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Independence in Higher-Order Subclassical Logic

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Introduction A formal logic can be distinguished in a variety of ways. In the case of intuitionistic logic or classical logic it may carry an intended interpretation. If that is lacking, however, it still can be distinguished by the theorems it generates. Indeed, between the intuitionistic and the classical there extends a whole hierarchy of uninterpreted logics (the so-called intermediate logics), generally known by their theorems alone.

A third point of view is possible. Within the logic certain interdefinabilities of connectives and quantifiers may arise and this would allow one to study the formalism for redundancy (or lack of it) among the primitives used to present it. First-order classical logic has a high degree of redundancy, typical examples being the dualities between $\vee$ and & and between $\exists$ and $\forall$. In higher-order classical logic we have the curious fact (reported by Henkin [2]) that all connectives and quantifiers are definable just from equality. Intuitionistic logic presents a quite different picture. Prawitz [3] reports a complete independence among connectives and quantifiers within first-order intuitionistic logic. Passing to higher-order intuitionistic logic, he finds a return to redundancy involving a distinguished pair of primitives. He shows that in higher-order intuitionistic logic all connectives and quantifiers are definable from $\rightarrow$ and $\forall$ alone.

A central purpose of this paper is to reexamine the $\rightarrow$, $\forall$ definabilities of Prawitz from the point of view of independence. The question to ask is obviously this: Are there other redundancies possible for higher-order intuitionistic logic and, if not, just how far through the hierarchy of intermediate (higher-order) logics does the resulting independence of connectives and quantifiers persist?

If one restricts attention to a reasonably standard list of primitives, namely $\neg$, $\vee$, &, $\rightarrow$, $\exists$, and $\forall$, then a fairly pleasant pattern emerges.[1] Starting

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with intuitionistic logic and extending considerably beyond through other intermediate logics the following are true:

(*) neither $\rightarrow$ nor $\forall$ is definable
(**) any definition of $\neg$, $\vee$, &, or $\exists$ requires use of both $\rightarrow$ and $\forall$.[2]

Here (see Prawitz [3]) a connective or quantifier $\otimes$ is to be definable in a logic L if for each formula A in the language of L which contains $\otimes$ there exists an L-equivalent formula A' which does not. To say a definition of $\otimes$ requires $\square$ means there exists a particular A containing $\otimes$, no L-equivalent A' of which lacks both $\otimes$ and $\square$.[3]

A related task of the paper will be to present certain proof techniques by which the independencies just named can be proved. The full pattern involves verifying ten separate items. In this paper we shall illustrate the proof ideas by demonstrating three of the ten, namely,

I. $\forall$ is not definable
II. any definition of & requires use of $\rightarrow$
III. any definition of $\vee$ requires use of $\forall$.

The remaining seven independencies have proofs modeled on the demonstrations given for I and II. We include III as it is a special case and deserves its own discussion.

The independence proofs are adaptable to higher-order logic in the broadest sense, namely, to arbitrary type theory. To simplify exposition we will present the three proofs in a highly restricted context. We will work in a weak form of second-order logic which omits use of λ-abstraction and has only two types of variables, both of which are "propositional" in the sense of Henkin [2]. Following the proofs we will discuss their generalization to arbitrary type theory. We will also indicate by examples how far beyond intuitionism and into intermediate logic the independencies persists intact.

Although technical in nature, the investigations reported in this paper do have broader philosophical implications. Historically, intuitionistic logic has been defined and marked out by an intended interpretation of its connectives and quantifiers. It is now apparent that above the first order this logic possesses an underlying structure (an independence pattern among primitives) which is preserved intact in a still larger domain not covered by such interpretation. Perhaps more of subclassical logic is capable of useful interpretation than is commonly realized. A final purpose of this paper will be to suggest such an interpretation.

