Notre Dame Journal of Formal Logic
Volume 26, Number 4, October 1985

Relevance and Conformity

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The Ackermann-Anderson-Belnap systems E of entailment and R of relevant implication possess three properties in virtue of which they are said to be "relevance" (or "relevant") logics.

First, whenever an entailment $A \rightarrow B$ is provable in E or R, the formulas A and B share at least one propositional variable. The relevantists maintain that in a true entailment or implication there must be some connection in meaning or content between antecedent and consequent, and they view the variable-sharing condition as a formal counterpart of this idea.

Second, E and R are *paraconsistent* in the sense that in these systems the deductive effects of inconsistency are minimized and, accordingly, there are theories based on these systems that are negation-inconsistent and yet not trivial.

Third, the theorems of R (or better, the nontheorems of R) reflect what relevantists believe to be a corrected conception of proof from hypotheses, according to which in a correct proof there can be no extraneous or unused hypotheses. It is for this reason that such principles as

(1) $A \rightarrow . B \rightarrow A$

are rejected. For how, the argument goes, does A follow *from B*, given only the assumption that A is true?

These three properties are not independent. For example, given the proof-theoretical motivation (the third property), paraconsistency (the second property) is inevitable in that the pure implicational fragment of R, which embodies the proof-theoretical motivation, cannot be conservatively extended by the addition of a theory of truth-functional inference that contains

(2) $\sim A \,\&\, (A \vee B) \rightarrow B$. (Disjunctive Syllogism, DS)

In such an extension it is possible to prove (1) and the like. But given the usual

*A version of this paper was read at the Spring meeting of the Association for Symbolic Logic, March 1983.

Received January 23, 1984; revised April 16, 1984

principles governing disjunction and conjunction, the rejection of (2) entails paraconsistency.

The study of E and R and related logics has proved quite rich both philosophically and technically. It is remarkable that apparently naive intuitions about proof from hypotheses have led to such a wealth of ideas, though perhaps this is less surprising if one observes that a concern for relevance as a component of inference bears at least a family resemblance to constructivist concerns.

Yet many people remain perennially dissatisfied with the Anderson-Belnap logics. They complain that the systems are not well-motivated, that the model theory is unilluminating, and the proof-theory opaque. A large part of the problem is simply that E and R are hard to understand. No one for example has to my knowledge yet devised a relational semantics for RQ, the first-order quantificational extension of R. Problems arise in the attempt to extend the relational semantics of R, due to Routley, Meyer, and Fine, to the case of quantification.[1]

I want to draw attention to one obvious source of the complexity and opacity of the Anderson-Belnap systems, and I want to suggest a way of constructing a theory of entailment and implication with strong relevance properties that possesses a simple and transparent structure.

A striking feature of the relevant logics is that they do not conform to the Boolean Order of Things—according to which nothing can be more true than a theorem of logic nor more false than the denial of a theorem of logic. Equivalently, E and R do not satisfy the principle of *conformity* (to the Boolean Order) that if A is a theorem, then the negation of A must entail A. Thus the models of R provide a way to give greater credence to the denials of theorems than is given to the theorems themselves, which makes mincemeat of Boolean priorities.[2]

But conformity guarantees a certain simplicity of structure as is apparent in the contrast between R and its conforming neighbor RM (R-Mingle) which is obtained from R by adding the axioms of the form

(3) $A \rightarrow . A \rightarrow A$.

The algebraic models of RM, for example, are simple chains with the truer points above and the more false points below, just as it should be according to Boolean law. RM, however, is not a true relevance logic since adding (3) to R makes it possible prove

(4) $\sim(A \rightarrow A) \rightarrow . B \rightarrow B$,

and this formula fails all tests of relevance.[3]

So it seems that relevance and conformity are somewhat at odds; and in the interests of a simple theory of entailment, it is useful to see to what extent they can be reconciled.

In the case of entailments between truth-functional formulas a full reconciliation is possible. Consider a language in which there are two kinds of formulas, truth-functional formulas constructed as usual from propositional variables and constants &, $\vee$, $\sim$ for conjunction, disjunction, and negation, and entailments $A \rightarrow B$, where A and B are truth-functional formulas. Thus entailment is here represented as a relation rather than an operation capable of iteration. We take as "truth-values" all of the subsets of a two-element set, $\{t, f\}$, so that we can now say that a proposition is *neither* true nor false (i.e., its value is the empty set), or even *both* true and false (i.e., its value is $\{t, f\}$), as well as just plain true ($\{t\}$) or false ($\{f\}$). A model is simply a mapping of the propositional variables into the set of four values, and the value of a formula in a model is determined by the following rules:

