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# NECESSARY TRUTH AS ANALYTICITY, AND THE ELIMINABILITY OF MONADIC *DE RE* FORMULAS

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Among formulas of modal predicate logic, the *de re* ones are those in which a variable occurs free within the scope of a modal operator; the *monadic* ones are those containing none but monadic predicates and no nested quantifiers (no quantifier within the scope of a quantifier).[1] Assuming that only *analytic* statements count as *necessary truths*—that necessary truths are somehow *logically* necessary—I will show that the *monadic de re formulas* are *eliminable* in this sense: there is an effective way of translating them into non-*de re* equivalents.

To prove this claim, I first recast it in a more precise, mathematically tractable form. In the process, the assumption that necessary truths are analytic gets weakened considerably. Note, by the way, that this assumption explicitly constrains the interpretation of modal operators only when prefixed to statements, not open sentences; it does not explicitly say anything about the interpretation of *de re* locutions.

1 *Syntax* Because monadic modal formulas contain none but monadic predicates and no nested quantifiers, they can be written with a single individual variable, which we may as well suppress, treating a predicate standing alone as though a variable followed it.[2] Such formulas can be built from a *vocabulary* comprising $\neg$, $\wedge$, $\exists$, $\Box$, parentheses and some (monadic) predicates. Although monadic modal formulas contain no nested quantifiers, hence no open subformulas with quantifiers inside, I find it convenient to work with a slightly more inclusive class of formulas, defined to allow nested $\exists$'s (though without distinct variables), and therewith open formulas with $\exists$ inside, so long as none has the form $\Box A$.

Let a *quasi-formula* be any member of the least class containing every predicate and containing $\neg A$, $(A \wedge B)$, $\exists A$, and $\Box A$ whenever it contains $A$, $B$. A quasi-formula is $\exists$-*free* if devoid of $\exists$, and *closed* if a member of the least class containing $\exists A$ for every quasi-formula $A$ and containing $\neg A$, $(A \wedge B)$ and $\Box A$ whenever it contains $A$, $B$. A *formula*, then, is any quasi-formula such that every quasi-formula of the form $\Box A$ occurring

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therein is either $\exists$-free or closed. The syntactic variables $A$, $B$, $C$ range over formulas, $F$ over predicates.

**2** *Semantics*[3] An *interpretation* is an ordered quadruple $(K, R, d, f)$ such that $K$ is a non-empty set (the set of 'possible worlds,' including those, if any, that are just possibly possible, possibly possibly possible, etc.), $R$ is a binary relation on $K$ (the 'accessability' or 'relative possibility' relation of a world $w$ to a world $v$ when everything true in $v$ is possible in $w$), $R$ is reflexive in $K$ (every world is possible relative to itself, at least), $d$ is a function that assigns to each $w \in K$ a non-empty set $d(w)$ (the 'domain' of $w$, comprising those 'possible things' that would actually exist if $w$ were the actual world), $d(w) \subseteq d(v)$ whenever $wRv$ (domains are cumulative), and $f$ is a function that assigns to each predicate $F$ and each $w \in K$ a subset $f(F, w)$ of $d(w)$ (the extension of $F$ at $w$).

When $(K, R, d, f)$ is an interpretation and $w \in K$, an object $x$ is defined as *satisfy*ing a given formula *at* $w$ *in* $(K, R, d, f)$ according to the following recursion:

$x$ satisfies $F$ at $w$ in $(K, R, d, f)$ iff $x \in f(F, w)$;
$x$ satisfies $(A \wedge B)$ at $w$ in $(K, R, d, f)$ iff $x$ satisfies both $A$ and $B$ at $w$ in $(K, R, d, f)$;
$x$ satisfies $\neg A$ at $w$ in $(K, R, d, f)$ iff $x \in d(w)$ and $x$ does not satisfy $A$ at $w$ in $(K, R, d, f)$;
$x$ satisfies $\exists A$ at $w$ in $(K, R, d, f)$ iff $x \in d(w)$ and some member of $d(w)$ satisfies $A$ at $w$ in $(K, R, d, f)$;
$x$ satisfies $\Box A$ at $w$ in $(K, R, d, f)$ iff $x \in d(w)$ and $x$ satisfies $A$ in $(K, R, d, f)$ at every $v$ for which $wRv$.