1 Overview of the proof ideas

The approach employs semantic structures featuring "truncated" extensional equality. With each type τ in the language is associated a set M^τ (the τ-objects) together with a binary map $\sim_\tau$ from M^τ to M^1 (the domain of truth values). The map $\sim_\tau$ resembles equality in that it is symmetric and transitive but since $\phi \sim_\tau \phi$ is not assumed to be $\top$ (truth) for given $\phi \in M^\tau$, it may fail to be reflexive. Nevertheless it will be extensional: if $\tau = (\sigma)\pi$ (the type of functions from σ-objects to π-objects) then for given $\phi \in M^\tau$ one has $(\theta \sim_\sigma \theta') \rightarrow (\phi(\theta) \sim_\pi \phi(\theta'))$ is $\top$ for every $\theta, \theta' \in M^\sigma$.[4] The independence proofs hinge on studying valuations of the formal language onto such semantic structures. These valuations amount to mappings V_τ (one for each τ) of τ-terms into the set M^τ, where the mappings satisfy certain obvious conditions. In particular, formulas (terms of type 1) get mapped into M^1 (truth values), which is taken to be a complete Heyting lattice H.[5] V-valid formulas (those mapped to $\top \in M^1$) form a deductively closed set which, if including the theorems of a logic L, can be used to gather information on independence in L. For example, given a formula A with occurrence of &, if we can show there exist no other formula A' which is V-equivalent to A (has the same V-value as A) and lacks both & and $\rightarrow$, we can conclude that for logic L any definition of & requires some use of $\rightarrow$. By construction, all theorems of intuitionistic logic are V-valid.

2 The language

Types The language has two types, 1 and (1)1.[6] The first is the type of truth values, the second the type of 1-placed maps from truth values to truth values. We use Greek letters τ, σ, π, etc. to denote arbitrary types in the language.

Variables For each type τ an infinite collection of free and bound variables is assumed. Free variables of type τ are written a^τ, b^τ, c^τ, etc.; bound variables are written x^τ, y^τ, z^τ, etc.

Terms Free variables of type τ are terms of the same type. If A and B are terms of type (1)1 and 1 respectively then (AB) is a term of type 1.[7] If A and B are terms of type 1, then so are $\neg(A)$, $(A \vee B)$, $(A \ \& \ B)$, and $(A \rightarrow B)$. If A is a term of type 1, if a^1, x^1 are free and bound variables of type 1 and A' is got from A by replacing a^1 everywhere with x^1 then $(\exists x^1 A')$ and $(\forall x^1 A')$ are terms of type 1.

Where confusion is not a problem parentheses will be omitted. Thus in terms formed by repeated disjunction or conjunction, omitted parentheses associate to the left: $A_0 \vee A_1 \vee \ldots \vee A_n$ denotes $((\ldots(A_0 \vee A_1) \vee \ldots) \vee A_n)$, and similarly for $A_0 \ \& \ A_1 \ \& \ldots \& \ A_n$. Following Prawitz [3] we shall call an expression A which would be a term except for the possible presence of bound variables not bound by quantifiers a *pseudo-term*. Terms and pseudo-terms of type 1 are called *formulas* and *pseudo-formulas*, respectively. If A is a pseudo-term with unbound bound variables among the distinct $x_1^{\tau_1}, \ldots, x_n^{\tau_n}$ and $B_1, \ldots, B_n$ are pseudo-terms of appropriate types, we use the notation $A_{x_j^{\tau_j}}[B_j]$ to denote the result of replacing each unbound occurrence of $x_j^{\tau_j}$ in A with B_j. The *degree* of a pseudo-term is the number of occurrences in it of the symbols $\neg$, $\vee$, &, $\exists$, and $\forall$ plus the number of times application is used in its formation.

3 Semantics We adopt *semantic structures* of the form $S = (H, \leq, h)$ where $(H, \leq)$ is any partially ordered set underlying a complete Heyting lattice and where $h \in H$ is a distinguished element. H is assumed to have at least two distinct members. When dealing with such a lattice $(H, \leq)$ we use the notations $\neg$, $\vee$, &, $\rightarrow$, and $\leftrightarrow$ to denote the corresponding operations on elements of

H. We let $\top$ and $\perp$ denote the maximum and minimum elements of H. Lub and glb denote the operations on nonempty subsets of H assigning least upper bounds and greatest lower bounds, respectively.

Given the structure $S = (H, \leq, h)$ we associate with each type τ a set M^τ and a binary map $\sim_\tau$ on M^τ with values in H as follows:

(1) $M^1 = H, x \sim_1 y = h \mathbin{\&} (x \leftrightarrow y)$

(2) $M^{(1)1} = \left\{ \begin{array}{l} \phi \in maps(M^1, M^1)\text{: for all } x, y \in M^1 \\ \qquad\qquad x \sim_1 y \leq \phi(x) \sim_1 \phi(y) \end{array} \right\}$

$\phi \sim_{(1)1} \psi = \text{glb}\{\phi(x) \sim_1 \psi(x)\colon x \in M^1\}$.

