if and only if* Π *satisfies* $\exists\wedge\mathcal{J}$.

To prove L1, let $v(i)$ be the number of produced variables which appear in incidences of $\mathcal{J}$ with exactly i producing variables, for $0 \leqslant i \leqslant 3$. If $v(3) > 0$, then P1 and P2 imply that $\mathcal{J}$ contains only one produced and three producing variables, and L1 follows immediately. If $v(3) = 0$, it is easy to see that the number of incidences in $\mathcal{J}$ is $\frac{1}{2}[5v(2) + 4v(1) + 3v(0)]$. But by P3, this is also equal to $1 + 2[v(2) + v(1) + v(0)]$, and equating these expressions gives that $v(2) = 2 + v(0)$. Thus each such productive set has $v(2) \geqslant 2$. Let $\mathcal{J}^*$ be any maximal proper subset of $\mathcal{J}$. Then $\mathcal{J}^*$ involves some produced variable appearing in $\mathcal{J}^*$ with exactly two producing variables. If say $\{A_i x_k, A_j x_k\} \subseteq \mathcal{J}^*$, we can introduce a variable a_k (not previously occurring in $\mathcal{J}$) interpreted as $A_i \vee A_j$. Let $\mathcal{J}_1 = \mathcal{J} - \{A_i x_k, A_j x_k\}$ with the remaining occurrence of x_k replaced by a_k. Then $\mathcal{J}_1$ will be a productive set with one fewer produced variable and two fewer incidences than $\mathcal{J}$. Also, the interpretation of the producing variables of $\mathcal{J}_1$ are $\mathcal{C}$-general. By P3, continuing in this fashion will eventually assign unique points and lines to all of $\mathcal{J}$'s variables so as to satisify $\wedge\mathcal{J}^*$.

Clause L2 of the Lemma now follows from L1 by exactly the same argument used to prove the corresponding clause of the Lemma in [3].

Theorem *If $\mathcal{J}_1, \ldots, \mathcal{J}_k$ are productive sets which correspond to a partition of the incidences of a regular configuration $\mathcal{C}$ and* Π *is any projective plane, then the following are equivalent:*

T1 *For each $\mathcal{C}$-general* Π*-interpretation of the producing variables of* $\mathcal{J}_1, \ldots, \mathcal{J}_k$, $[\forall[\mathcal{J}_1], \ldots, \forall[\mathcal{J}_k]]$.

T2 *For each $\mathcal{C}$-general* Π*-interpretation of the producing variables of* $\mathcal{J}_1, \ldots, \mathcal{J}_k$, $\forall[\mathcal{J}_1 \cup \ldots \cup \mathcal{J}_k]$.

The Theorem follows from the Lemma by exactly the same argument used to prove the corresponding Theorem in [3].

3 *Applications to configuration theorems* Using the Theorem, any partition of the incidences of a regular configuration into productive sets produces a paraphrase of the corresponding configuration theorem. While we shall restrict our applications to the theorems of Pappus and Desargues and to those paraphrases resulting from the simplest partitions, any such partition of any suitable configuration of [1] or [2] could be used as well.

The most primitive formulation of Pappus' theorem asserts that, for every nine points and nine lines (usually assumed to satisfy some sort of generality condition), if any 26 of the appropriate set of 27 incidences occur, then so must the 27th.

The more traditional formulation essentially asserts that for every nine (suitably general) points, if eight particular triples of the points are collinear triples, then a ninth triple (the "Pappus line") must be collinear. Moreover, because of the regularity of the Pappus configuration, the collinearity of any eight of the nine triples implies the collinearity of the ninth. Thus Pappus' theorem is expressible as a universally-quantified generalized equivalence of nine statements. And each of the nine is itself a universally-quantified generalized equivalence, since three distinct points are collinear if and only if each line through two of them passes through the third.

The first formulation above corresponds to T2 of the Theorem, and the second corresponds to T1 where $\mathscr{A}_i$ (for $i \leqslant 9$) is identified with the ith column of the incidence matrix (Figure 2). The equivalence of the formulations follows from the Theorem after noting that the $\mathcal{O}$-general assumption agrees with the customary "general case" assumption of Pappus' theorem.