If $(K, R, d, f)$ is an interpretation and $w \in K$, $A$ is *true at* $w$ *in* $(K, R, d, f)$ iff every member of $d(w)$ satisfies $A$ at $w$ in $(K, R, d, f)$.
$A$ and $B$ are *equivalent in* an interpretation $(K, R, d, f)$ iff, for every $w \in K$, all and only those things that satisfy $A$ at $w$ in $(K, R, d, f)$ satisfy $B$ at $w$ in $(K, R, d, f)$.
$A$ is *valid in* an interpretation $(K, R, d, f)$ iff, for every function $g$ such that $(K, R, d, g)$ is also an interpretation, $A$ is true in $(K, R, d, g)$ at each $w \in K$.

**3** *Analyticity* My thesis is that if only analytic statements count as necessary truths, then the *de re* formulas are eliminable—effectively translatable into non-*de re* equivalents. Somewhat more precisely: There is an effective way of transforming formulas into non-*de re* formulas so that each formula is equivalent to its non-*de re* transform in every interpretation that counts only analytic statements as necessary truths.

Every analytic statement is either a formal logical truth, like "Every bachelor is a bachelor," or a statement that can be turned into a formal logical truth by replacing some occurrences therein of simple expressions with (possibly complex) synonyms, as "Every bachelor is a male" can be turned into "Every unmarried male is a male" by replacing "bachelor" with "unmarried male."

In particular, a statement of the form $\exists A$ with $\exists$-free $A$ is analytic only if it can be turned into a formal logical truth by replacing some occurrences of predicates with (possibly complex) synonyms, which I take to be $\exists$-free formulas of some sort or other.[4] The formal logical truth got thereby has the form $\exists B$. Because synonym-replacements turned $A$ into $B$, $A$ and $B$ are themselves synonymous, making $\neg\exists(B \wedge \neg A)$ necessarily true.

So an interpretation $(K,R,d,f)$ that construes necessary truth as analyticity must, *at the very least*, fulfill this condition: If $A$ is $\exists$-free, $w \in K$ and $\Box\exists A$ is true at $w$ in $(K,R,d,f)$, there is a $\exists$-free $B$ such that (1) $\exists B$ is a formal logical truth, and (2) $\neg\exists(B \wedge \neg A)$ is necessarily true, i.e., $\Box\neg\exists(B \wedge \neg A)$ is true, at $w$ in $(K,R,d,f)$.

Although it is debatable what to count as formal logical truths in modal logic, the formulas *construed* as formal logical truths *by a given interpretation* must at least be *valid in that interpretation.* This means we can replace (1) with the clause: $\exists B$ is valid in $(K,R,d,f)$.

The condition we end up with defines what I call an *analytic interpretation.* That is, an *analytic interpretation* is an interpretation $(K,R,d,f)$ fulfilling this condition:

> If $A$ is $\exists$-free, $w \in K$ and $\Box\exists A$ is true at $w$ in $(K,R,d,f)$, there is an $\exists$-free $B$ such that $\exists B$ is valid in $(K,R,d,f)$ and $\Box\neg\exists(B \wedge \neg A)$ is true at $w$ in $(K,R,d,f)$.

What I will prove is that there is an effective transformation of formulas into non-*de re* formulas such that every formula is equivalent to its non-*de re* transform in every analytic interpretation.

In one respect, the defining condition for an analytic interpretation is a bit stronger than some might have wished: Like $A$, $B$ is said to be $\exists$-free (intuitively, the synonym-replacements that turned $A$ into $B$ are supposed to have added no quantifiers). But in other important respects, this condition is much weaker than the intuitive requirement that $(K,R,d,f)$ count only analytic statements as necessary truths. It applies, after all, only to necessary truths of the form $\exists A$ for $\exists$-free $A$, and it does not say that $B$ can be got from $A$ by synonym-replacements, or even that $A$ and $B$ are coextensive.