We say that a model m is defined at a variable p iff $m(p) \neq \varnothing$, and that m is defined at a truth-functional formula A iff m is defined at each variable occurring in A. m is then extended to a valuation m' defined for all truth-functional formulas as follows:

(5) (*i*) $m'(A) = \varnothing$ iff m is not defined at A.
(ii) $t \in m'(\sim B)$ iff $f \in m'(B)$
$f \in m'(\sim B)$ iff $t \in m'(B)$.
(iii) $t \in m'(B \,\&\, C)$ iff $t \in m'(B)$ and $t \in m'(C)$
$f \in m'(B \,\&\, C)$ iff $f \in m'(B)$ or $f \in m'(C)$.
(iv) $t \in m'(B \vee C)$ iff $t \in m'(B)$ or $t \in m'(C)$
$f \in m'(B \vee C)$ iff $f \in m'(B)$ and $f \in m'(C)$.

An entailment $A \rightarrow B$ is valid if in each model, A is at least true only if B is; that is, $A \rightarrow B$ is valid if $t \in m'(A)$ only if $t \in m'(B)$, for every model m.[4]

Notice first that if an entailment $A \rightarrow B$ is valid, then every propositional variable appearing in B also appears in A, and so the variable-sharing criterion of relevance is satisfied. Second, the semantics is paraconsistent since (2) is not valid. And third, if $A \rightarrow B$ is valid, then B really does follow *from* A in the favored sense of Anderson and Belnap.

This last claim requires a brief defense, since, for example,

(6) $A \vee B \rightarrow B \vee \sim B$

is valid and (6) is just the sort of thing the relevantists find objectionable. However, according to these semantics disjunction is, as I would put it, "explicitly truth-functional": a disjunction is at least true if and only if one or other of A and B is at least true and the other is defined. We must have information about *both* A and B in order to affirm $A \vee B$. Given this understanding, (6) should be acceptable even to those with the strictest standards of relevance, for $A \vee B$ affirms one or other of $A \,\&\, B$, $A \,\&\, \sim B$, and $\sim A \,\&\, B$, and from each of these $B \vee \sim B$ follows relevantly.[5]

Thus for the case of entailment between truth-functions this simple four-valued logic provides a way to obtain as much relevance among entailments as anyone could want, and at a much cheaper price than is paid by the Anderson-Belnap logics. For notice that the semantics is conforming: A theorem of ordinary (two-valued) propositional logic is never simply false according to the semantics and the negation of a theorem is never simply true. The moral is that it is possible to have a simple theory of relevant entailment between truth-functions that makes no appeal to the philosophically dubious concept of an impossible possible world – a world in which what is logically undeniable is flatly denied, and what logically cannot be affirmed is flatly affirmed. Philosophically this is important because a common philosophical objection to the enterprise of relevance logic is its apparent dependence on the concept of impossible worlds. From another though equivalent viewpoint, conformity means that the four-valued semantics contains the ordinary two-valued semantics in a certain direct way. In fact, a truth-functional formula A is valid in the two-valued sense if and only if $\sim A \rightarrow A$ is a valid entailment in the four-valued sense.[6]

These ideas can be extended to many-degree systems in which entailment and implication are full-fledged connectives capable of iteration. In this case, the models for entailment are more complicated. They consist of a set K of points or worlds, with a distinguished point $G \in K$, a binary transitive and reflexive relation R defined on K, a function C that associates a semilattice $C_H = (C, o)$ with each point in K, and a function f that assigns to each propositional variable p and point $H \in K$ an element $f_H(p)$ of the associated semilattice C_H.

Thus each variable is given a content or meaning at each world in K, and the semilattices provide that the content of a compound formula is a function of the content of its parts. In addition, each formula has a truth value at each world in K as recursively determined, in the case of truth-functional formulas, by rules similar to those mentioned earlier except that now there are only three truth values: *true*, *false*, and *both* true and false. The semantical clause for entailment is as follows:

(7)[7] $t \in \|A \rightarrow B, H\|$ iff for each $H' \in K$ such that HRH' both:
(a) $f_{H'}(B) \leq f_{H'}(A)$
(b) $t \in \|A, H'\|$ only if $t \in \|B, H'\|$.

$f \in \|A \rightarrow B, H\|$ iff either
(a) $t \notin \|A \rightarrow B, H\|$ or
(b) $t \in \|A, H\|$ and $f \in \|B, H\|$.

