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ON THE NECESSITY OF S4

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In [4] I pointed out that Lewis' verbal definitions of necessity and impossibility in [2], pp. 248-249 constitute an essential part of his famous "Independent" proofs. For ease of reference I quote Lewis' words again:

> To say 'p is necessary' means 'p is implied by its own denial' or 'the denial of p is not self-consistent' . . . . To say 'p is impossible' means 'p implies its own denial' or 'p is not self-consistent'. Necessary truths so defined coincide with the class of tautologies or truths which can be certified by logic alone; and impossible propositions coincide with the class of those which deny some tautology.
>
> Every tautology is expressible as some proposition of the general form $p \vee -p$. . . . The negation of any proposition of the form $p \vee -p$ is a corresponding proposition of the form $p \,.\, -p$.

I extracted the following symbolic definition of impossibility from this passage:

(i) $-\Diamond p =_{df} [p = (r \,\&\, -r)]$, see [4], p. 545.

By substituting $-p$ for p we obtain the corresponding definition for *necessity*:

(ii) $-\Diamond -p =_{df} [-p = (r \,\&\, -r)]$

If in (i) we negate both sides of the definitional equality and apply *Double Negation* (strong version) and *Substitution*, we have

(iii) $\Diamond p =_{df} -[p = (r \,\&\, -r)]$

It is obviously feasible to treat (iii) as primitive and the others as derivative. The sign '=' is used here in the *strict* sense of Lewis in which

$$p = q =_{df} [(p \prec q) \,\&\, (q \prec p)]$$

I wish in this paper to prove that if (iii) is added to Lewis' system S1, then, using a certain strong but very plausible version of the principle of the substitutivity of strict equivalents to be discussed below, we can obtain S4. (See [2], pp. 123-153, where S1 is established and developed). In the impending exercise, I write '*M*' for "it is possible that" and '*L*' for "it is

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necessary that". Thus (iii) becomes $Mp =_{df} .-[p = (r \& -r)]$ and (ii) becomes $-M-p =_{df} [-p = (r \& -r)]$. We shall, as is customary, abbreviate '$-M-$' as 'L', so that the two expressions are interchangeable. I shall indicate the use of this convention in proof work by writing "Modality Interchange" against the appropriate line. I should point out that propositional variables are used throughout as metalogical variables.

We first establish the following simple lemma: $-(p = p) = (r \& -r)$. [Call it Lemma 0].

(a) $-(p = p) \prec (r \& -r)$

→ 1. $-(p = p)$	
2. $p = p$	Thesis 12.11
3. $(p = p) \vee (r \& -r)$	2 Addition
4. $r \& -r$	1, 3, Disjunctive Syllogism.
5. $-(p = p) \prec (r \& -r)$.	1-4 **C.P.**

(b) $(r \& -r) \prec -(p = p)$.

The Lewis-type proof of this is obvious and is left to the reader.

From (a) and (b) and the definition of '=' we obtain $-(p = p) = (r \& -r)$. The annotation "Thesis 12.11" on the right hand side of line 2 in the proof of (a) above indicates that the assertion made at that line is a thesis in the system S1, proved as a theorem with that number in [2]. Note that in the above proof no principle is used that is not contained in S1. *Addition* viz., $p \prec (p \vee q)$, is Theorem 13.2 in [2], p. 135. Its use as a rule here is merely for economy; one can always introduce a suitable form of it as an asserted thesis and use *modus ponens* to obtain the disjunction sought after. The law of disjunctive syllogism is also deducible in S1. Further, the strict conditionalisation at line 5 is within the capabilities of S1. One can obtain the same result by first conditionalising to a material implication, namely, $-(p = p) \supset (r \& -r)$ which, since it would be logically true, being the conclusion of a *categorical* argument, would yield $-(p = p) \prec (r \& -r)$ on Lewis' own showing that strict implication is identical with a logically true material implication. The same remarks are true of the omitted proof of $(r \& -r) \prec -(p = p)$, *mutatis mutandis*.

