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POINT MONADS AND P-CLOSED SPACES

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1 *Introduction** Let P be any topological property. Recall that a space (X,τ) is *P-closed* if X is a P-space and a closed subset of every P-space in which it is embedded. As is well known [1] for P = completely regular, normal, paracompact, metric, completely normal, locally compact, zero-dimensional, a P-space is P-closed iff it is compact. Robinson [12] was the first to show that a space (X,τ) is compact iff ${}^*X = \bigcup\{\mu(p) \mid p \in X\}$, where $\mu(p) = \bigcap\{{}^*G \mid p \in G \in \tau\}$. In [7], [9], it is shown that a space (X,τ) is Hausdorff-closed (henceforth called H-closed) iff ${}^*X = \bigcup\{\mu_\theta(p) \mid p \in X\}$, where $\mu_\theta(p) = \bigcap\{{}^*(\mathrm{cl}_X G) \mid p \in G \in \tau\}$. A space (X,τ) is almost completely regular [13] if for each regular-closed $A \subset X$ (i.e., $A = \mathrm{cl}_X\mathrm{int}_X A$) and $x \notin A$ there exists a real valued continuous map $f{:}X \to [0,1]$ such that $f[A] = \{0\}$ and $f(x) = 1$. In [9], it is shown that an almost completely regular Hausdorff space (X,τ) is almost completely regular-closed iff ${}^*X = \bigcup\{\mu_\alpha(p) \mid p \in X\}$, where $\mu\ (p) = \bigcap\{{}^*(\mathrm{int}_X\mathrm{cl}_X G) \mid p \in G \in \tau\}$. The monad $\mu(p)$, α-monad $\mu_\alpha(p)$ and θ-monad $\mu_\theta(p)$, in addition to characterizing various P-closed spaces, are extensively employed to investigate numerous other important topological properties. Of particular interest is the result in [6] which shows that a filter base $\mathfrak{F}$ on X is Whyburn [resp. Dickman] iff $\mathrm{Nuc}\ \mathfrak{F} = \bigcap\{{}^*F \mid F \in \mathfrak{F}\} \subset \mathrm{ns}({}^*X) = \bigcup\{\mu(p) \mid p \in X\}$ [resp. $\mathrm{Nuc}\ \mathfrak{F} \subset \mathrm{ns}_\theta({}^*X) = \bigcup\{\mu_\theta(p) \mid p \in X\}$]. For other recent results using these monads, we direct the reader to references [6], [7], [8], [10], [11]. Elementary applications of the α and θ-monads and simple basic propositions may be found in [7].

The major goal of this paper is to define a new monad, the w-monad, and show that it characterizes the completely Hausdorff-closed spaces in the usual nonstandard manner. A space (X,τ) is *completely Hausdorff* (sometimes called Urysohn or functional Hausdorff) if for distinct p, $q \in X$ there exists a map $f \in \mathrm{C}(X)$, the set of all real valued continuous functions

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on X, such that $f(p) \neq f(q)$. Example 4.13, [1] is that of a nontrivial completely Hausdorff-closed space which is not H-closed while example 4.19 is that of a nontrivial completely Hausdorff-closed, regular-closed space which is not compact. Thus a slight weakening of the completely regular axiom yields an important P-structure which has been subjected to intense investigations [5], [15]. In the last section of this paper, certain applications of our major result are given which indicate the inherent economy of effort associated with nonstandard methods.

As far as the structure of the nonstandard model $^*\mathfrak{M}$ is concerned, we shall assume that the superstructure $\mathfrak{M} = \langle \mathfrak{U}, \epsilon, \mathrm{pr}, \mathrm{ap}\rangle$ is that constructed by Machover and Hirschfeld [11], even though any appropriate set theoretic superstructure will suffice. We shall, unless otherwise indicated, assume that $^*\mathfrak{M} = \langle {}^*\mathfrak{U}, {}^*\epsilon, {}^*\mathrm{pr}, {}^*\mathrm{ap}\rangle$ is an enlargement of $\mathfrak{M}$ [10], [11], [12]. As usual, if $A \in {}^*\mathfrak{U}$, then $^*A = \{x \mid [x \in {}^*\mathfrak{U}] \wedge [x \,{}^*\!\epsilon\, A]\}$ and for $\mathfrak{F} \subset \mathcal{P}(X)$, the power set of X, we let $\mathrm{Nuc}\ \mathfrak{F} = \bigcap\{{}^*F \mid F \in \mathfrak{F}\}$.

