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A MORE RELEVANT RELEVANCE LOGIC

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Relevant implication or entailment is designed to convey the notion of "logical consequence". While $A \supset B$, where $\supset$ is the weak implication usually specified by a truth table, tells us that either A is false or B is is true, $A \rightarrow B$, where $\rightarrow$ is entailment, tells us that A is actually used, and perhaps necessary, in the proof of B. In this paper we show that, in a certain sense, the entailment systems of Anderson and Belnap ([1]) are still not fully relevant and we describe a new system which is at least more so.

The simplest notion of relevance can be expressed in terms of the following deduction theorem:

If there is a proof of B using all of $A_1, \ldots, A_n$,
then $\vdash A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_n \rightarrow B$.

(The A_is, B, and all capital Roman letters used below range over (well formed) formulas as in [1]).

This deduction theorem together with modus ponens or $\rightarrow$E ($\rightarrow$elimination) is equivalent to the system $\mathbf{R}_{\rightarrow}$ of [1]. This system can be axiomatized as follows:

$R_{\rightarrow}1$ $\vdash A \rightarrow A$
$R_{\rightarrow}2$ $\vdash A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$
$R_{\rightarrow}3$ $\vdash (A \rightarrow . B \rightarrow C) \rightarrow . B \rightarrow . A \rightarrow C$
$R_{\rightarrow}4'$ $\vdash (A \rightarrow . B \rightarrow C) \rightarrow . A \rightarrow B \rightarrow . A \rightarrow C$[1]

The instance

$$\vdash A \rightarrow A \rightarrow . A \rightarrow A \tag{1}$$

of $R_{\rightarrow}1$ and $R_{\rightarrow}2$ lead directly to

$$\vdash A \rightarrow . (A \rightarrow A) \rightarrow A. \tag{2}$$

within which the $A \rightarrow A$ still seems to be irrelevant. Anderson and Belnap also claim (2) to be irrelevant, but perhaps for other reasons. They

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develop new systems $\mathbf{E}_{\rightarrow}$, $\mathbf{T}_{\rightarrow}$, and others which involve other connectives, quantifiers and modal operators, in which (2) is not provable, but which retain $R_{\rightarrow}1$ and hence (1). In (1), however, the first $A \rightarrow A$ seems just as irrelevant to the proof of the final A as does the $A \rightarrow A$ in (2). There are in fact instances of all the axiom schemes of $\mathbf{E}_{\rightarrow}$ (and probably all those of the other systems in [1]) which are irrelevant in this sense, for example:

$$\vdash (A \rightarrow . B \rightarrow A) \rightarrow . (A \rightarrow B) \rightarrow (A \rightarrow A)$$

is an instance of Axiom Scheme $E_{\rightarrow}4$ of $\mathbf{E}_{\rightarrow}$.

To help us to eliminate such formulas as relevant theorems we introduce the following definition:

D.T *If Δ is a (possibly empty) set of formulas and the formula A_n, which is not of the form $B \rightarrow C$, can be deduced using* $E\rightarrow$ *from $\Delta \cup \{A_1, \ldots, A_{n-1}\}$ and not from a proper subset of this set, then $\Delta \vdash A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_n$ is a relevant theorem.*

Thus $\vdash p \rightarrow p$ is relevant, but as was shown above not all substitution instances of this (such as (1)) are. We therefore require axiom schemes that are strongly restricted. The axioms we choose are in fact all the relevant (in the sense of the above definition) instances of $R_{\rightarrow}1$-$R_{\rightarrow}4'$. Taking for example $R_{\rightarrow}1$ with $A = A_1 \rightarrow \ldots \rightarrow A_n$, where A_n is not of the form $B \rightarrow C$, it is clear that the number of ways in which $A, A_1, \ldots, A_{n-1}$ can be combined by $E\rightarrow$ is finite. We can therefore always decide effectively whether an instance of $R_{\rightarrow}1$ is relevant. Similarly we can effectively decide whether instances of $R_{\rightarrow}2$, $R_{\rightarrow}3$, and $R_{\rightarrow}4'$ are relevant.

