Notre Dame Journal of Formal Logic
Volume XX, Number 3, July 1979
NDJFAM

A MATRIX DECISION PROCEDURE FOR THREE MODAL LOGICS

ADRIAN LARNER

The decision procedure described below was reached by an attempt to define a 'decidability' operator, 'U', that would clarify the interpretation of nested modalities. '$U\alpha$' was intended to mean:

(the truth value of) α is (non-referentially) decidable.

I.e. We need not know what proposition (statement) α is in order to know whether it is true or false. Note that I use metalogical variables such as 'α' as abbreviations for (not names of) logical formulae, but I use logical formulae such as 'p' as names of propositions; I therefore treat operators such as 'K' as generators of names of (complex) propositions from names of propositions.

The 'necessity' operator, 'L' is defined thus:

(Def L) $L\alpha =_{df} K\alpha U\alpha$.

If 'L' is taken as primitive, 'U' may be defined thus:

(Def U) $U\alpha =_{df} AL\alpha\ LN\alpha$.

The system **U11** **U11** is intended to imitate the propositional calculus (**PC**) in regarding as decidable just what is formally decidable within the system itself, the concept of decidability being imported from the metalogic of **PC** into the logic of **U11**.

For any wff, f, we may generate a matrix (truth table) having c columns and r lines, where f has c letters including n distinct variables and $r = 2^n$. Each line contains a distinct assignment of truth values to the variables (0-false, or 1-true). We may express the internal relations of the propositions named by the variables of f as a vector, **v**, of length r, containing the values 0 and 1, the k'th element of **v** indicating that the internal relations of the propositions do (1) or do not (0) permit the assignment of truth values in the k'th line of the matrix. E.g. for some wff '$\ldots p \ldots q$', if Epq is true then of the lines:

Received February 20, 1978

(1) . . . 0 . . . 0
(2) . . . 0 . . . 1
(3) . . . 1 . . . 0
(4) . . . 1 . . . 1,

(1) and (4) are permitted, (2) and (3) precluded, giving a vector: 1 0 0 1. Clearly, we may generate 2^r distinct vectors, of which we may ignore one (the all zero vector). For each of these vectors we process the matrix: we first eliminate the lines not permitted by the vector, **v**; we then apply the operator rules (working from right to left along the wff)—those of **PC** for the non-modal operators, and for the modal operators as follows:

(U1) For '$U\alpha$', if the α-column is uniform (all ones or all zeroes) set the U-column to all ones, otherwise to all zeroes.

(L1) For '$L\alpha$', if the α-column is all ones, set the L-column to all ones, otherwise to all zeroes.

(M1) For '$M\alpha$', if the α-column is all zeroes, set the M-column to all zeroes, otherwise to all ones.

A thesis of the system is just a wff of which the matrix has a leftmost column of all ones for each **v**. Note that 'UUp' is a thesis—intuitively, we can always decide whether or not a proposition is decidable. **U11** is just S5. If we consider the matrices of a wff having just one variable, the possible values in any column are:

0 (or 0), 0, 1, and 1 (or 1);
0 1 0 1

calling these 4, 3, 2, and 1 respectively, we may generate characteristic matrices for single variable wffs thus:

(11)

α	$N\alpha$	$K\alpha$	1	2	3	4	$L\alpha$
1	4		1	2	3	4	1
2	3		2	2	4	4	4
3	2		3	4	3	4	4
4	1		4	4	4	4	4

which are just the Lewis Group III matrices. In a similar manner, characteristic matrices for wffs of 2, 3, . . . variables could also be generated.

The system **U01** **U01** is designed for logics sufficiently complex to be completable but not decidable. Its decision procedure is like that of **U11** except for the interpretation of the value 0 in **v**. A 1 in **v** indicates that the line of values is permitted in the sense that some interpretation of the variables of the wff can be found to allow those values, a 0 indicates that no such interpretation can be found—but, as there is no consistency proving procedure, such an interpretation might exist: we do not eliminate these lines. The rules for the modal operators are as follows:

(U0) For '$U\alpha$', take the α-subcolumn, i.e., those elements of the α-column in a permitted (1 in **v**) line, and apply (U1) to it: if the result is zeroes then set the U-column to all zeroes, otherwise set the U-subcolumn to ones and the remaining entries in the U-column thus:

if the α-entry is the same as those in the α-subcolumn set the U-entry to 1, otherwise set the U-entry to 0.

(L0) For '$L\alpha$', if the α-subcolumn is all ones, set the L-column to the same values, line by line, as the α-column, otherwise to all zeroes.

(M0) For '$M\alpha$', if the α-subcolumn is all zeroes, set the M-column to the same values, line by line, as the α-column, otherwise to all ones.

Note that 'UUp' is not a thesis, but '$CUpUUp$' is a thesis—intuitively, if a proposition is decidable then we can tell that it is decidable. I believe that **U01** is Sobociński's S4.4, *cf.* [3]. If we consider the matrices of a wff having just one variable, as above, for **v** = 1 1 the characteristic matrices are as for **U11**, for **v** = 1 0, the characteristic matrix for 'L' is:

(10)

α	$L\alpha$
1	1
2	2
3	4
4	4

and for **v** = 0 1:

(01)

α	$L\alpha$
1	1
2	4
3	3
4	4

the latter being in Lewis Group II.

The system **U00** **U00** is designed for non-completable systems. Its decision procedure is like that of **U01**, except that the all zeroes vector is also used: under this vector 'U' (and so 'L') always has an all zeroes column, 'M' always has an all ones column. I know of no axiomatization of **U00**. Its theses include all those of the propositional calculus and the axioms of Sobociński's S4.1.1: those of S4—*CLpp, CLCpqCLpLq, CLpLLp*, and (M1) *CLCLCpLpLpCMLpLp* (Sobociński uses the strict version—*LCLCLCpLpLpCMLpLp*—but of course no thesis of **U00** begins with 'L'). Its additional characteristic matrix for 'L' is

(00)

α	$L\alpha$
1	4
2	4
3	4
4	4

Nested modalities in **U11**, **U01**, *and* **U00** From the four characteristic matrices for 'L' we may calculate the distinct nested modalities in these systems:

	α	$L\alpha$	$M\alpha$	$LM\alpha$	$ML\alpha$	$LML\alpha$	$MLM\alpha$
(11)	1	1	1	as M	as L	as L	as M
	2	4	1				
	3	4	1				
	4	4	4				
(10)	1	1	1	1	as LM	as LM	as LM
	2	2	1	1			
	3	4	3	4			
	4	4	4	4			
(01)	1	1	1	1	as LM	as LM	as LM
	2	4	2	4			
	3	3	1	1			
	4	4	4	4			
(00)	1	4	1	as L	as M	as L	as M
	2	4	1				
	3	4	1				
	4	4	1				

Thus **U11** has three positive modalities (α, $L\alpha$, $M\alpha$), **U01** has five (**U11**'s $+ LM\alpha, ML\alpha$), and **U00** has seven (**U01**'s + $LML\alpha$, $MLM\alpha$).

REFERENCES

[1] Lewis, C. I. and C. H. Langford, *Symbolic Logic*, 2nd Ed., Dover Publications, Inc. (1959).

[2] Hughes, G. E. and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen and Co., Ltd., London (1968).

[3] Sobociński, B., "Modal system S4.4," *Notre Dame Journal of Formal Logic*, vol. V (1964), pp. 305-312.

Data Processing Division
I.B.M. United Kingdom Ltd.
England