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A FORMAL INTERPRETATION OF ŁUKASIEWICZ' LOGICS

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One way of coming to understand the meaning of the concepts in a non-standard logic is by seeing how that logic can be interpreted within some more familiar, more readily understood, logic. A case in point is intuitionism. Given the intuitionistic account of the nature of mathematics, it is natural to expect that, from a classical point of view, intuitionistic logic is concerned with the assertability or provability of sentences, and not merely with their truth. Here interpretation can sharpen insight. For, as McKinsey and Tarski [1] show, intuitionistic propositional logic is exactly interpretable in the modal system S4 under a number of very natural mappings.

In the present note, I shall use the method of interpretation, and indeed one of the McKinsey-Tarski mappings, as a way of sharpening some recent views about the meaning of Łukasiewicz' many-valued logics. Several logicians have contended that these systems can plausibly be regarded as logics of exactness. The various values represent degrees of truth, with the top value being complete truth, the bottom value complete falsity, and the intermediate values degrees of partial truth. If this idea stands scrutiny, then Łukasiewicz' logics may be appropriate tools to use in developing logics for vague terms. (For discussion, see [2] and [3]).

The following model suggests itself as a formal representation of the view just mentioned. Suppose n judges are asked to decide if a certain claim (e.g., "Gerald Ford is bald.") is true or false. Suppose the only responses permissible are true and false and that the judges are a fair cross-section of the population. Then the verdict of the judges can be represented by an n-place sequence of $\mathbf{t}$'s and $\mathbf{f}$'s. How can we model the degree to which this claim is true? Given that the judges are a fair sample, a reasonable view is that the degree of truth is shown by the number of judges who say that the claim is true. And this number k is naturally represented by the n-place sequence of truth-values which begins with k $\mathbf{t}$'s and has $\mathbf{f}$'s from then on. So, I propose that the degree of truth $\mathrm{d}(A)$ of A be construed as the sequence $\langle \mathbf{t}, \mathbf{t}, \ldots, \mathbf{t}, \mathbf{f}, \ldots \mathbf{f}\rangle$ where the sequence associated with A contains exactly k $\mathbf{t}$'s.

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To provide the required formal interpretation, consider the logic which assigns as values n-place sequences of **t**'s and **f**'s, and in which the values of the various connectives are computed componentwise according to the classical truth tables. This logic has 2^n values and, following Rescher [4], I call it the 2^n valued product logic $\mathbf{P}_{2^n}$. In this logic, the one-place connective $\mathrm{d}(A)$ has the value $\langle \mathbf{t}, \mathbf{t}, \ldots, \mathbf{t}, \mathbf{f}, \ldots, \mathbf{f} \rangle$ in just those cases where there are k **t**'s in the sequence assigned to A. Now, map the $n + 1$-valued Łukasiewicz logic $\mathbf{L}_{n+1}$, with values $0, \ldots, n$, into the logic $\mathbf{P}_{2^n}$, as follows:

$f(p) = \mathrm{d}(p)$, where p is an atomic sentence letter.
$f(-A) = \mathrm{d}(-fA)$
$f(A \mathbin{\&} B) = f(A) \mathbin{\&} f(B)$ (or $\mathrm{d}(f(A) \mathbin{\&} f(B))$)
$f(A \vee B) = f(A) \vee f(B)$ (or $\mathrm{d}(f(A) \vee f(B))$)
$f(A \supset B) = \mathrm{d}(f(A) \supset f(B))$

It is now a straightforward matter to prove that A is an $\mathbf{L}_{n+1}$ tautology iff $f(A)$ is a $\mathbf{P}_{2^n}$ tautology, where by "tautology" is meant always taking the top value. To see this, suppose first that A is not an $\mathbf{L}_{n+1}$ tautology. Let the atomic sentence letters of A be $p_1, \ldots, p_c$. Define v^* on $\mathbf{P}_{2^n}$ by setting $v^*(p_i) = \langle \mathbf{t}, \mathbf{t}, \ldots \mathbf{t}, \mathbf{f}, \ldots \mathbf{f} \rangle$, where this begins with k **t**'s, iff $v(p_i) = k$ on the assignment v falsifying A. We now show that, for any subformula B of A, $v(B) = k$ iff $v^*(f(B))$ is the sequence beginning with exactly k **t**'s. The basis case is guaranteed by definition. So suppose that B has the form $-C$ and that $v(-C) = k$. Then $v(C) = n - k$, and by the **IH**, $v^*(f(C))$ is the sequence beginning with $n - k$ **t**'s. Thus $v^*(-f(C))$ is the sequence beginning with $n - k$ **f**'s and ending with k **t**'s. So $v^*(d(-f(C)))$ is the sequence beginning with exactly k **t**'s as required. Second, suppose that B has the form $D \supset E$, and that $v(D \supset E) = n$. This implies that $v(D) \leqslant v(E)$, where $v(D) = k$ and $v(E) = j$. Then by **IH**, $v^*(f(D))$ and $v^*(f(E))$ are the sequences beginning with k and j **t**'s respectively. Since $k \leqslant j$, $v^*(f(D) \supset f(E))$ is all **t**'s, as is $v^*(d(f(D) \supset f(E)))$. Suppose, however, that $v(C) = k$, where $k \neq n$. Then $v(D) = j > m = v(E)$ and $k = n - j + m$. By **IH**, $v^*(f(D))$ begins with j **t**'s and $v^*(f(E))$ with m **t**'s. So $v^*(f(D) \supset f(E))$ will have **f**'s in the $j - m$ places where $v^*(f(D))$ has **t** and $v^*(f(E))$ has **f**. So $v^*(f(D) \supset f(E))$ will have **t** in $n - (j - m) = n - j + m$ places. Whence $v^*(d(f(D) \supset f(E)))$ will begin with the right number of **t**'s. The cases for & and v are similar.

For the converse, assume that $f(A)$ is not a tautology in $\mathbf{P}_{2^n}$. Let $p_1, \ldots, p_c$ be the atomic sentence letters in A, and suppose that on the falsifying assignment v, $v(p_i)$ is the sequence s_i. Define the valuation v^* on $\mathbf{L}_{n+1}$ by setting $v^*(p_i) = k$ iff there are exactly k **t**'s in the sequence s_i. Using the same arguments as above, it can be shown that, for any subformula B of A, $v^*(B) = k$ iff $v(f(B))$ is the sequence beginning with k **t**'s.

The relation described between Łukasiewicz' logics and product logics can be thought of in more pictorial terms. To obtain an $n + 1$-valued Łukasiewicz logic, take a 2^n-valued product logic. First erase all columns and rows not headed by sequences constituted by an unbroken string of **t**'s followed by an unbroken string of **f**'s. Rewrite the resulting table identifying two values in the result if and only if they have the same number of **t**'s. The result of $\mathbf{L}_{n+1}$.

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