($a \leq b$ iff $a \circ b = b$, for a, b, C_H.) So an entailment $A \rightarrow B$ is true only if the content of A contains that of B, a condition somewhat reminiscent of Kant's notion of an analytic proposition.[8]

Similar models for a nonmodal form of implication are obtained by requiring that the accessibility relation be linear, and that there be a single semilattice of contents and a single content assignment function, the same for each world in K. In addition, the models must satisfy a familiar sort of hereditary condition to the effect that the truth value of a proposition variable at a given world H in K is contained in the value of the formula at any world accessible from H. This condition extends to all formulas.

It is straightforward to axiomatize the classes of valid formulas determined by these two sorts of models, and proofs of strong completeness are more or less routine though some extra work is required to deal with the concept of the content of a formula and with the fact that the systems are paraconsistent and many-valued (see [3] and [5]).

It is worthwhile to look briefly at the way the concept of the content of a formula appears in the axioms. If A is a formula, $\phi(A)$ is the formula $(P_1 \rightarrow P_1) \& \ldots \& (P_n \rightarrow P_n)$, where $P_1, \ldots, P_n$ lists all the propositional variables in A (in alphabetical order and without repetitions). It is intended that formulas of the form

(8) $A \to \phi(B)$

say that the content of B is contained in that of A, so that $A \to \phi(B)$ will be provable iff each variable in B is in A. For the system of entailment the axioms involving this notion are the following:

(9) (a) $A \to \phi(A)$
(b) $A \to B \to. \phi(A) \to \phi(B)$
(c) $A \,\&\, \phi(C) \to B \to. A \,\&\, \phi(B) \to B$
(d) $(A \to B) \to. C \to (A \to B)$, provided $\phi(C) = \phi(A \to B)$.

The first two axioms guarantee that in any theory T based on the system (i.e., any set of formulas containing all of the axioms of the system and closed under its rules of inference), $A \to B$ will be provable in T only if, according to T, the content of B is contained in that of A; that is, only if $A \to \phi(B)$ is provable in T. The restriction on (d) is needed to ensure that (d) is compatible with this requirement. (c) says simply that the content of $\phi(B)$ is the maximal content of B. This is needed in the proof of an appropriate deduction theorem.

For the system of nonmodal implication, (d) is strengthened to

(e) $A \to (B \to A)$, provided $\phi(A) = \phi(B)$,

and the following unwieldy formula

(f) $(A \to \phi(B)) \vee [(A \to \phi(B) \,\&\, \phi(C) \to C]$

is added to ensure that the relation of content containment is two-valued, so that there need be in effect only one semilattice of contents per model.

It happens that these systems are weakly complete for their respective classes of consistent models—models wherein no formula is both true and false at the distinguished point $G \in K$, and hence that Disjunctive Syllogism and detachment for the material conditional are admissible rules. By somewhat modified filtrations arguments, these logics are decidable, and the theory of entailment contains the Lewis system $S4$ on translation.[9] Various quantificational extensions of these "conforming relevance logics"—as they might as well be called—can be formulated and completely characterized. In the most straightforward of these, the variable-sharing condition is extended to predicate constants so that every predicate appearing in the consequent of a provable entailment also appears in its antecedent.

Admittedly, the most ardent proponent of the Anderson-Belnap approach won't find these conforming relevance logics entirely satisfactory, for the models invalidate some principles that hold in R or in E and they validate others that do not hold in R or in E. For example, transitivity in the form

(10) $A \to B \to (B \to C \to. A \to C)$

is not valid on these semantics. However, transitivity in the form

(11) $(A \to B) \,\&\, (B \to C) \to. A \to C$

is valid; and (11) is all that is really needed to ensure that one can get from one end of a chain of inferences to the other. This sort of trade-off is common in the relevance logics; thus, for example,

(12) $A \to (B \to. A \& B)$

does not hold in E or R (nor in the systems I am proposing) and so adjunction is taken as a primitive rule.

Similarly, contraposition

(13) $A \to B \to. \sim B \to \sim A$

fails in the conforming relevance logics though it holds in the Anderson-Belnap systems. This may seem to be a serious flaw, since contraposition certainly corresponds to an intuitive and useful form of proof. Yet, as mentioned, in order to preserve the fundamental motivation concerning a "corrected" concept of proof from hypotheses, the Anderson-Belnap logics must omit *DS*; and as Burgess has convincingly argued, *DS* clearly corresponds to an intuitive and useful form of proof.[10] It seems inevitable that in the attempt to make logical sense of somewhat hazy intuitions about a desired connection between the premises and conclusion of an inference *some* otherwise plausible principles are lost. After all, it is these losses that give structure to the technical enterprise, which in turn serves to clarify the original intuitions. Of course, the losses cannot be too great lest the founding intuitions be pronounced incoherent. In any case, in the conforming relevance logics both *DS* and contraposition are available whenever they are really needed, namely in any negation-consistent theory based on these systems.

