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THE AXIOMATISATION OF THEORIES OF MATERIAL NECESSITY

J. P. CLEAVE

A distinction is frequently made between empirical truths, the truths of logic and a third class of truths—the "materially necessary". In this latter category one might place arithmetic and geometry (considered as *a priory* exact sciences—see Kamlah, Lorenzen [5] and Lorenzen [8]) and also some of the propositions of set theory such as the axiom of infinity and the axiom of choice.[1] In the realm of mathematical-physical theories materially necessary propositions frequently occur as explicitly stated or tacitly assumed qualitative restrictions on the class of models to be used. For instance, it is customary to formulate a physical law by a set of ordinary or partial differential equations (Newton's equations of motion for gravitating bodies, Maxwell's equations etc.), it being assumed that the differential calculus is the "correct" mathematical tool. One could externalise this procedural decision by expressing it as an axiom: "the universe is a differentiable manifold". This is a materially necessary proposition—it is not forced on one by logical considerations only; no amount of empirical evidence can verify or refute it. It can be justified on pragmatic grounds and by *a priori* conceptions of space and time. Similarly the "dogma of structural stability" (Abraham [1], Thom [19]), which requires that models of a physical theory shall have certain qualitative features, finds its justification in some conceptions of the nature of physical enquiry.

The exact borderlines between the three types of truth is a matter of philosophical dispute. Körner [6] observes however that all the various definitions of material necessity have this feature in common:

> "To assert the material necessity of a proposition is to assert that the proposition is true and that it follows logically from a conjunction of principles which for some reason or other have a privileged status and confer it on their logical consequences."

1. Exactly which propositions are to be classed as principles of logic is a matter of the continuing controversy over the "logistic thesis", the fortunes of which have been clearly exposed by Lakatos ("Infinite regress and the foundations of mathematics," *Aristotelian Society Proceedings*, Supplementary, vol. 36, (1962)).

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In this view of material necessity, the privileged principles define the "possible worlds" as a subclass of the totality of structures, they impose an ontological constraint on the totality of worlds. Thus necessity is predicated of properties of possible worlds rather than properties of arbitrary individuals within a given world. This leads to the view that material necessity is a *de dicto* modality rather than *de re* and hence to the formal requirement that the "necessity operator", N, be applied only to sentences (*cf.* [20], pp. 26-28).

The basic conception behind the construction of theories of material necessity, below, is that given a language $\mathbf{L}$ for which a notion of logical consequence is defined, the system $\mathbf{L}$ can be extended by incorporating this notion (*cf.* Scott [14], [15]). In particular, we take a first order predicate calculus $\mathbf{L}$ and a principle $\mathbf{P}$ of material necessity formulated as a set of sentences of $\mathbf{L}$. The language is then extended to $\mathbf{L}_{\mathsf{N}}$ by adjoining the necessity operator N, treating it as a unary connective applied to *closed* formulas only. The truth definition of $\mathbf{L}_{\mathsf{N}}$ is then obtained by adjoining to the truth definition for $\mathbf{L}$ a clause declaring that $\mathsf{N}A$ is true in a given structure if A is a logical consequence of $\mathbf{P}$. The set $\mathbf{P}^{\mathsf{N}}$ of sentences of $\mathbf{L}_{\mathsf{N}}$ which are true in all structures satisfying $\mathbf{P}$ is the *theory of necessity in* $\mathbf{P}$. It is clear that the conception of necessity occurring here is based entirely on the well-known principles of classical semantics, the class of "possible worlds" being simply an elementary class in the wider sense, to use the standard terminology of model theory (e.g., [17]). Our approach is therefore closely related, in spirit at least, to the work of Montague [9] and Löb [7].

We present some results on the axiomatisation of theories of necessity. The principle result of section 2 (i.e., Theorem 3) implies that a theory of necessity $\mathbf{P}^{\mathsf{N}}$ is axiomatisable (and decidable) if, and only if, the theory $\mathbf{P}$ is decidable. It follows from this that the theory of logical necessity—obtained by taking $\mathbf{P}$ to be the set of logical truths, i.e., the set of universally valid sentences of the predicate calculus—is non-axiomatisable. In section **2** it is shown that the *general theory of necessity*—the set of sentences of $\mathbf{L}_{\mathsf{N}}$ common to all theories of necessity—can be axiomatised in a S5-like system, as is already suggested by results in [6], [9], [14], [15]. It follows that the general theory of necessity is a proper subsystem of the theory of logical necessity.

1 *The notion of validity in the extended language* Our treatment of theories of necessity is based on a first order predicate calculus extended to include the necessity operator applied to sentences. The question of which rules of inference preserve validity in the extended language depends on the original language. Usually it matters little how one formulates a first order theory—with or with or without equality, function symbols, individual constants etc. The same mathematical facts can be expressed in a language without function symbols and individual constants (*cf.* [11]). Now if the original language $\mathbf{L}$ contained individual constants, a theory of necessity could contain a sentence $\mathsf{N}A(a)$, where a is an individual constant in the sentence A. The familiar rules of deduction of the predicate calculus permit the deduction of $(\exists x)\mathsf{N}A(x)$ from $\mathsf{N}A(a)$. But $(\exists x)\mathsf{N}A(x)$ is not a well-formed formula of $\mathbf{L}_{\mathsf{N}}$ as $A(x)$ is not a sentence. We therefore have two choices: either restrict the application of the usual rules of inference, or abandon the use of individual constants in the original language. We adopt the latter alternative, taking the view that any individuals must be defined by the individuating principles (universals) available in the language, and by so doing avoid problems of transmundane identification of individuals.