By an *assignment* we mean a mapping $\mathfrak{A}$ which assigns to each free variable a^τ (τ arbitrary) an element of the set M^τ. Given specific a^τ and $\phi^\tau \in M^\tau$, we let $\mathfrak{A}^{\phi}_{a^\tau}$ denote the assignment got from $\mathfrak{A}$ by altering its value at a^τ to ϕ. A *valuation* is a mapping on terms and sends a τ-term A (τ arbitrary) to an element in M^τ. It is defined inductively from an assignment $\mathfrak{A}$:

(1) $V(a^\tau, \mathfrak{A}) = \mathfrak{A}(a^\tau)$

(2) $V(\neg A, \mathfrak{A}) = \neg V(A, \mathfrak{A})$

(3) $V(A \otimes B, \mathfrak{A}) = V(A, \mathfrak{A}) \otimes V(B, \mathfrak{A})$ where $\otimes = \vee, \&,$ or $\rightarrow$

(4) $V((AB), \mathfrak{A}) = V(A, \mathfrak{A})(V(B, \mathfrak{A}))$

(5) $V(Qx^\tau A, \mathfrak{A}) = \Gamma(\{V(A_{x^\tau}[a^\tau], \mathfrak{A}^{\phi}_{a^\tau})\colon \phi \in M^\tau\})$ where $Q, \Gamma = \exists$, lub or $\forall$, glb.

In the last clause a^τ is to be an arbitrary free variable of type τ not already occurring in A.

We write $V(A)$ when the assignment $\mathfrak{A}$ is understood. $V^{\phi}_{a^\tau}$ is the valuation induced by $\mathfrak{A}^{\phi}_{a^\tau}$ and $V^{\boldsymbol{\phi}}_{\boldsymbol{a}^\tau}$ denotes the obvious generalization,[8] the entries of $\boldsymbol{a}^\tau$ being assumed to be distinct. If the pseudo-term A has unbound bound variables among the (distinct) entries of $\boldsymbol{x}^\tau$ and if the entries of $\boldsymbol{\phi}$ are in the appropriate M^{τ_j}'s then $V(A)_{\boldsymbol{x}^\tau}[\boldsymbol{\phi}]$ (or $V(A)[\boldsymbol{\phi}]$ if $\boldsymbol{x}^\tau$ is understood) denotes $V^{\boldsymbol{\phi}}_{\boldsymbol{a}^\tau}(A_{\boldsymbol{x}^\tau}[\boldsymbol{a}^\tau])$ where the entries of $\boldsymbol{a}^\tau$ are arbitrary distinct free variables of appropriate type not occurring in A.

Valuations behave well with substitutions since $V(A_x[B]) = V(A)_x[V(B)]$ holds generally. A pseudo-term A of type σ becomes, under valuation, an "extensional" function: If $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ have entries in the appropriate M^{τ_j}'s and if $k \leq \phi_j \sim_{\tau_j} \psi_j$ for each $j (k \in H)$ then $k \leq V(A)[\boldsymbol{\phi}] \sim_\sigma V(A)[\boldsymbol{\psi}]$.

Given valuation V on structure $S = (H, \leq, h)$ and pseudo-terms A, B both of the same type, we say A and B are *V-equivalent* if for all appropriate $\boldsymbol{\phi}$, $V(A)[\boldsymbol{\phi}] = V(B)[\boldsymbol{\phi}]$. Pseudo-formula A is *V-valid* if for all appropriate $\boldsymbol{\phi}$, $V(A)[\boldsymbol{\phi}] = \top$. All theorems of first-order intuitionistic logic written in the type variables of our chosen language are V-valid. The collection of V-valid formulas is always deductively closed.

4 Statement of results In the context we have set up we shall prove:

I. There is a valuation V on a structure S such that $V(A) \neq V(\forall x^1(a^{(1)1}x^1))$ for every formula A lacking $\forall$.

II. There is a valuation V on a structure S such that $V(A) \neq V(a^1 \mathbin{\&} b^1)$ for every formula A lacking both & and $\rightarrow$.

III. There is a valuation V on a structure S such that $V(A) \neq V(a^1 \vee b^1)$ for every formula A lacking both $\vee$ and $\forall$.