1	0	0	1	0	1	0	0	0
0	1	0	1	0	0	0	1	0
0	0	1	0	0	0	0	1	1
0	0	0	1	0	0	1	0	1
0	1	0	0	1	0	1	0	0
0	1	1	0	0	1	0	0	0
1	0	1	0	0	0	1	0	0
1	0	0	0	1	0	0	1	0
0	0	0	0	1	1	0	0	1

Figure 2

A more interesting partition of the Pappus configuration has $\mathscr{A}_1$ identified with row 1 and columns 1, 4, 6, $\mathscr{A}_2$ with row 3 and columns 3, 8, 9, and $\mathscr{A}_3$ with row 5 and columns 2, 5, 7. The generalized equivalence of T1 is then

$$\begin{bmatrix} \forall X_1 \forall x_1, x_4, x_6 [A_7 x_1, A_8 x_1, X_1 x_1, A_2 x_4, A_4 x_4, X_1 x_4, A_6 x_6, A_9 x_6, X_1 x_6], \\ \forall X_3 \forall x_3, x_8, x_9 [A_6 x_3, A_7 x_3, X_3 x_3, A_2 x_8, A_8 x_8, X_3 x_8, A_4 x_9, A_9 x_9, X_3 x_9], \\ \forall X_5 \forall x_2, x_5, x_7 [A_2 x_2, A_6 x_2, X_5 x_2, A_8 x_5, A_9 x_5, X_5 x_5, A_4 x_7, A_7 x_7, X_5 x_7] \end{bmatrix} .$$

Corollary 1 *Pappus' theorem is equivalent to the following: For all distinct points* $A_2, A_4, A_6, A_7, A_8, A_9$, *the following are generalized equivalent:*

C1 *The lines* $A_7 \vee A_8$, $A_2 \vee A_4$, $A_6 \vee A_9$ *are concurrent*

C2 *The lines* $A_6 \vee A_7$, $A_2 \vee A_8$, $A_4 \vee A_9$ *are concurrent*

C3 *The lines* $A_2 \vee A_6$, $A_8 \vee A_9$, $A_4 \vee A_7$ *are concurrent,*

whenever, in each of the concurrences, the three lines are distinct.

This is merely the application of the Theorem to the above partition, after translating $\mathcal{O}$-general into more colloquial language (see Figure 3). Note that C1, C2, C3 are generalized equivalences since three distinct lines are concurrent if and only if each point on two is also on the third. Corollary 1 corresponds to the well-known restatement of Pappus' theorem: Any two triangles ($A_2A_7A_9$ and $A_4A_6A_8$ in our notation) which are doubly perspective must be triply perspective.

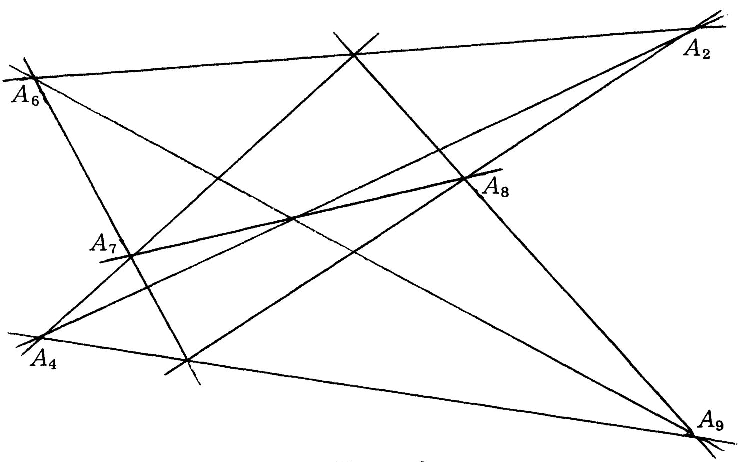

Figure 3

Turning now to Desargues' theorem, a primitive version can be expressed as a universally-quantified generalized equivalence of thirty incidences. Or, partitioning the incidence matrix into its ten rows or columns, as a universally-quantified generalized equivalence of ten universally-quantified generalized equivalences of three incidences each.