By a *structure* we mean an ordered pair $w = \langle |w|, R\rangle$ where $|w|$ is a non-empty domain and R is a sequence of finitary relations. We assume the notion of *similarity* of structures; henceforth "μ" denotes a fixed similarity class of structures. "$\mathbf{L}$" denotes a first order language appropriate for defining elementary properties of structures in μ. $\mathbf{L}$ is equipped with a countable sequence $x_0, x_1, x_2, \ldots$ of *individual variables*, the *falsity sign* $\mathbf{f}$, the *implication sign* $\supset$, the *universal quantifier* $\forall$, and a sequence of *predicate constants* appropriate to μ. The other familiar connectives $\neg$, &, $\vee$, $\equiv$, etc., the truth sign $\mathbf{t}$ and the existential quantifier $\exists$ can be introduced as abbreviations in the usual way. The usual syntactical notions of *formula* and *sentence* (i.e., closed formula) are assumed. $\mathcal{F}$, $\mathcal{S}$ denote the sets of formulas and sentences of $\mathbf{L}$. To obtain a language for expressing necessity relative to some assumed principles we extend the language $\mathbf{L}$ to $\mathbf{L}_{\mathsf{N}}$ by adjoining a one place connective N and assuming, in addition to the formation rules for the formulas of $\mathbf{L}$, the rule:

"If A is a sentence then so is $\mathsf{N}A$".

(Note: "N" can be prefixed to *sentences* only—if A contains a free variable then $\mathsf{N}A$ is not counted as a formula of $\mathbf{L}_{\mathsf{N}}$.) $\mathcal{F}_{\mathsf{N}}$, $\mathcal{S}_{\mathsf{N}}$ denote the sets of formulas and sentences, respectively, of $\mathbf{L}_{\mathsf{N}}$. It will be convenient below to use the following notation of Schütte [13]. Let $\mathsf{F}(A)$ denote a formula with a distinguished occurrence of a subformula A. Then for any formula Q, $\mathsf{F}(Q)$ denotes the formula obtained from F by substituting Q for that occurrence of A in F.

Some standard notions of first order semantics will be assumed, principally the notion of satisfaction of sentences in structures of μ. The phrase "the system w satisfies the sentence A (of $\mathbf{L}$)" is written as "$w \models A$". For a subclass W of μ $W^* = \{A \in \mathcal{S};\, w \models A \text{ for all } w \in W\}$. For a set $\mathbf{P} \subset \mathcal{S}$ $\mathbf{P}^* = \{w \in \mu;\, w \models A \text{ for all } A \in \mathbf{P}\}$. Thus $W^* = W^{***}$ and $\mathbf{P}^* = \mathbf{P}^{***}$; $\mathbf{P}^{**}$ is the set of logical consequences of $\mathbf{P}$.

The semantics of $\mathbf{L}_{\mathsf{N}}$ can now be formulated. Suppose that a principle $\mathbf{P}$ of material necessity is prescribed. "$\mathsf{N}A$" shall mean that the sentence A is true, not just as a matter of fact, but because A follows logically from $\mathbf{P}$. Thus, using the Tarski-Bolzano conception of logical consequence we have for a sentence A and $w \in \mathbf{P}^*$

$$w \models \mathsf{N}A \longleftrightarrow u \models A \text{ for all } u \in \mathbf{P}^* \tag{1}$$

More formally, the truth definition for $\mathbf{L}$ is extended to a truth-definition for $\mathbf{L}_{\mathrm{N}}$ by appending (1) to the defining clauses of the definition of the relation $\models$ in $\mathbf{L}$.[2] To stress that the notion of satisfaction is relative to $\mathbf{P}$ we henceforth write "$\models_{\mathbf{P}}$" instead of "$\models$" when referring to the satisfaction of $\mathbf{L}_{\mathrm{N}}$ sentences, reserving the latter sign when concerned only with sentences of the original language $\mathbf{L}$. $\mathcal{S}_{\mathrm{N}}$ includes $\mathcal{S}$; the above notion of satisfaction relative to $\mathbf{P}$ coincides with the standard notion of satisfaction when we consider the N-free sentences of $\mathbf{L}_{\mathrm{N}}$ (i.e., $\mathcal{S}$). Thus

$$\text{for } w \in \mathbf{P}^*, A \in \mathcal{S}, w \models A \text{ iff } w \models_{\mathbf{P}} A \tag{2}$$

A sentence $A \in \mathcal{S}_{\mathrm{N}}$ is $\mathbf{P}$-*valid* if $w \models_{\mathbf{P}} A$ for all $w \in \mathbf{P}^*$.

From the notion of truth one derives in the usual way a notion of logical consequence. Thus, let $A \in \mathcal{S}_{\mathrm{N}}$ and $M \subset \mathcal{S}_{\mathrm{N}}$. A is a $\mathbf{P}$-*consequence* of M (written "$M \underset{\mathbf{P}}{\rightarrow} A$") if for all $w \in \mathbf{P}^*$, $w \models_{\mathbf{P}} A$ whenever $w \models_{\mathbf{P}} M$. The symbol "$\rightarrow$" denotes the usual consequence relation on the sentences of $\mathbf{L}$. It is clear that if $\mathbf{P}$, $M \subset \mathcal{S}$ and $A \in \mathcal{S}$ then

$$M \underset{\mathbf{P}}{\rightarrow} A \text{ iff } \mathbf{P} \cup M \rightarrow A. \tag{3}$$