There is a basic difference between the strategy embodied in *E* and *R* for rejecting the unwanted, irrelevant entailments, and the strategy for doing the same that is embodied in the alternative systems sketched here. The strategy of E and R is direct and blunt: To reject an entailment one must find a model in which the antecedent is true and the consequent is false. Thus to reject the entailment

(14) $B \to (A \to A)$

requires an "impossible world" in which B is true and $A \to A$ is simply false. In the models for conforming entailment and implication there are no such impossible worlds, and $A \to A$ is never simply false. Instead, to reject (14) it is sufficient to note that the content of $A \to A$ may not form a part of the content of B.

It seems to me that this second strategy is more in accord with the intuitions that led to the development of relevance logic. In arguing their case relevance logicians often advert to the following quite convincing sort of example: Suppose a mathematician succeeds in proving that some significant mathematical statement B follows from some other significant mathematical statement A, but that neither A nor B is known to be true. It is subsequently discovered that A is false and B is true. It is doubtful whether anyone would seriously claim that the original proof that A entails B has now been supplanted by two entirely trivial proofs of the same result, one of which relies on the idea that B, being true and hence incontrovertible, follows, according to elementary logic, from any statement whatsoever, including A; and the other of which relies on the idea that A, being false, is incontrovertibly false, and hence, according to elementary logic, entails anything whatsoever, including B.

Examples such as this are quite forceful and they do show that there is some butterfly of entailment worth pursuing with some sort of net. But now the Anderson-Belnap strategy seems to be that the two trivial proofs are not legitimate because A, though incontrovertibly false, might be true, and B, though incontrovertibly true, might be false! And this certainly seems to be the wrong explanation.

A final point. The Anderson-Belnap strategy not only succeeds in rejecting (14), it also provides a counterexample to (3), and this, it seems to me, throws out the baby with the bathwater. For as Meyer has observed ([1], Section 29.3), what the rejection of (3) seems to suggest is that it matters how many times one assumes that A is true. Assuming A once is sufficient to prove A, since A entails A. But assuming A twice is not enough, since (3) is not valid.

NOTES

1. Cf. [12]. Kit Fine has, I believe, proved that EQ is not complete with respect to the relational semantics.

2. This characterization of conformity depends on the properties of negation in the system; and it may be argued that the fundamental proof-theoretic motivation for the relevant logics is independent of the properties of negation. But for negation-free pure implicational formulas and positive extensions thereof, the question of conformity amounts to the question whether the formulas of the form $A \rightarrow . A \rightarrow A$ are theorems. The system is conforming if and only if these formulas are theorems (cf. [4]).

3. See [1] and [7] for the facts about RM.

4. The semantics closely resembles that given by Dunn in [8] for tautological entailment. The main difference lies in the treatment of truth value "gaps" (clause (i) of (5)). The two semantical schemes nevertheless determine very different classes of valid entailments. For example, (6) is not a valid tautological entailment, though it is valid in the sense here defined; and $A \rightarrow A \vee B$ is not valid in the sense here defined, though it is a valid tautological entailment.

5. The reason for calling disjunction, as here characterized, "explicitly truth-functional" is that classically a disjunction $A \vee B$ is true if one or other of A and B is true; but in stating these truth conditions it is tacitly assumed that each of A and B is either true or false. By making this assumption explicit we obtain explicitly truth-functional disjunction.

6. See [4] for various facts about this four-valued logic, including an interesting form of the Craig-Lyndon interpolation theorem.

7. Thus the semantics somewhat resembles that given by Dunn in [9] and [10] for the systems E+Mingle (E+M) and R-Mingle (RM). There are two main differences: first, the semilattices of contents do not play a role in Dunn's semantics; and second, Dunn requires that true implicative formulas preserve both truth *and* falsity. Here it is only required that they preserve truth.

8. See the reference in [3] and [5] to the work of Dunn, Fine, and Urquhart on "analytic" implication and entailment. Epstein's work (in [11] and elsewhere) on relatedness and dependency logic is also of interest in this regard.

9. See [3], [5], and [6] for proofs of decidability and admissibility of rules. The result about entailment and *S*4 is proved in [4].

10. See [2]. However, Burgess's arguments do not refute the relevantists, for the latter would agree that *in the right circumstances*, *DS* represents a correct form of reasoning. The "right circumstances" are those in which we are reasoning with negation-consistent information. Burgess also fails to note that, as mentioned (p. 455), the rejection of *DS* is more or less inevitable once one accepts the pure implicational principles representing the relevantists' altered conception of proof from hypothesis.

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