2 *The* w-*monad* Let (X,τ) be a topological space. An open filter base $\mathfrak{F}$ is *completely Hausdorff* if for each $p \in X$ which is not a cluster point of $\mathfrak{F}$ there exist an open neighborhood G of p, $F \in \mathfrak{F}$ and $f \in \mathrm{C}(X)$ such that $f[G] = \{1\}$ and $f[F] = \{0\}$.

Definition 2.1 For each $p \in X$, let the w-monad of p be

$$\mu_{\mathrm{w}}(p) = \bigcap\{{}^*f^{-1}[\mu(f(p))] \mid f \in \mathrm{C}(X)\}.$$

Observe that for each $p \in X$, $\mu(p) \subset \mu_{\mathrm{w}}(p)$ and that if τ_{w} is the weak topology generated by $\mathrm{C}(X)$, then $\mu'(p) = \mu_{\mathrm{w}}(p)$, where $\mu'(p)$ is the monad generated by the topology τ_{w}. The following result is obtained directly from Definition 2.1 and the nonstandard theory of filter bases on a meet-semilattice of sets.

Theorem 2.1 *For a space (X,τ), an open filter base $\mathfrak{F}$ is completely Hausdorff iff $\mu(p) \cap \mathrm{Nuc}\ \mathfrak{F} = \emptyset$ implies that there exists an $f \in \mathrm{C}(X)$ such that ${}^*f[\mu(p)] = \{1\}$ and ${}^*f[\mathrm{Nuc}\ \mathfrak{F}] = \{0\}$.*

Theorem 2.2 *Assume that $\mathfrak{F}$ is a completely Hausdorff filter base on X. Then*

$$\mu(p) \cap \mathrm{Nuc}\ \mathfrak{F} \neq \emptyset \textit{ iff } \mu_{\mathrm{w}}(p) \cap \mathrm{Nuc}\ \mathfrak{F} \neq \emptyset.$$

Proof: The necessity is obvious. For the sufficiency assume that $\mu(p) \cap \mathrm{Nuc}\ \mathfrak{F} = \emptyset$. Then Theorem 2.1 implies that there exists a map $f \in \mathrm{C}(X)$ such that ${}^*f[\mu(p)] = \{1\}$ and ${}^*f[\mathrm{Nuc}\ \mathfrak{F}] = \{0\}$. Since $f(p) = 1$, then $\mu(f(p)) \cap {}^*f[\mathrm{Nuc}\ \mathfrak{F}] = \emptyset$. Hence ${}^*f^{-1}[\mu(f(p))] \cap \mathrm{Nuc}\ \mathfrak{F} = \emptyset$ implies that $\mu_{\mathrm{w}}(p) \cap \mathrm{Nuc}\ \mathfrak{F} = \emptyset$.

Corollary 2.2.1 *A completely Hausdorff space (X,τ) is completely Hausdorff-closed iff for each completely Hausdorff filter base $\mathfrak{F}$ there exists some $p \in X$ such that $\mu_{\mathrm{w}}(p) \cap \mathrm{Nuc}\ \mathfrak{F} \neq \emptyset$.*

Proof: This follows from Theorem 4.9 in [1].

Recall that if $\mathfrak{F}$ is an ultrafilter on X, then $\mathrm{Nuc}\ \mathfrak{F}$ is called an *ultramonad*. Also, if we let $\mathrm{Fil}\{q\} = \{x \mid [x \subset X] \wedge [q \in {}^*x]\}$, $q \in {}^*X$, then we know that $\mathrm{Nuc}\ \mathfrak{F}$ is an ultramonad iff there exists some $q \in {}^*X$ such that $\mathrm{Nuc}\ \mathfrak{F} = \mathrm{Nuc}\ \mathrm{Fil}\{q\} = \mathrm{NF}\{q\}$.

Theorem 2.3 *For any* $q \in {}^*X$ *and* $p \in X$, $\mathrm{NF}\{q\} \cap \mu_{\mathrm{w}}(p) \neq \emptyset$ *iff for each* $f \in \mathrm{C}(X)$, $\mu(f(p)) \cap {}^*f\,[\mathrm{NF}\{q\}] \neq \emptyset$.