It was assumed above that the principle of necessity was expressed by a set of sentences in the language $\mathbf{L}$; the theory of necessity obtained from such a principle will be called an elementary theory of necessity. Thus, for any $\mathbf{P}$ the set $\mathbf{P}^{\mathrm{N}} = \{A \in \mathcal{S}_{\mathrm{N}}; A \text{ is } \mathbf{P}\text{-valid}\}$ is the *theory of necessity in* $\mathbf{P}$. If $\mathbf{P}$ is a finite set of sentences of $\mathbf{L}$ then $\mathbf{P}^{\mathrm{N}}$ is said to be a *finite theory of necessity*. We remark that for any $\mathbf{P}$, $\mathbf{P}^{\mathrm{N}} = \mathbf{P}^{**\mathrm{N}}$. A particular case of interest is the finite theory obtained from the empty set of sentences; ϕ^{N} is the *theory of logical necessity*. This is to be distinguished from $\mathbf{N}$, the set of sentences common to all elementary theories of necessity. $\mathbf{N} = \bigcap \{\mathbf{P}^{\mathrm{N}}; \mathbf{P} \subset \mathcal{S}\}$ is the *general theory of necessity*. Later it will be shown that $\mathbf{N}$ is the set of sentences common to all *finite* theories of necessity.

In order to investigate the problems of axiomatisation of theories of necessity we first record some simple facts. The following relations follow directly from the definitions of $\mathbf{P}$-validity and $\mathbf{P}$-consequence. Their proofs are omitted.

(4) For any μ-structure w and $F \in \mathcal{S}_{\mathrm{N}}$

$$w \models_{\mathbf{P}} \mathrm{N}F \text{ iff } F \in \mathbf{P}^{\mathrm{N}}.$$

(5) For any set $\mathbf{P} \subset \mathcal{S}$

$$\mathbf{P}^{**} = \mathbf{P}^{\mathrm{N}} \cap \mathcal{S}$$

2. This general method of introducing the necessity operator semantically is not confined to first order logic, but could also be applied to higher order logic or type theory—in fact, any logic for which an inductive definition of truth is available, i.e. the truth value of a formula in a structure depends on truth values of its proper subformulas in a class of structures.

(6) If $\mathbf{P} \subset \mathcal{S}$ and $A, B \in \mathcal{S}_{\mathrm{N}}$ the following sentences are in $\mathbf{P}^{\mathrm{N}}$: $\mathrm{N}A \supset A$, $\neg\mathrm{N}A \supset \mathrm{N}\neg\mathrm{N}A$, $\mathrm{N}(A \supset B) \supset (\mathrm{N}A \supset \mathrm{N}B)$, $\mathrm{N}A \mathbin{\&} \mathrm{N}B \equiv \mathrm{N}(A \mathbin{\&} B)$.
(7) If $\mathbf{P} \subset \mathcal{S}$ and $A, A \supset B \in \mathbf{P}^{\mathrm{N}}$ then $B \in \mathbf{P}^{\mathrm{N}}$.
(8) If $\mathbf{P} \subset \mathcal{S}$ then $A \in \mathbf{P}^{\mathrm{N}}$ iff $\mathrm{N}A \in \mathbf{P}^{\mathrm{N}}$.
(9) If $\mathbf{P} \subset \mathcal{S}$ then $\neg\mathrm{N}A \in \mathbf{P}^{\mathrm{N}}$ iff $A \notin \mathbf{P}^{\mathrm{N}}$.

From (7) it easily follows that for any $\mathbf{P} \subset \mathcal{S}$ and $A, B \in \mathcal{S}_{\mathrm{N}}$

(10) $$A \underset{\mathbf{P}}{\rightarrow} B \text{ iff } A \supset B \in \mathbf{P}^{\mathrm{N}},$$

(11) $$A \underset{\mathbf{P}}{\leftrightarrow} B \text{ iff } A \equiv B \in \mathbf{P}^{\mathrm{N}}.$$

Further, if $F(A)$ is an $\mathbf{L}_{\mathrm{N}}$-sentence and A, B are closed formulas then

(12) $$\text{if } A \underset{\mathbf{P}}{\leftrightarrow} B \text{ then } F(A) \underset{\mathbf{P}}{\leftrightarrow} F(B).$$

The basic fact upon which our analysis of theories of necessity depends is

Lemma 1 *Let* $\mathrm{F}(\mathrm{N}B) \in \mathcal{S}_{\mathrm{N}}$. *Then for all* $\mathbf{P} \subset \mathcal{S}$

$$\mathrm{F}(\mathrm{N}B) \equiv .\ \mathrm{N}B \mathbin{\&} \mathrm{F}(\mathbf{t}) \vee \neg\mathrm{N}B \mathbin{\&} \mathrm{F}(\mathbf{f}) \in \mathbf{P}^{\mathrm{N}}.$$