We now deduce $Lp \prec LLp$, the characteristic thesis of S4:

→ 1. Lp	
→ 2. $-LLp$	
3. $-p = (r \& -r)$	1, Definition of L
4. $--M- -M-p$	2, Modality Interchange.
5. $MM-p$	4, Double Negation.
6. $MM(r \& -r)$	3, 5, Strong Substitution.
7. $M-[(r \& -r) = (r \& -r)]$	6, Definition of M.
8. $M(r \& -r)$	7, Lemma 0; Substitution.
9. $-[(r \& -r) = (r \& -r)]$	8, Definition of M.

10. $r \& -r$	9, Lemma 0.
11. $-LLp \supset (r \& -r)$	2-10, **C.P.**
12. $- -LLp$	11, RAA.
13. LLp	12, Double Negation.
14. $Lp \prec LLp$	1-13, **C.P.**

Except for the strong rule of substitution and the definition of possibility with the associated abbreviative convention on the interchange of modalities, this proof uses only principles available in S1. At line 10 Lemma 0 has been used as a rule rather than as a thesis merely for the sake of economy. Regarding the strict conditionalisation at line 13, exactly the same remarks as were made in connection with the strict conditionalisation at line 5 in the proof of Lemma 0 apply.

It remains to justify our use of the strong rule of substitution. The first of Lewis' two rules of substitution for S1 is stated by him as follows: "Either of two equivalent expressions may be substituted for each other", [2], p. 125). This statement might lead one to suppose that Lewis is offering a rule of the following form:

I *If* $p = q$, *then* $A(p) = A(q/p)$

(Here, as below, $A(q/p)$ is like $A(p)$ except for containing q in place of p in all or some of the occurrences of p in A).

However, Lewis continues as follows: "Thus if an expression of the form $p = q$ has been assumed, or subsequently established, what precedes the sign of equivalence may be substituted for what follows it; or vice versa" (*loc. cit*). This suggests that what Lewis has in mind, at any rate, in the actual development of his system which is of the axiomatic type, is a rule that may be formulated as follows:

II *If* $\vdash p = q$, *then* $\vdash A(p) = A(q/p)$

This is the rule that I have used at line 8 in the proof of $Lp \prec LLp$ above. It is the rule that is standardly used in the formulation of S1 and of the Lewis systems, generally. What I now wish to show is that the first, stronger, rule, which I have used at line 6, is very plausible and intuitively acceptable. The following argument is very informal. In any model system of classical propositional logic a metatheorem of the substitutivity of material equivalents will be available in the following form:

III *If* $p \equiv q$, *then* $A(p) \equiv A(q/p)$

Now, it is of course permissible to define '$p = q$' as Lt$(p \equiv q)$ where "Lt" means "Logically true in the Propositional Calculus (**P.C.**)". Suppose that '$p \equiv q$' is logically true in **P.C.**, then $A(p) \equiv A(q/p)$ must also be logically true, since p and q, the only parts in which $A(p)$ and $A(q/p)$ differ, are, by hypothesis, logically true. That is to say, we have

IV *If* $\mathrm{Lt}(p \equiv q)$, *then* $\mathrm{Lt}[A(p) \equiv A(q/p)]$

which, in virtue of the definition of strict implication in terms of material implication, yields

V *If* $(p = q)$, *then* $A(p) = A(q/p)$

One might point out also that Lewis' four-valued matrices for S1 (group 5) show that $p = q$ has a designated value only when p and q have the same value. It follows that given $p = q$, substitution of q for p in A must preserve the original value of A.

Since, as indicated by Lewis, ([2], pp. 500-501) S1 plus $Lp \prec LLp$ yields S4, the deduction of $Lp \prec LLp$ given above establishes that the addition of the definition $Mp =_{df} -[p = (r \& -r)]$ and the strong rule of the substitutivity of strict equivalents just discussed yields a system of at least the strength of S4. It should be noted that in view of the fact that $-Mp = [p = (r \& -r)]$, which is equivalent to $Mp = -[p = (r \& -r)]$, is contained in S2 as a thesis (with the number 19.89; see [2], p. 506), the present result shows that (at least) S4 becomes available by the mere strengthening of the rule for the substitution of strict equivalents.