Instead of using $R_{\rightarrow}1$-$R_{\rightarrow}4'$ we could also state the corresponding axioms:

A1 $\vdash p \rightarrow p$ etc.

and add a substitution rule allowing only substitutions that lead to relevant theorems. In what follows we will refer to the relevant instances of $R_{\rightarrow}1$-$R_{\rightarrow}4'$ as instances of A1-A4 respectively.

It is obvious that A1-A4 can be derived from our new deduction theorem **D.T** and $E\rightarrow$, we now show that **D.T** can be proved from A1-A4 and $E\rightarrow$.

Proof of **D.T**. We prove by induction that each step

$$\Delta, A_i \vdash B_{i,t}{}^{2} \tag{1}$$

in a proof of

$$A_1, \ldots, A_i \vdash B_i \tag{2}$$

where $B_i = A_{i+1} \rightarrow \ldots \rightarrow A_n \rightarrow B$, $1 \leq i \leq n$ and $\Delta \subseteq \{A_1, \ldots, A_{i-1}\}$, can be replaced by

$$\Delta \vdash A_i \rightarrow B_{i,t}. \tag{3}$$

Clearly such a $B_{i,t}$ cannot be an instance of an axiom scheme as then A_i would not be used in its proof. If $B_{i,t} = A_i$, Δ is empty and as A_i must be relevant in the proof of $B_{i,t}$,

$$\vdash A_i \rightarrow A_i$$

is an instance of A1. Thus (3) holds in this case.

If (1) was derived in the proof of (2) by E$\rightarrow$, we must have had, as a pair of previous steps:

$$\Delta_1, A_i \vdash C \rightarrow B_{i,t} \tag{4}$$

and

$$\Delta_2, A_i \vdash C, \tag{5}$$

(5) and

$$\Delta_1 \vdash C \rightarrow B_{i,t} \tag{6}$$

or (4) and

$$\Delta_2 \vdash C \tag{7}$$

for some C, where $\Delta_1 \cup \Delta_2 = \Delta$.

We can assume that by an inductive step we have for (4) and (5)

$$\Delta_1 \vdash A_i \rightarrow .\ C \rightarrow B_{i,t} \tag{8}$$

and

$$\Delta_2 \vdash A_i \rightarrow C. \tag{9}$$

If we had (4) and (5) in the proof of (2) we will have (8) and (9) in a new proof.

$$\vdash (A_i \rightarrow .\ C \rightarrow B_{i,t}) \rightarrow .\ (A_i \rightarrow C) \rightarrow (A_i \rightarrow B_{i,t})$$

will under the relevance restrictions on (4) and (5), be a relevant instance of A2, so by E$\rightarrow$ we can using (8) and (9), conclude (3).

Similarly if we had (5) and (6) in the proof of (2) we can use (6), (9) and A3 to deduce (3) and if we had (4) and (7) in the proof of (2) we can use (7), (8), and A4 to deduce (3).

Thus (3) can be proved in all cases and we have in particular

$$A_1, \ldots, A_{i-1} \vdash A_i \rightarrow B_i \text{ for } 1 < i \leqslant n$$

so that we have

$$\vdash A_1 \rightarrow \ldots \rightarrow A_n \rightarrow B.$$

NOTES

1. This is the formulation of $R'_{\rightarrow}$ of [1]. $R_{\rightarrow}$ has $R_{\rightarrow}4'$ replaced by

$$R_{\rightarrow}4 \vdash (A \rightarrow . A \rightarrow B) \rightarrow \quad A \rightarrow B$$

but the systems are equivalent.

2. Every step such as this is of course, a relevant theorem.

REFERENCE

[1] Anderson, A. R. and N. D. Belnap, Jr., *Entailment*, vol. I, Princeton University Press, Princeton (1975).

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