Proof: Since the necessity is obvious, then assume that $\mu(f(p)) \cap {}^*f\,[\mathrm{NF}\{q\}] \neq \emptyset$ for each $f \in \mathrm{C}(X)$. Now for each $f \in \mathrm{C}(X)$ it follows that ${}^*f\,[\mathrm{NF}\{q\}] = \mathrm{NF}\{{}^*f(q)\}$. Consequently, $\mu(f(p)) \cap {}^*f\,[\mathrm{NF}\{q\}] \neq \emptyset$ implies that ${}^*f\,[\mathrm{NF}\{q\}] \subset \mu(f(p))$ for $f \in \mathrm{C}(X)$. Hence $\mathrm{NF}\{q\} \subset \mu_{\mathrm{w}}(p)$ and the proof is complete.

We now come to the major result:

Theorem 2.4 *Let* (X,τ) *be a topological space. Then every completely Hausdorff filter base on* X *has a cluster point in* X *iff* ${}^*X = \bigcup\{\mu_{\mathrm{w}}(p) \mid p \in X\}$.

Proof: For the necessity, assume that every completely Hausdorff filter base on X has a cluster point in X and ${}^*X \neq \bigcup\{\mu_{\mathrm{w}}(p) \mid p \in X\}$. Thus there exists some $q \in {}^*X$ such that for each $p \in X$, $\mathrm{NF}\{q\} \cap \mu_{\mathrm{w}}(p) = \emptyset$ since $\mu_{\mathrm{w}}(p)$ is a filter monad. Consequently, for each $p \in X$ there exists $f_p \in \mathrm{C}(X)$ such that ${}^*f_p[\mathrm{NF}\{q\}] \cap \mu(f_p(p)) = \emptyset$. Therefore, for each $p \in X$, there exist $f_p \in \mathrm{C}(X)$ and $U_p \in \mathrm{Fil}\{q\}$ such that

$${}^*(f_p[U_p]) \cap \mu(f_p(p)) = {}^*(\mathrm{cl}_{\mathcal{R}} f_p[U_p]) \cap \mu(f_p(p)) = \emptyset,$$

where, as usual, "$\mathrm{cl}_{\mathcal{R}}$" denotes the closure in $\mathcal{R}$. Regularity now implies that ${}^*(\mathrm{cl}_{\mathcal{R}} f_p[U_p]) \cap \mu_\theta(f_p(p)) = \emptyset$ [6], p. 163. Hence for the set monad $\mu(\mathrm{cl}_{\mathcal{R}} f_p[U_p])$, we have that $\mu(\mathrm{cl}_{\mathcal{R}} f_p[U_p]) \cap \mu_\theta(f_p(p)) = \emptyset$. Normality of $\mathcal{R}$ now yields, $\mu_\theta(\mathrm{cl}_{\mathcal{R}} f_p[U_p]) \cap \mu_\theta(f_p(p)) = \emptyset$ [6], p. 163. From continuity, we have that $\mu(p) \cap {}^*f_p^{-1}[\mu_\theta(\mathrm{cl}_{\mathcal{R}} f_p[U_p])] = \emptyset$. Observe that there exists a continuous map $h_p : \mathcal{R} \to \mathcal{R}$ (the so-called Urysohn map) such that ${}^*h_p[\mu_\theta(f(p))] = \{1\}$ and ${}^*h_p[\mu_\theta(\mathrm{cl}_{\mathcal{R}} f_p[U_p])] = \{0\}$. Hence we have that ${}^*(h_p f_p)\,[\mu(p)] = \{1\}$ and ${}^*(h_p f_p)\,[{}^*f_p^{-1}[\mu(\mathrm{cl}_{\mathcal{R}} f_p[U_p])]] = \{0\}$ since $\mu_\theta(\mathrm{cl}_{\mathcal{R}} f_p[U_p]) = \mu(\mathrm{cl}_{\mathcal{R}} f_p[U_p])$. Now by simply observing that for each $p \in X$, $U_p \subset f_p^{-1}[f_p[U_p]] \subset f_p^{-1}[G]$ for any open set $G \supset \mathrm{cl}_{\mathcal{R}} f_p[U_p]$, we have, using the fact that $\mathrm{Fil}\{q\}$ is a filter, that $A = \bigcap\{{}^*f_p^{-1}[\mu(\mathrm{cl}_{\mathcal{R}} f_p[U_p])] \mid p \in X\} \neq \emptyset$. Moreover, the continuity of each f_p and the nonstandard theory of filters on a meet-semilattice of sets imply that $A = \mathrm{Nuc}\ \mathfrak{G}$, where $\mathfrak{G}$ is an open filter base.