*5 **Preliminaries*** Assume a fixed structure $S = (H, \leq, h)$. Given a nonempty subset $H' \subseteq H$ we define for each type τ the *H'-portion* of M^τ, written $M^\tau(H')$, as follows:

(1) $M^1(H') = H'$
(2) $M^{(1)1}(H') = \{\phi \in M^{(1)1}: \textit{image}\ (\phi) \subseteq M^1(H')\}$.

If the set H' has only one element, say $k \in H$, it is clear that each $M^\tau(H')$ also has a single element which we denote as k^τ. Thus $k^1 = k$ and $k^{(\tau)\sigma}$ is a constant function with k^σ as its only value. The use of H'-portions in the proofs below hinges on the following triviality: If A is of type (1)1 and $V(A) \in M^{(1)1}(H')$, then for every B of type 1, $V(AB) \in M^1(H')$.

An element $k \in H$ is *(finitely) $\exists$-simple* if for every nonempty $H' \subseteq H$ (non-empty finite $H' \subseteq H$) whenever $\text{lub}(H') \geq k$, then $k' \geq k$ for some $k' \in H'$. We say k is *(finitely) $\forall$-simple* if for every nonempty $H' \subseteq H$ (nonempty finite $H' \subseteq H$) whenever $\text{glb}(H') \leq k$ then $k' \leq k$ for some $k' \in H'$. Note that whenever $k \in H$ is $\exists$-simple, yet distinct from $\bot$, the set $\{k' \in H: k' < k\}$ contains a unique maximum which we denote by k_*. If $k \in H$ is $\forall$-simple, yet distinct from $\top$, the set $\{k' \in H: k' > k\}$ also contains a unique minimum which we denote by k^*. The use of $\exists$- and $\forall$-simplicity is recurrent in the independence proofs. For example, from $V(A \vee B) \geq k$ and finite $\exists$-simplicity of k we can conclude that $V(A) \geq k$ or $V(B) \geq k$; from $V(\forall x^\tau A) \leq k$ and $\forall$-simplicity of k we can conclude $V(A)[\phi] \leq k$ for some $\phi \in M^\tau$.

Suppose $\bot$ is $\forall$-simple and that $h \geq \bot^*$. Then for each $k \in H$ for which $k \geq \bot^*$ and for each type τ we define a special map $\Delta_{\tau,k}: M^\tau \to M^\tau\ (\{\bot, k\})$ as follows:

$$(1)\quad \Delta_{1,k}(\phi) = \begin{cases} k \text{ if } \phi \neq \bot \\ \\ \bot \text{ if } \phi = \bot \end{cases} \qquad \phi \in M^1 = H$$

$$(2)\ \Delta_{(1)1,k}(\phi) = \quad \Delta_{1,k} \circ \phi \qquad\qquad \phi \in M^{(1)1}.$$

One verifies that for all $\phi, \psi \in M^\tau$ $\phi \sim_\tau \psi \leq \Delta_{\tau,k}(\phi) \sim_\tau \Delta_{\tau,k}(\psi)$,[9] and that for all $\phi \in M^\tau$, $\bot^* \leq \phi \sim_\tau \Delta_{\tau,k}(\phi)$. If M is the product set $\prod_{j=0}^{n} M^{\tau_j}$ we let $\Delta_{\tau,k}$: $M \to M$ denote the map such that $\Delta_{\tau,k}(\phi)_j = \Delta_{\tau_j,k}(\phi_j)$ for $j = 0, \ldots, n$. In the independence proofs the $\Delta_{\tau,k}$ maps are used to form from a given $\phi \in M^\tau$ a "thinned out" version $\Delta_{\tau,k}(\phi)$ which is "sufficiently" equivalent, with respect to $\sim_\tau$, to the original.

For given valuation V a formula A will in general not be V-equivalent to any of its prenex normal forms. However by choosing the lattice $(H, \leq)$ carefully and noting the structure of the particular A we're dealing with, a V-equivalent prenex form can be assumed where needed. Obviously if $(H, \leq)$ is Boolean, all prenex forms are possible. If A does not contain occurrence of $\to$, if $\bot$ is $\forall$-simple, and H has at most finitely many incomparable elements, then a prenex form for A is also assured.