**4** *A de re eliminability theorem* Define the transformation e on the set of formulas as follows:

$$
\begin{aligned}
\mathrm{e}(F) &= F\\
\mathrm{e}((A \wedge B)) &= (\mathrm{e}(A) \wedge \mathrm{e}(B))\\
\mathrm{e}(\neg A) &= \neg\,\mathrm{e}(A)\\
\mathrm{e}(\exists A) &= \exists\,\mathrm{e}(A)\\
\mathrm{e}(\Box A) &= \begin{cases} \Box\,\mathrm{e}(A), & \text{if } A \text{ is closed,}\\ \Box\exists\,\mathrm{e}(A), & \text{if } A \text{ is } \exists\text{-free.}\end{cases}
\end{aligned}
$$

It is obvious that e is effective and e$(A)$ is never *de re*. I will prove that $A$ is always equivalent to e$(A)$ in every analytic interpretation.

Lemma *Let $(K,R,d,f)$ and $(K,R,d,g)$ be interpretations and $x$ an arbitrary object. Suppose, for every $w \in K$ and every $F$, that $g(F,w) = d(w)$ if*

*$x \in f(F,w)$, while $g(F,w)$ is empty if $x \notin f(F,w)$. And suppose $A$ is $\exists$-free, $v \in K$ and $x \in d(v)$. Then if $x$ satisfies $A$ at $v$ in $(K,R,d,f)$, every member of $d(v)$ satisfies $A$ at $v$ in $(K,R,d,g)$, while if $x$ does not satisfy $A$ at $v$ in $(K,R,d,f)$, nothing satisfies $A$ at $v$ in $(K,R,d,g)$.*

*Proof:* Straightforward induction on the complexity of $A$.

Theorem *If $(K,R,d,f)$ is an analytic interpretation, $A$ is equivalent to $e(A)$ in $(K,R,d,f)$.*

*Proof:* By induction on the complexity of $A$. Suppose $w \in K$ and $x \in d(w)$; to prove that $x$ satisfies $A$ at $w$ (in $(K,R,d,f)$) iff $x$ satisfies $e(A)$ at $w$. Three cases:

Case 1 $A$ is a predicate. Then $e(A) = A$ and the proof is trivial.

Case 2 $A$ has the form $\neg B$, $(B \wedge C)$ or $\exists B$, or the form $\Box B$ for closed $B$. Then $e(A)$ is either $\neg e(B)$, $(e(B) \wedge e(C))$, $\exists e(B)$ or $\Box e(B)$, respectively, and it is a straightforward consequence of the inductive hypothesis (applied to $B$ and $C$) that $x$ satisfies $A$ at $w$ iff $x$ satisfies $e(A)$ at $w$.

Case 3 $A$ has the form $\Box B$ for $\exists$-free $B$. Then $e(A) = \Box\exists e(B)$. By inductive hypotheses, at every $v$ for which $wRv$, something satisfies $B$ iff something satisfies $e(B)$, whence $x$ satisfies $\exists B$ iff $x$ satisfies $\exists e(B)$. Therefore, $x$ satisfies $\exists B$ at every $v$ for which $wRv$ iff $x$ satisfies $\exists e(B)$ at every such $v$, i.e., $x$ satisfies $\Box\exists B$ at $w$ iff $x$ satisfies $\Box\exists e(B)$ at $w$. So, to show that $x$ satisfies $A = \Box B$ at $w$ iff $x$ satisfies $e(A) = \Box\exists e(B)$ at $w$, it suffices to show that $x$ satisfies $\Box B$ at $w$ iff $x$ satisfies $\Box\exists B$ at $w$.

If $x$ satisfies $\Box B$ at $w$, then $x$ satisfies $B$ at every $v$ for which $wRv$, whence $x$ satisfies $\exists B$ at every such $v$, and thus $x$ satisfies $\Box\exists B$ at $w$.