We now show that $\mathfrak{G}$ is a completely Hausdorff filter base. Let $\mu(p) \cap \mathrm{Nuc}\,\mathfrak{G} = \emptyset$. Then there exists a map $h_p f_p \in \mathrm{C}(X)$ such that ${}^*h_p {}^*f_p[\mu(p)] = \{1\}$ and

$${}^*h_p {}^*f_p[\mathrm{Nuc}\ \mathfrak{G}] \subset {}^*h_p {}^*f_p[{}^*f_p^{-1}[\mu(\mathrm{cl}_{\mathcal{R}} f_p[U_p])]] = \{0\}.$$

Therefore, from Theorem 2.1 we have that $\mathfrak{G}$ is a completely Hausdorff filter base. However, since $\mu(p) \cap \mathrm{Nuc}\ \mathfrak{G} = \emptyset$ for each $p \in X$, then we have a contradiction and the necessity follows.

The sufficiency is immediate from Theorem 2.2 and this completes the proof.

Corollary 2.4.1 *A completely Hausdorff space X is completely Hausdorff-closed iff* $*X = \bigcup\{\mu_w(p) \mid p \in X\}$.

Corollary 2.4.2 *A completely Hausdorff space X is completely Hausdorff-closed iff for each filter base* $\mathfrak{F}$ *on X there exists some* $p \in X$ *such that* $\mu_w(p) \cap \mathrm{Nuc}\ \mathfrak{F} \neq \emptyset$.

We say that a space (X,τ) is *Urysohn* if for distinct $p, q \in X$ there exist neighborhoods N_p, N_q of p and q respectively such that $\mathrm{cl}_X N_p \cap \mathrm{cl}_X N_q = \emptyset$. It is well known that a space X is Hausdorff [resp. Urysohn] iff for distinct $p, q \in X$, $\mu(p) \cap \mu(q) = \emptyset$ [resp. $\mu_\theta(p) \cap \mu_\theta(q) = \emptyset$ [7]].

Theorem 2.5 *A space* (X,τ) *is completely Hausdorff iff for distinct* $p, q \in X$, $\mu_w(p) \cap \mu_w(q) = \emptyset$.

Proof: Assume that X is completely Hausdorff. Then there exists $f \in C(X)$ such that $f(q) \neq f(p)$. Hence $\mu(f(p)) \cap \mu(f(q)) = \emptyset$. Thus $*f^{-1}[\mu(f(p))] \cap *f^{-1}[\mu(f(q))] = \emptyset$ implies that $\mu_w(p) \cap \mu_w(q) = \emptyset$. Conversely, assume that X is not completely Hausdorff. Then there exist distinct $p, q \in X$ such that for each $f \in C(X)$, $f(p) = f(q)$. Consequently, $\mu(f(p)) = \mu(f(q))$ for each $f \in C(X)$ implies that $\mu_w(p) = \mu_w(q)$ and the proof is complete.

3 *Applications* It should not be construed from the fact that the following results are easily verified that they are essentially trivial. On the contrary, it is an indication of the strength of Theorem 2.4 and the inherent economy of effort associated with the nonstandard monadic method.

Let τ_w be the weak topology generated by $C(X)$. As previously observed, for each $p \in X$, $\mu'(p) = \mu_w(p)$, where $\mu'(p)$ is the monad generated by τ_w. Thus one obtains as an immediate consequence of Theorem 2.5 that a space (X,τ) is completely Hausdorff iff (X,τ_w) is Hausdorff. It is well known that τ_w is a completely regular topology.

Theorem 3.1 *If* (X,τ) *is a topological space, then* $*X = \bigcup\{\mu_w(p) \mid p \in X\}$ *iff* (X,τ_w) *is compact.*

Proof: Simply apply Robinson's [12], p. 93 criterion for compactness.

Corollary 3.1.1 *A completely Hausdorff space* (X,τ) *is completely Hausdorff-closed iff* (X,τ_w) *is compact.*

Corollary 3.1.2 *For a completely Hausdorff space* (X,τ), *let* $\mathfrak{G} = \{f^{-1}[G] \mid [f \in C(X)] \wedge [G \text{ is open in } \mathcal{R}]\}$. *Then* (X,τ) *is completely Hausdorff-closed iff every cover* $\mathfrak{C} \subset \mathfrak{G}$ *of X has a finite subcover.*

Proof: Either use the fact that $\mathfrak{G}$ is a subbase for τ_w or apply Theorems 4.1 and 4.3 [8], where we note that Theorem 4.1 holds in any enlargement for the nonstandard extension of any standard subset of X.

Corollary 3.1.3 *A completely Hausdorff-closed space is compact iff it is completely regular.*

Proof: A space (X,τ) is completely regular iff $\tau = \tau_w$.