Each independence proof begins with an initial reduction. By careful choice of the lattice $(H, \leq)$ and the assignment used to form the valuation V, it can be assumed that the presumed counterexample A, taken to have minimal degree, has the form $Q_0 x_0^{\tau_0} \ldots Q_n x_n^{\tau_n} B$, the Q_j being quantifiers $\exists$ or $\forall$, the $x_j^{\tau_j}$ being distinct and B being some pseudo-formula. Some criteria for estimating $V(A)$ given the Q_j's and B become necessary and thus we develop the following notion: Let $M_0, \ldots, M_n$ be given sets. Write their product $M = \prod_{j=0}^{n} M_j$. Suppose $\alpha \subseteq \{0, \ldots, n\}$ is any subset. Then call a mapping Φ: $M \to M$ a *choice functional for M relative to α* if

(1) for all $\phi \in M$, $\Phi(\phi)|_\alpha = \phi|_\alpha$[10]
(2) for all ϕ, $\psi \in M$ and $0 \leq j \leq n$ if $\alpha_j = \{0, \ldots, j\}$ then $\phi|_{\alpha_j} = \psi|_{\alpha_j}$ implies $\Phi(\phi)|_{\alpha_j} = \Phi(\psi)|_{\alpha_j}$.

Returning to the formula $A = Q_0 x_0^{\tau_0} \ldots Q_n x_n^{\tau_n} B$, let $M = \prod_{j=0}^{n} M^{\tau_j}$ and let $\alpha^+ = \{0 \leq j \leq n \colon Q_j = \forall\}$, $\alpha^- = \{0 \leq j \leq n \colon Q_j = \exists\}$. Then the following is easily verified: If a choice functional Φ for M relative to α^+ exists such that for all $\phi \in M$, $V(B)[\Phi(\phi)] \geq k$, then $V(A) \geq k$; conversely if $V(A) \geq k$ and k is $\exists$-simple, such Φ exists. Similarly, if a choice functional Φ for M relative to α^- exists where $V(B)[\Phi(\phi)] \leq k$ for each $\phi \in M$, then $V(A) \leq k$; conversely if $V(A) \leq k$ and k is $\forall$-simple, such Φ exists.

To speed up description of Heyting lattices used in the independence proofs, we take each $(H, \leq)$ to be a substructure of $(\mathbf{C}, \leq)$ where $\mathbf{C}$ is the complex numbers and $\leq$ is the partial ordering of $\mathbf{C}$ whereby $z \leq w$ means $Re(z) \leq Re(w)$ in the ordinary sense. Following custom we use i to denote $\sqrt{-1}$.

6 *Proofs of independence*

I. Let $S = (H, \leq, \perp^*)$ where $H = \{-1, 0\} \cup \{1/p \colon p = 1, 2, \ldots\}$. Assume of V only that for each τ the V-images of free variables of type τ include all of $M^\tau(\{\perp, \top\})$ (a finite set) yet are restricted to $M^\tau(H - \{\perp^*\})$ and that $V(a^{(1)1}) = g$ where for $k \in H$ $g(k) = \top$ if $k \in \{\perp, \perp^*\}$ and $g(k) = k$ if otherwise. Thus $V(\forall x^1(a^{(1)1}x^1)) = \perp^*$. One notes that $\perp^*$ is not of the form $\neg(r)$ for any $r \in H$, that $\perp^*$ is $\exists$-simple and finitely $\forall$-simple and that whenever $r \to s = \perp^*$ for some $r, s \in H$ one has $s = \perp^*$. Assume a formula A exists which lacks $\forall$ yet for which $V(A) = \perp^*$. Picking A of minimal degree, we can infer that A has the form $\exists x^\tau B$. Since $\perp^*$ is $\exists$-simple we can pick $\phi \in M^\tau$ such that $V(B)[\phi] = \perp^*$. Let $\phi' = \Delta_{\tau,\top}(\phi)$. Since $\perp^* \leq \phi \sim_\tau \phi'$ we have $\perp^* \leq V(B)[\phi] \sim_1 V(B)[\phi']$ and hence $V(B)[\phi'] \geq \perp^*$. But for every $\psi \in M^\tau$, $V(B)[\psi] \leq \perp^*$ so in fact $V(B)[\phi'] = \perp^*$. On the other hand, $\phi' \in M^\tau(\{\perp, \top\})$ so we must have for some free variable a^τ that $V(a^\tau) = \phi'$. But then $A' = B_{x^\tau}[a^\tau]$ has lower degree yet satisfies $V(A') = V(B_{x^\tau}[a^\tau]) = V(B)[V(a^\tau)] = V(B)[\phi'] = \perp^*$—a contradiction.