Conversely, suppose $x$ satisfies $\Box\exists B$ at $w$, so that $\Box\exists B$ is true at $w$; to prove that $x$ satisfies $\Box B$ at $w$. Since $(K,R,d,f)$ is an analytic interpretation and $B$ is $\exists$-free, there is an $\exists$-free $C$ such at $\exists C$ is valid in $(K,R,d,f)$ and $\Box\neg\exists(C \wedge \neg B)$ is true at $w$. So $\neg\exists(C \wedge \neg B)$ is true at every $v$ for which $wRv$, whence nothing satisfies $(C \wedge \neg B)$ at any such $v$, and thus nothing that satisfies $C$ fails to satisfy $B$ at such a $v$. That is, at every $v$ for $wRv$, whatever satisfies $C$ satisfies $B$. So if $x$ satisfies $C$ at every $v$ for which $wRv$, then $x$ satisfies $B$ at every $v$ for which $wRv$, which means that $x$ satisfies $\Box B$ at $w$. Therefore, it suffices to show that $x$ satisfies $C$ at every $v$ for which $wRv$.

Suppose, then, that $wRv$; to prove that $x$ satisfies $C$ at $v$ in $(K,R,d,f)$. Since $x \in d(w)$ and $wRv$, $x \in d(v)$. Let $g$ be the function on $\{P|P \text{ is a predicate}\} \times K$ defined as follows:

$$g(F,u) = \begin{cases} d(u), & \text{if } x \in f(F,u), \\ \phi, & \text{if } x \notin f(F,u). \end{cases}$$

Then $(K,R,d,g)$ is an interpretation. Since $\exists C$ is valid in $(K,R,d,f)$, $\exists C$ is true at $v$ in $(K,R,d,g)$, and thus something satisfies $C$ at $v$ in $(K,R,d,g)$. But by the Lemma and the definition of $g$, since $x \in d(v)$, if $x$ did not satisfy

$C$ at $v$ in $(K, R, d, f)$ then nothing would satisfy $C$ at $v$ in $(K, R, d, g)$. Consequently, $x$ satisfies $C$ at $v$ in $(K, R, d, f)$. Q.E.D.

**5** *Philosophic significance of the theorem* You can read the theorem this way: Let *necessity* be so interpreted that only *analytic* statements count as necessary truths. Then to say of a given object that it has a given monadically formulated property by necessity is tantamount to saying that it is necessarily true (hence analytic) that something or other has that property. So the former locution can be paraphrased by the latter, hence is eliminable from discourse.

Depending on your general attitude toward *de re* eliminability theorems, you can regard this result either as a partial explication—and vindication—of a philosophically interesting and problematic kind of assertion (the *de re* kind), or as proof that we cannot make monadic *de re* assertions of any really interesting (= non-eliminable) kind unless we are prepared to attribute necessary truth to synthetic statements.

## NOTES

1. By *modal predicate logic* I mean one-sorted first-order modal predicate logic without equality, individual constants or operation symbols.

   The formulas of monadic predicate logic, especially the non-modal variety, often are characterized, as I have characterized the monadic modal formulas, in a way that proscribes nested quantifiers; see, for example, the well-known textbook treatment of W. V. Quine, *Methods of Logic*, Holt, Rinehart and Winston, New York (1972), Part II; or G. H. von Wright, "On the Idea of Logical Truth," in *Logical Studies*, Routledge and Kegan Paul, London (1957), pp. 1-21. Sometimes, of course, the formulas of monadic predicate logic, modal and non-modal, are characterized, more broadly, so as to allow nested quantifiers.

2. As do Quine and von Wright, *loc. cit.*

3. My approach to modal semantics is essentially the 'orthodox' one of Saul Kripke, "Semantical Considerations on Modal Logic," *Acta Philosophica Fennica*, vol. 16 (1963), pp. 83-94.

4. If $\exists A$ is already a formal logical truth, then, trivially, we get a formal logical truth (viz., $\exists A$) by replacing some predicate with itself. If we wish to count *only* formal logical truths as 'analytic,' we can do so by construing predicate-*synonymy* as predicate-*identity*—by insisting that no two predicates are synonymous.

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