Remark: In [15], Stephenson shows that a completely Hausdorff space is Stone-Weierstrass iff it is completely Hausdorff-closed. Consequently, Corollary 3.1.1 also follows from the known result, which has been previously established by application of the Stone-Čech compactification, that a completely Hausdorff space is Stone-Weierstrass iff (X,τ_w) is compact.

Theorem 3.2 *A completely Hausdorff-closed space is* H*-closed iff it is nearly-compact iff it is almost completely regular.*

Proof: Observe that completely Hausdorff implies Urysohn and an H-closed Urysohn space is nearly-compact. Moreover, a nearly-compact space is almost completely regular. Now assume that completely Hausdorff-closed (X,τ) is almost completely regular. It is easily verified from the definitions that for any almost completely regular space X, $\mu_w(p) \subset \mu_\alpha(p)$ for each $p \in X$. Moreover, if $f \in C(X)$ and $p \in X$, then $*f[\mu_\theta(p)] \subset \mu_\theta(f(p))$ (i.e., f is θ-continuous) [8]. Since $\mathcal{R}$ is regular, then $\mu_\theta(p) \subset *f^{-1}[\mu(f(p))]$ for each $p \in X$. Consequently, $\mu_\theta(p) = \mu_w(p)$ for each $p \in X$. Theorem 4.2 now implies that X is H-closed [6], [7], [8] and this completes the proof.

Corollary 3.2.1 *A space (X,τ) is almost completely regular [resp. completely regular] iff for each $p \in X$, $\mu_w(p) = \mu_\alpha(p)$ [resp. $\mu_w(p) = \mu(p)$].*

Corollary 3.2.2 *For any space (X,τ) and $p \in X$, we have that $\mu_\theta(p) \subset \mu_w(p)$.*

Theorem 3.3 *If $*X = \bigcup\{\mu_w(p) \mid p \in X\}$, then X is pseudocompact.*

Proof: Let $q \in *X$. Then $q \in \mu_w(p)$ for some $p \in X$. Hence for each $f \in C(X)$, $*f(q) \in \mu(f(p)) \subset M_0$, the set of finite elements in $*\mathcal{R}$. Consequently, $*(f[X]) \subset M_0$ implies that $f[X]$ is bounded.

Theorem 3.4 *A completely Hausdorff-closed space X is homeomorphic to a quotient of $*X$.*

Proof: This is obvious since $\{\mu_w(p) \mid p \in X\}$ is a partition of $*X$.

The Q-topology $\mathfrak{T}$ on $*X$ is the topology generated by $\{*G \mid G \in *\tau\}$ as a base. Notice that since Theorem 2.1 [6] does not require a highly saturated enlargement, then $\mu_w(p)$ is Q-open for each $p \in X$. Consequently, if (X,τ) is completely Hausdorff-closed, then the surjection $h:(*X,\mathfrak{T}) \to (X,\tau)$ defined by $h[\mu_w(p)] = \{p\}$, $p \in X$, is Q-continuous. Furthermore, $\mu_w(p)$ is Q-closed for each $p \in X$ since $\mathcal{R}$ is regular.

In [3], Button establishes the following interesting results. A space (X,τ) is regular [resp. Hausdorff, discrete] iff $(*X,\mathfrak{T})$ is regular [resp. Hausdorff, discrete]. If (X,τ) is completely Hausdorff, then $(*X,\mathfrak{T})$ is totally separated. In an $\aleph_1$-saturated enlargement, (X,τ) is regular iff $(*X,\mathfrak{T})$ is completely regular and each $G_\delta \in \mathfrak{T}$ (i.e., a P-space). Thus any nonregular completely Hausdorff-closed space is the continuous image of a totally separated nonregular space. Moreover, any completely Hausdorff-closed, regular, noncompact space (Y,τ), such as example 4.19 [1], is the continuous image of a nondiscrete, totally separated, completely regular

P-space $(^*Y,\mathfrak{T})$. Indeed, if there are no measurable cardinals, then we may also assume that $(^*Y,\mathfrak{T})$ is not extremally disconnected.

Finally, we point out that the w-monads may be used to characterize c-maps [5]. It is not difficult to show that a map $f:X \to Y$ is a c-map iff $^*f[\mu(p)] \subset \mu_w(f(p))$ for each $p \in X$. Further, Lemma 2.2 in [5] may be improved upon. Simply observe that if $f:X \to Y$ is a c-map and Y is almost completely regular, then Corollary 3.2.1 implies that f is almost-continuous; which implies that f is θ-continuous.

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