Proof: Case (i). $B \in \mathbf{P}^{\mathrm{N}}$. By (8) $\mathrm{N}B \in \mathbf{P}^{\mathrm{N}}$ so $\mathrm{N}B \underset{\mathbf{P}}{\leftrightarrow} \mathbf{t}$ and $\neg\mathrm{N}B \underset{\mathbf{P}}{\leftrightarrow} \mathbf{f}$. Then by (12), $\mathrm{F}(\mathrm{N}B) \underset{\mathbf{P}}{\leftrightarrow} \mathrm{F}(\mathbf{t}) \underset{\mathbf{P}}{\leftrightarrow} \mathrm{N}B \mathbin{\&} \mathrm{F}(\mathbf{t}) \vee \neg\mathrm{N}B \mathbin{\&} \mathrm{F}(\mathbf{f})$. Again, by (12), $\mathrm{F}(\mathrm{N}B) \equiv .\ \mathrm{N}B \mathbin{\&} \mathrm{F}(\mathbf{t}) \vee \neg\mathrm{N}B \mathbin{\&} \mathrm{F}(\mathbf{f}) \in \mathbf{P}^{\mathrm{N}}$.
Case (ii). $B \notin \mathbf{P}^{\mathrm{N}}$. By (9), $\neg\mathrm{N}B \in \mathbf{P}^{\mathrm{N}}$. Hence $\mathrm{N}B \underset{\mathbf{P}}{\leftrightarrow} \mathbf{f}$ and $\neg\mathrm{N}B \underset{\mathbf{P}}{\leftrightarrow} \mathbf{t}$. Then by (12), $\mathrm{F}(\mathrm{N}B) \underset{\mathbf{P}}{\leftrightarrow} \mathrm{F}(\mathbf{f}) \underset{\mathbf{P}}{\leftrightarrow} \mathrm{N}B \mathbin{\&} \mathrm{F}(\mathbf{t}) \vee \neg\mathrm{N}B \mathbin{\&} \mathrm{F}(\mathbf{f})$. Again, by (12), $\mathrm{F}(\mathrm{N}B) \equiv .\ \mathrm{N}B \mathbin{\&} \mathrm{F}(\mathbf{t}) \vee \neg\mathrm{N}B \mathbin{\&} \mathrm{F}(\mathbf{f}) \in \mathbf{P}^{\mathrm{N}}$. Q.E.D.

It is obvious that the above Lemma can be used to "unnest" the occurrences of the operator N in any given sentence, and, moreover, remove them from within the scope of a quantifier, to obtain a propositional combination of sentences of the form S or $\mathrm{N}S$, where $S \in \mathcal{S}$. Using standard methods of the propositional calculus we then have

Corollary 2 *There exists a recursive function* β: $\mathcal{S}_{\mathrm{N}} \rightarrow \mathcal{S}_{\mathrm{N}}$ *such that for every* $\mathrm{F} \in \mathcal{S}_{\mathrm{N}}$ *and* $\mathbf{P} \subset \mathcal{S}$, $F \underset{\mathbf{P}}{\leftrightarrow} \beta(F)$ *and* $\beta(\mathrm{F})$ *is a conjunction* $F_1 \mathbin{\&} \ldots \mathbin{\&} F_n$ *where each conjunct* F_i *is a disjunction of sentences of the form* S, $\mathrm{N}S$, *or* $\neg\mathrm{N}S$, *where* $S \in \mathcal{S}$.

Once the principles of necessity $\mathbf{P}$ have been fixed, the truth-values of formulas $\mathrm{N}S$, where $S \in \mathcal{S}$, are by (4) independent of the structure w in which the above sentences F_i are evaluated. Hence each $\mathrm{F} \in \mathcal{S}_{\mathrm{N}}$ is $\mathbf{P}$-equivalent to a sentence F' in $\mathbf{L}$. This seems to imply that $\mathbf{P}^{\mathrm{N}}$ is *merely* a transcription of $\mathbf{P}^{**}$. This conclusion, however, is not quite accurate because it follows from Theorem 3 (below) that if $\mathbf{P}^{**}$ is undecidable then the transformation of F to F' is non-recursive.

2 *Axiomatisability of theories of necessity* The possibility of axiomatising theories of necessity rests upon the following relations of reducibility between $\mathbf{P}^{\mathrm{N}}$ and $\mathbf{P}^{**}$. Below "$\underset{\mathrm{T}}{\equiv}$" denotes recursive equivalence. The reader is referred to Rogers [12] for the basic notions of recursive function theory.

Theorem 3 *Let* $\mathsf{P} \subset \mathcal{S}$. *Then* (i) $\mathsf{P}^{\mathsf{N}} \underset{T}{\equiv} \mathsf{P}^{**}$, *and*

(ii) *if* P^{N} *is recursively enumerable, then* P^{**} *is recursive.*

Before embarking on the proof of Theorem 3 we draw some conclusions from it and give some examples.

Corollary 4 *If* P^{**} *is not decidable the* P^{N} *is non-axiomatisable.*

Proof: If P^{**} is not decidable it is not recursive so by Theorem 3 (ii) P^{N} is not recursively enumerable. Hence P^{N} is not axiomatisable.

Corollary 5 P^{N} *is either non-axiomatisable or decidable.*

Proof: Suppose P^{N} is axiomatisable. Then it is recursively enumerable. By Theorem 3 (ii) P^{**} is recursive. By Theorem 3 (i) P^{N} is recursive, i.e., decidable. Q.E.D.

Example 1: The theory of logical necessity, ϕ^{N}. Let μ be a similarity class of structures with at least one binary relation. Then ϕ^{**} is the set of valid sentences of the predicate calculus with at least one two-place predicate letter. By theorems of Kalmar [4] and Church [2], ϕ^{**} is recursively enumerable but non-recursive. By Corollary 4 ϕ^{N} is non-axiomatisable.

Example 2: The theory of necessity in groups. Let $\mathfrak{G}$ be the (finite) set of group axioms (*see* Robinson [11]). The elementary theory of groups is undecidable [18]. By Corollary 4, $\mathfrak{G}^{\mathsf{N}}$ is not axiomatisable. (Being non-axiomatisable does not make a theory unusable, as any number theorist can testify.)