The traditional statement—that two triangles are perspective in a point if and only if they are perspective in a line—corresponds to partitioning the incidence matrix (Figure 1) into the productive sets $\mathscr{I}_1$ identified with row 1 and columns 1, 4, 5 and $\mathscr{I}_2$ with rows 3, 8, 9 and columns 2, 3, 6, 7, 8, 9, 10. Thus, Desargues' theorem can be expressed as a universally-quantified equivalence of two universally-quantified generalized equivalences of 9 and 21 incidences.

Inspection of the incidence matrix shows several other partitions into productive sets. For instance, $\mathscr{I}_1$ identified with rows 1, 4, 5, 6 and columns 1, 4, 5, and $\mathscr{I}_2$ with rows 3, 8, 9 and columns 2, 3, 7, 10. This again expresses Desargues' theorém as a universally-quantified equivalence of two universally-quantified generalized equivalences, but now each has 15 incidences. Stated more conventionally (see Figure 4):

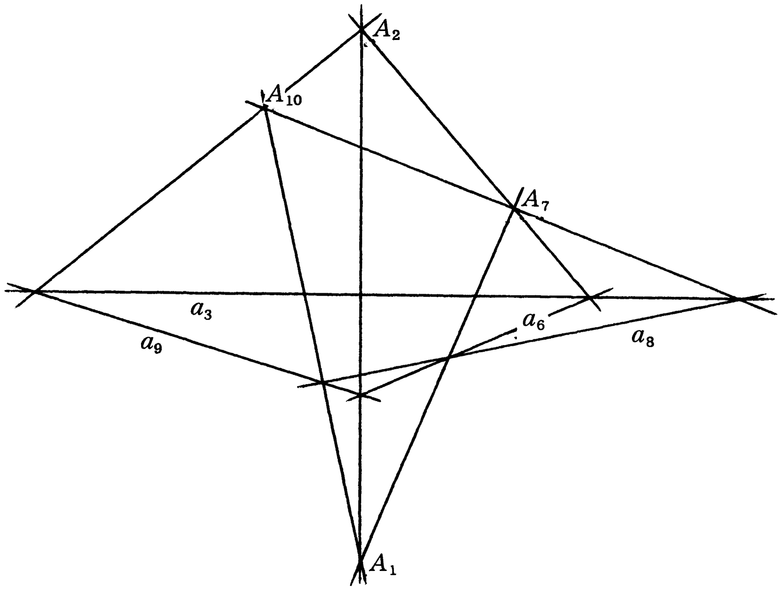

Figure 4

Corollary 2 *Desargues' theorem is equivalent to the following: For all $\mathcal{O}$-general points A_2, A_7, A_{10} and lines a_6, a_8, a_9, the following are equivalent:*

Points $A_2 \vee A_7 \wedge a_6$, $A_2 \vee A_{10} \wedge a_9$, $A_7 \vee A_{10} \wedge a_8$ are collinear

Lines $a_6 \wedge a_8 \vee A_7$, $a_6 \wedge a_9 \vee A_2$, $a_8 \wedge a_9 \vee A_{10}$ are concurrent.

Another variation is given by the partition with $\mathscr{A}_1$ identified with row 6 and column 1, $\mathscr{A}_2$ with row 9 and column 2, $\mathscr{A}_3$ with row 4 and column 4, $\mathscr{A}_4$ with row 5 and column 5, $\mathscr{A}_5$ with row 3 and column 7, and $\mathscr{A}_6$ with row 8 and column 10. Desargues' theorem now becomes a universally-quantified generalized equivalence of six universally-quantified generalized equivalences, each in the form of Menger's "umlaut" relation of [4]. Stated more conventionally (see Figure 4):

Corollary 3 *Desargues' theorem is equivalent to the following: For all $\mathcal{O}$-general points A_1, A_2, A_7, A_{10} and lines a_3, a_6, a_8, a_9, the following are equivalent:*

Line $A_2 \vee A_{10}$ is incident with point $a_3 \wedge a_9$.
Line $A_7 \vee A_{10}$ is incident with point $a_3 \wedge a_8$.
Line $A_2 \vee A_7$ is incident with point $a_3 \wedge a_6$.
Line $A_1 \vee A_2$ is incident with point $a_6 \wedge a_9$.
Line $A_1 \vee A_7$ is incident with point $a_6 \wedge a_8$.
Line $A_1 \vee A_{10}$ is incident with point $a_8 \wedge a_9$.

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