The question naturally arises whether the system obtained by adding $Mp =_{df} -[p = r \& -r]$ is strong enough to yield $Mp \prec LMp$, the characteristic thesis of S5. The following considerations suggest a negative answer: If this formula were provable, one would expect the conjunction of Mp and $-LMp$ to lead to a contradiction. In fact, however, they are consistent. By definition and our interchangeability convention $-LMp$ reduces to $--M-Mp$ which by *Double Negation* equals $M-Mp$. Hence Mp & $-LMp$ becomes Mp & $M-Mp$. In terms of the definition of possibility this amounts to $-[p = (r \& -r)]$ & $M[p = (r \& -r)]$. This latter merely both envisages the falsity of a certain proposition (viz. $p = (r \& -r)$) and its logical possibility; which is surely non-contradictory.

It is important to remark that this unavailability of S5 is relative to the adoption of a strictly *logical* interpretation of the modalities, that is to say, an interpretation in which necessity is construed as *logical* necessity and possibility as logical possibility. I have pointed out in [5] that when necessity is construed as *conceptual* necessity and possibility, correspondingly, as conceptual possibility, S5 is easily available.

In the last mentioned paper the suggestion is made (without proof) that if the definition $Lp =_{df} [-p = (r \& -r)]$ is added to a model system of propositional calculus, then a system of at least the strength of S4 can be obtained by reasoning that employs meta-logical methods of a rather simple sort. I will substantiate this conjecture below as a way of establishing the logical inevitability of S4 alternative to the one already given in this article.

Gödel, in [1], was the first to construct a modal logic by adding modal postulates to the primitive basis of a classical propositional calculus. The system he obtained was equivalent to S4. In our present symbolism Gödel's modal principles may be formulated as follows:

I $Lp \supset p$
II $Lp \supset [L(p \supset q) \supset Lq]$
III $Lp \supset LLp$
IV *From* $\vdash p$ *to infer* Lp

To show that the addition of $Lp =_{df} [-p = (r \& -r)]$ (the definition of necessity which corresponds to $Mp =_{df} -[p = (r \& -r)]$) to a suitable system of propositional logic yields at least S4 it suffices to establish Gödel's modal principles as theorems in the new system.

Theorem I $Lp \supset p$

Proof:		
	1. Lp	
	2. $-p = (r \& -r)$	1, Definition of Necessity.
	3. $p = -(r \& -r)$	By negating both sides of (3) and applying Double Negation
	4. $-(r \& -r)$	P.C. Thesis
	5. p	3, 4 Affirmation of an equivalent.
	6. $Lp \supset p$	1-5, C.P.

In the present context we understand $p = q$ as an abbreviation for "$p \equiv q$ is logically true (i.e., tautological)". In virtue of line 3 of this derivation Lp, as defined, may be taken to mean "p is equivalent to a tautology"; which, again, is in conformity with Lewis [2], pp. 248-249 (i.e., passage quoted at the opening of this paper).

Theorem II $Lp \supset [L(p \supset q) \supset Lq]$

This formula is now easily seen to be logically equivalent to the familiar metatheorem that the property of being tautological is hereditary with respect to the principle of *modus ponens*. I omit the repetition of its proof here.

Theorem III $Lp \supset LLp$

The proof of this is essentially the same as the one already given for $Lp \dashv LLp$ on page 690 of this paper except that here the conclusion is a material implication instead of a strict implication. (We carry over from the earlier system the definition of possibility (now as a derivative concept) together with the abbreviative convention on the interchange of modalities.)

IV The rule: "*From* $\vdash p$ *to infer* Lp"

is directly derivable from the fact that our base system is tautological. That is to say, if p is a theorem, then p is tautological and hence necessary, on the above showing.

This way of obtaining S4 has the merit of showing that a modal logic predicated upon a logical interpretation of the modalities is really metalogical. Quine has often stressed the meta-logical character of (at any rate, propositional) modal logic, for example in [3]. It seems to me that the preceding derivation confirms his point. It should be remarked, however, that the point holds only when a modal system is based on a logical interpretation of the modalities. Where a broader interpretation is in question, in the expression '*Lp*', *p* may be logically simple. And such propositions would call for a primary, rather than a metalogical, systematisation.

A point of interest with respect to the foregoing methods of deriving S4 is that they show clearly that by Lewis' own logical interpretation of the modalities he was committed to S4. Lewis himself was to the last cautious of any system stronger than S2. Our result shows that he could have been more adventurous modally with perfect logicality.

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