Example 3: The theory of necessity in real-closed fields. Let $\mathfrak{R}$ denote the set of first order axioms for real-closed fields. In contrast to $\mathfrak{G}^{\mathsf{N}}$, $\mathfrak{R}^{\mathsf{N}}$ is not a finite theory of necessity. By a result of Tarski [16], $\mathfrak{R}^{**}$ is decidable. By Theorem 3 (i), $\mathfrak{R}^{\mathsf{N}}$ is also decidable.

Proof of Theorem 3 (ii): For any $A \in \mathcal{S}$, by (5),

$$A \in \mathsf{P}^{\mathsf{N}} \text{ iff } A \in \mathsf{P}^{**}, \tag{13}$$

and by (9)

$$\neg \mathsf{N} A \in \mathsf{P}^{\mathsf{N}} \text{ iff } A \notin \mathsf{P}^{**}. \tag{14}$$

Now suppose P^{N} is recursively enumerable. From any recursive enumeration of P^{N} one can construct a recursive enumeration of those sentences of P^{N} of the form $\neg \mathsf{N} A$, where $A \in \mathcal{S}$. By (14) this is an enumeration of $\mathcal{S}$ - P^{**}. Similarly, from a recursive enumeration of P^{N} one can construct an enumeration of the N-free sentences in P^{N}. By (13) this is precisely a recursive enumeration of P^{**}. Thus, both P^{**} and $\mathcal{S}$ - P^{**} are recursively enumerable. Hence P^{**} is recursive. Q.E.D.

Proof of Theorem 3 (i): First we show that P^{N} is Turing-reducible to P^{**}. Let $\mathsf{F} \in \mathcal{S}_{\mathsf{N}}$. $\beta(\mathsf{F}) = F_1 \,\&\, \ldots \,\&\, F_n$, say, where F_i is $\neg \mathsf{N} G_1^i \vee \ldots \vee \neg \mathsf{N} G_{m_i}^i \vee$

$\mathsf{N}H_1^i \vee \ldots \vee \mathsf{N}H_{n_i}^i \vee H_{n_i+1}^i$ and $G_1^i, \ldots, H_1^i, \ldots$ are sentences of **L** (*see* Corollary 2). Then

$$\mathsf{F} \in \mathbf{P}^{\mathsf{N}} \text{ iff } F_i \in \mathbf{P}^{\mathsf{N}},\ 1 \leqslant i \leqslant n. \tag{15}$$

and, by (4), $F_i \in \mathbf{P}^{\mathsf{N}}$ iff for all structures $w \in \mathbf{P}^*$, $w \models_{\mathbf{P}} \neg\mathsf{N}G_1^i$ or . . . or $w \models_{\mathbf{P}} \mathsf{N}H_1^i$ or . . . or $w \models_{\mathbf{P}} H_{n_i+1}^i$, i.e., $F_i \in \mathbf{P}^{\mathsf{N}}$ iff $G_1^i \notin \mathbf{P}^{\mathsf{N}}$ or . . . $H_1^i \in \mathbf{P}^{\mathsf{N}}$ or . . . or $H_{n_i+1}^i \in \mathbf{P}^{**}$. As $G_1^i, \ldots, H_{n_i+1}^i \in \mathcal{S}$, by (5), $F_i \in \mathbf{P}^{\mathsf{N}}$ iff $G_1^i \notin \mathbf{P}^{**}$ or . . . or $H_{n_i+1}^i \in \mathbf{P}^{**}$. Then, by (15),

$$\mathsf{F} \in \mathbf{P}^{\mathsf{N}} \text{ iff for } 1 \leqslant i \leqslant n,\ G_1^i \notin \mathbf{P}^{**} \text{ or } \ldots \text{ or } H_{n_i+1}^i \in \mathbf{P}^{**} \tag{16}$$

(16) defines the required reduction of $\mathbf{P}^{\mathsf{N}}$ to $\mathbf{P}^{**}$. Conversely, $\mathbf{P}^{**}$ is reducible to $\mathbf{P}^{\mathsf{N}}$ by Theorem 3(i). Hence as each of $\mathbf{P}^{**}$ and $\mathbf{P}^{\mathsf{N}}$ is reducible to the other, they are recursively equivalent. Q.E.D.

3 *Axiomatisation of the general theory of necessity* Although individual theories of necessity may not be axiomatisable (examples 1, 2) the general theory is. (6) and (8) suggest some version of Lewis' S5. The aim of this section is to define a formal system π and to prove that it gives a complete axiomatisation of **N**. π is based on the system of predicate calculus as defined in [10]. The formal system π is defined as follows:

(a) Well-formed formulas. $\mathcal{S}_{\mathsf{N}}$.
(b) Axioms: all formulas in $\mathcal{S}_{\mathsf{N}}$ of one of the forms

A1. $A \supset . B \supset A$
A2. $A \supset (B \supset C) \supset . (A \supset B) \supset (A \supset C)$
A3. $\neg\neg A \supset A$
A4. $\forall x A \supset \mathsf{S}_a^x A|$, *where x and a are individual variables*
A5. $\forall x(A \supset B) \supset . A \supset \forall x B$, *where x is an individual variable*
A6. $\mathsf{N}A \supset A$
A7. $\neg\mathsf{N}A \supset \mathsf{N}\neg\mathsf{N}A$
A8. $\mathsf{N}(A \supset B) \supset (\mathsf{N}A \supset \mathsf{N}B)$,

where, in A6, A7, and A8 A and B are closed.

(c) Rules of inference:

R1. From A and $A \supset B$ to infer B,
R2. From A, if x is an individual variable, to infer $\forall x A$,
R3. From A, where A is closed, to infer $\mathsf{N}A$.

This completes the definition of π. "$\vdash_\pi A$" means that A is a theorem of π.

Theorem 6 *Let* $A \in \mathcal{S}_{\mathsf{N}}$. *If* $\vdash_\pi A$ *then* $A \in \mathbf{N}$.

Proof: Let "$\forall A$" denote the universal closure of the formula A, so, that if A is already closed $\forall A = A$. Let $\mathbf{P} \subset \mathcal{S}$. Then it follows directly from the definition of validity that $\mathbf{P}^{\mathsf{N}}$ contains all the universal closures of the axioms of π. Again, directly from the definition of validity it can be verified that

(i) if $\forall A$, $\forall(A \supset B) \in \mathbf{P}^{\mathsf{N}}$, then $\forall B \in \mathbf{P}^{\mathsf{N}}$,

(ii) if $\forall A \in \mathbf{P}^{\mathsf{N}}$, and x is an individual variable, then $\forall(\forall x A) \in \mathbf{P}^{\mathsf{N}}$,
(iii) if A is closed and $A \in \mathbf{P}^{\mathsf{N}}$, then by (8) $\mathsf{N}A \in \mathbf{P}^{\mathsf{N}}$.

Hence by induction on the length of proof that if $\vdash_{\pi} A$, then $\forall A \in \mathbf{P}^{\mathsf{N}}$. Thus if A is closed and $\vdash_{\pi} A$ then $A \in \mathbf{P}^{\mathsf{N}}$. Q.E.D.

The converse of Theorem 6 is based on a syntactical version of Lemma 1.

Lemma 7 *Let* $\mathsf{F}(\mathsf{N}B) \in \mathcal{S}_{\mathsf{N}}$. *Then*

$$\vdash_{\pi} \mathsf{F}(\mathsf{N}B) \equiv . \ \mathsf{N}B \ \& \ \mathsf{F}(\mathbf{t}) \vee \neg \mathsf{N}B \ \& \ \mathsf{F}(\mathbf{f}) \tag{17}$$

Proof: By induction on the number n of logical symbols in $\mathsf{F}(\mathbf{t})$.

Basis: $n = 0$. Then $\mathsf{F}(\mathsf{N}B)$ is $\mathsf{N}B$. Using the propositional calculus rules of π, $\vdash_{\pi} A \equiv . \ A \ \& \ \mathbf{t} \vee \neg A \ \& \ \mathbf{f}$ for all $A \in \mathcal{S}_{\mathsf{N}}$. In particular, $\vdash_{\pi} \mathsf{N}B \equiv . \ \mathsf{N}B \ \& \ \mathbf{t} \vee \neg \mathsf{N}B \ \& \ \mathbf{f}$. Thus (17) holds for $n = 0$.

Inductive step. Suppose (17) holds for all formulas $\mathsf{F}(\mathsf{N}B)$ for which $\mathsf{F}(\mathbf{t})$ has $\leqslant n$ logical symbols. Let $\mathsf{F}(\mathsf{N}B)$ be a formula of $\mathcal{S}_{\mathsf{N}}$ such that $\mathsf{F}(\mathbf{t})$ has $n+1$ logical symbols. Then $\mathsf{F}(\mathbf{t})$ has one of the four forms: (i) $H(\mathbf{t}) \supset J$, (ii) $J \supset H(\mathbf{t})$, (iii) $\forall x H(\mathbf{t})$, (iv) $\mathsf{N}H(\mathbf{t})$, where $H(\mathbf{t})$ and J are formulas of $\mathcal{S}_{\mathsf{N}}$ with at most n logical symbols each. By the inductive hypothesis

$$\vdash_{\pi} H(\mathsf{N}B) \equiv . \ \mathsf{N}B \ \& \ H(\mathbf{t}) \vee \neg \mathsf{N}B \ \& \ H(\mathbf{f}) \tag{18}$$

case (i). $\mathsf{F}(\mathsf{N}B)$ is $H(\mathsf{N}B) \supset J$. (17) then follows from (18) by propositional calculus.

case (ii). $\mathsf{F}(\mathsf{N}B)$ is $J \supset H(\mathsf{N}B)$. Similar to case (i).

case (iii). $\mathsf{F}(\mathsf{N}B)$ is $\forall x H(\mathsf{N}B)$. By propositional calculus

$$\vdash_{\pi} \mathsf{N}B \ \& \ H(\mathbf{t}) \vee \neg \mathsf{N}B \ \& \ H(\mathbf{f}) \equiv . \ (\mathsf{N}B \supset H(\mathbf{t})) \ \& \ (\neg \mathsf{N}B \supset H(\mathbf{f})).$$

Hence using (18),

$$\vdash_{\pi} \forall x H(\mathsf{N}B) \equiv . \ \forall x(\mathsf{N}B \supset H(\mathbf{t})) \ \& \ \forall x(\neg \mathsf{N}B \supset H(\mathbf{f})). \tag{19}$$

But B is closed so that $\vdash_{\pi} \forall x(\mathsf{N}B \supset H(\mathbf{t})) \equiv . \ \mathsf{N}B \supset \forall x H(\mathbf{t})$ and $\vdash_{\pi} \forall x(\neg \mathsf{N}B \supset H(\mathbf{f})) \equiv . \ \neg \mathsf{N}B \supset \forall x H(\mathbf{f})$. Hence, by (18)

$$\vdash_{\pi} \forall x H(\mathsf{N}B) \equiv . \ (\mathsf{N}B \supset \forall x H(\mathbf{t})) \ \& \ (\neg \mathsf{N}B \supset \forall x H(\mathbf{f})).$$

Then by propositional calculus,

$$\vdash_{\pi} \forall x H(\mathsf{N}B) \equiv . \ \mathsf{N}B \ \& \ \forall x H(\mathbf{t}) \vee \neg \mathsf{N}B \ \& \ \forall x H(\mathbf{f}).$$

Thus (17) holds for $\forall x H(\mathsf{N}B)$.

case (iv). $\mathsf{F}(\mathsf{N}B)$ is $\mathsf{N}H(\mathsf{N}B)$. One can readily see from the rules and axioms of π (see S5 results in [3]) that for all $A, B \in \mathcal{S}_{\mathsf{N}}$ $\vdash_{\pi} \mathsf{NN}A \supset \mathsf{N}A$, $\vdash_{\pi} \mathsf{N}(A \ \& \ B) \equiv . \ \mathsf{N}A \ \& \ \mathsf{N}B$, $\vdash_{\pi} \mathsf{N}(\mathsf{N}A \supset B) \supset . \ \mathsf{N}A \supset \mathsf{N}B$, and $\vdash_{\pi} \mathsf{N}(\mathsf{N}A \vee B) \equiv . \ \mathsf{N}A \vee \mathsf{N}B$. Now, as in case (iii), using propositional calculus we have from (18) that

$$\vdash_{\pi} \mathsf{N}H(\mathsf{N}B) \equiv . \ \mathsf{N}((\mathsf{N}B \supset H(\mathbf{t})) \ \& \ (\neg \mathsf{N}B \supset H(\mathbf{f}))). \tag{20}$$

Using (20) and the theorems preceding it we have

$$\vdash_\pi \mathsf{N}H(\mathsf{N}B) \equiv. (\mathsf{N}B \supset \mathsf{N}H(\mathbf{t})) \& (\mathsf{N}B \vee \mathsf{N}H(\mathbf{f})).$$

Hence, $\vdash_\pi \mathsf{N}H(\mathsf{N}B) \equiv. \mathsf{N}B \& \mathsf{N}H(\mathbf{t}) \vee \neg\mathsf{N}B \& \mathsf{N}H(\mathbf{f})$. Thus (17) holds for $\mathsf{N}H(\mathsf{N}B)$.

The case analysis for the inductive step is now complete. The theorem may follows by induction. Q.E.D.

Since the system π contains the full classical propositional calculus, the process of unnesting the modal operators and putting the result in normal form as in Corollary 2 can actually be effected in π.

Corollary 8 *For each* $\mathsf{F} \in \mathcal{S}_{\mathsf{N}}$, $\vdash_\pi \mathsf{F} \equiv \beta(\mathsf{F})$.

Lemma 9 *Let* $G_1, \ldots, G_n, H_1, \ldots, H_{m+1} \in \mathcal{S}$. *Let* F *be the formula* $\neg\mathsf{N}G_1 \vee \ldots \vee \neg\mathsf{N}G_n \vee \mathsf{N}H_1 \vee \ldots \vee \mathsf{N}H_m \vee H_{m+1}$ *and* $\mathbf{P} = \{G_1, \ldots, G_n\}$. *Then:*

(i) *If* $\mathsf{F} \in \mathbf{P}^{\mathsf{N}}$, *then for some* j, $1 \leqslant j \leqslant m+1$, $G_1 \& \ldots \& G_n \supset H_j$ *is a universally valid sentence of* $\mathbf{L}$.
(ii) *If for some* j, $1 \leqslant j \leqslant n$, $G_1 \& \ldots \& G_n \supset H_j$ *is universally valid, then* $\vdash_\pi \mathsf{F}$.

Proof: (i) Suppose $\mathsf{F} \in \mathbf{P}^{\mathsf{N}}$. Then for all $w \in \mathbf{P}^*$, $w \models_{\mathbf{P}} \neg\mathsf{N}G_1 \vee \ldots \vee \mathsf{N}H_m \vee H_{m+1}$. As $G_1, \ldots, H_{m+1}$ are N-free, by (2) and the definition of validity, for all $w \in \mathbf{P}^*$, $G_1 \notin \mathbf{P}^{**}$ or $\ldots$ or $G_n \notin \mathbf{P}^{**}$ or $H_1 \in \mathbf{P}^{**}$ or $\ldots$ or $H_m \in \mathbf{P}^{**}$ or $w \models H_{m+1}$. Thus one of the $n+m+1$ conditions $G_i \notin \mathbf{P}^{**}$, $1 \leqslant i \leqslant n$, $H_j \in \mathbf{P}^{**}$, $1 \leqslant j \leqslant m+1$ must hold. But $G_i \in \mathbf{P} \subset \mathbf{P}^{**}$. Hence one of the conditions $H_j \in \mathbf{P}^{**}$ holds. Thus $\mathbf{P} \rightarrow H_j$ for some j, $1 \leqslant j \leqslant m+1$. Hence $G_1 \& \ldots \& G_n \supset H_j$ is universally valid for some j, $1 \leqslant j \leqslant m+1$.
(ii) Suppose $G_1 \& \ldots \& G_n \supset H_j$ is universally valid. By the completeness theorem of the predicate calculus $\vdash_\pi G_1 \& \ldots \& G_n \supset H_j$. By R3, $\vdash_\pi \mathsf{N}(G_1 \& \ldots \& G_n \supset H_j)$. By R1, A8 and the results quoted in Lemma 7, $\vdash_\pi \mathsf{N}G_1 \& \ldots \& \mathsf{N}G_n \supset \mathsf{N}H_j$. If $j \leqslant m$ then $\vdash_\pi \mathsf{N}G_1 \& \ldots \& \mathsf{N}G_n \supset. \mathsf{N}H_1 \vee \ldots \vee \mathsf{N}H_m \vee H_{m+1}$. Hence $\vdash_\pi \mathsf{F}$. But if $j = m+1$, by A6, $\vdash_\pi \mathsf{N}H_{m+1} \supset H_{m+1}$. Hence $\vdash_\pi \mathsf{N}G_1 \& \ldots \& \mathsf{N}G_n \supset H_{m+1}$. From this, $\vdash_\pi \mathsf{N}G_1 \& \ldots \& \mathsf{N}G_n \supset. \mathsf{N}H_1 \vee \ldots \vee \mathsf{N}H_m \vee H_{m+1}$. Hence $\vdash_\pi \mathsf{F}$. Q.E.D.

The converse of Theorem 6 can now be proved.

Theorem 10 *Let* $\mathsf{F} \in \mathcal{S}_{\mathsf{N}}$. *If* $\mathsf{F} \in \mathbf{N}$, *then* $\vdash_\pi \mathsf{F}$.

Proof: Suppose $\mathsf{F} \in \mathbf{N}$. By Corollary 2 $\beta(\mathsf{F}) \in \mathbf{N}$. $\beta(\mathsf{F}) = F_1 \& \ldots \& F_n$, where F_i is $\neg\mathsf{N}G_1^i \vee \ldots \vee \neg\mathsf{N}G_{n_i}^i \vee \mathsf{N}H_1^i \vee \ldots \vee \mathsf{N}H_{m_i}^i \vee H_{m_i+1}^i$, say, where $G_1^i, \ldots, H_{m_i+1}^i \in \mathcal{S}$. As $\beta(\mathsf{F}) \in \mathbf{N}$, $F_i \in \mathbf{N}$ for $1 \leqslant i \leqslant n$. By the definition of $\mathbf{N}$, $F_i \in \{G_1^i \ldots G_{n_i}^i\}^{\mathsf{N}}$, $1 \leqslant i \leqslant n$. By Lemma 9 (i), (ii), $\vdash_\pi F_i$ for $1 \leqslant i \leqslant n$. Hence $\vdash_\pi \beta(\mathsf{F})$. By Corollary 8, $\vdash_\pi \mathsf{F}$. Q.E.D.

Corollary 11 *Let* $\mathsf{F} \in \mathcal{S}_{\mathsf{N}}$. $\mathsf{F} \in \mathbf{N}$ *iff* $\vdash_\pi \mathsf{F}$.

Corollary 12 $\mathbf{N}$ *is recursively enumerable.*

Corollary 12 contrasts with examples 1 and 2.

A slight refinement of Theorem 10, which is based ultimately on the compactness theorem of the predicate calculus, shows that if a sentence of $\mathcal{S}_{\mathsf{N}}$ is not provable in π then there exists a finite "counterexample".

Theorem 13 *Let* $\mathsf{F} \in \mathcal{S}_{\mathsf{N}}$. *If* $\nvdash_{\pi} \mathsf{F}$, *then there exists* $S \in \mathcal{S}$ *such that* $\mathsf{F} \notin \{S\}^{\mathsf{N}}$.

Proof: $\beta(\mathsf{F}) = \bigwedge_{i=1}^{n} F_i$ where F_i is $\neg \mathsf{N} G_1^i \vee \ldots \vee \neg \mathsf{N} G_{n_i}^i \vee \mathsf{N} H_1^i \vee \ldots \vee \mathsf{N} H_{m_i}^i \vee H_{m_i+1}^i$. Suppose $\nvdash_{\pi} \mathsf{F}$. By Corollary 8 $\nvdash_{\pi} \beta(\mathsf{F})$. Hence for some i, $1 \leqslant i \leqslant n$, $\nvdash_{\pi} F_i$. Let S be $G_1^i \;\&\; \ldots \;\&\; G_{n_i}^i$. By Lemma 9 (ii), for all j, $1 \leqslant j \leqslant m_i$, $G_1^i \;\&\; \ldots \;\&\; G_{n_i}^i \supset H_j^i$ is not universally valid. By Lemma 9 (i), $F_i \notin \{G_1^i, \ldots, G_{n_i}^i\}^{\mathsf{N}} = \{S\}^{\mathsf{N}}$. Hence $\beta(\mathsf{F}) \notin \{S\}^{\mathsf{N}}$ so $\mathsf{F} \notin \{S\}^{\mathsf{N}}$. Q.E.D.

The general theory of necessity is defined by quantifying over *all* subsets of $\mathcal{S}$. Theorem 14 (below) enables us to reduce this definition to first order form in a natural way.

Theorem 14 $\mathbf{N} = \bigcap \{\{S\}^{\mathsf{N}};\ S \in \mathcal{S}\}$.

Proof: Clearly, $\mathbf{N} \subset \bigcap \{\{S\}^{\mathsf{N}};\ S \in \mathcal{S}\}$. Suppose $\mathsf{F} \in \mathcal{S}_{\mathsf{N}}$. If $\mathsf{F} \notin \mathbf{N}$, then, by Theorem 6, $\nvdash_{\pi} \mathsf{F}$. By Theorem 13, for some $S \in \mathcal{S}$, $\mathsf{F} \notin \{S\}^{\mathsf{N}}$. Hence $\mathbf{N} \supset \{\{S\}^{\mathsf{N}};\ S \in \mathcal{S}\}$. Q.E.D.

Thus, the general theory of necessity is the set of sentences common to all finite theories of necessity.

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University of Bristol
Bristol, England