II. Let $S = (H, \leq, \top)$ where $H = \{-3, -2, -1, -i, i, 1\}$ and pick any valuation V on S such that $V(a^1 \,\&\, b^1) = -1$ and for free variable a^τ, $V(a^\tau) \in \{-i, i\}$ if $\tau = 1$ and $V(a^\tau) = i^\tau$ otherwise.[11] One notes that -1 is not of the form $\neg(r)$ for $r \in H$ and that -1 is finitely $\exists$-simple. Assume a formula A exists lacking both & and $\rightarrow$ yet for which $V(A) = -1$. Picking A of minimal degree we can infer that for some $n \geq 0$, A is of the form $Q_0 x_0^{\tau_0} \ldots Q_n x_n^{\tau_n} B$ where each Q_j is either $\exists$ or $\forall$, the $x_j^{\tau_j}$ are distinct and B is a pseudo-formula not itself of the form $Qx^\tau B'$. Fix $M = \prod_{j=0}^{n} M^{\tau_j}$, $\alpha^+ = \{0 \leq j \leq n\colon Q_j = \forall\}$ and $\alpha^- = \{0 \leq j \leq n\colon Q_j = \exists\}$. Since $\perp = -3$ is $\forall$-simple, H is finite and A contains no $\rightarrow$, we can in fact assume A is in prenex normal form so that B lacks $\exists$ and $\forall$ altogether. Passing to V-equivalent we can write $B = C_0 \vee C_1 \vee \ldots \vee C_k$ where each C_e is either a negation $\neg(E)$ or a variable (of type 1) or else an applied pseudo-term $(E_0 E_1)$ where E_0 is again a variable (of type (1)1). In the latter two cases it turns out that the variable in question must be bound and in fact be $x_j^{\tau_j}$ for some $j \in \alpha^+$. This is seen by noting that $V(A) \leq -i, i$ both of which are $\forall$-simple so one can find suitable choice functionals on M relative to α^- to produce $\phi, \psi \in M$ such that $V(B)[\phi] \leq -i$ and $V(B)[\psi] \leq i$ and yet for any $j \in \alpha^-$, $\phi_j = \psi_j = \top^{\tau_j}$. If the variable in question (within C_e) were *not* one of the $x_j^{\tau_j}$ for $j \in \alpha^+$ then it would be the case that $V(C_e)[\phi] = V(C_e)[\psi] \geq$ one of $-i$ or i which would violate one of $V(B)[\phi] \leq -i$ or $V(B)[\psi] \leq i$. Continuing, one notes that $V(A) \geq -1$ which is $\exists$-simple, so a choice functional Φ' on M relative to α^+ should exist for which $V(B)[\Phi'(\phi)] \geq -1$ for each $\phi \in M$. Define map $\Phi\colon M \rightarrow M$ by

$$\Phi(\phi)_j = \begin{cases} \phi_j & \text{if } j \in \alpha^+ \\ \Phi'(\Delta_{\tau,\perp^*}(\phi))_j & \text{if otherwise} \end{cases} .$$

Then Φ is a choice functional on M relative to α^+. We shall derive a contradiction by showing that $V(B)[\Phi(\phi)] = \top$ for arbitrary $\phi \in M$ which in turn implies $V(A) = \top \neq -1$. Let $\phi \in M$ be arbitrary. We have $V(B)[\Phi'(\Delta_{\tau,\perp^*}(\phi))] \geq -1$ and since -1 is finite $\exists$-simple we must have $V(C_e)[\Phi'(\Delta_{\tau,\perp^*}(\phi))] \geq -1$ for some C_e. If C_e is not a negation then it contains (see above) a crucial variable $x_j^{\tau_j}$ for $j \in \alpha^+$ and with $\Phi'(\Delta_{\tau,\perp^*}(\phi))_j = \Delta_{\tau,\perp^*}(\phi)_j = \Delta_{\tau_j,\perp^*}(\phi_j)$ which lies in $M^{\tau_j}(\{\perp, \perp^*\})$ this forces $V(C_e)[\Phi'(\Delta_{\tau,\perp^*}(\phi))]$ into the set $\{\perp, \perp^*\}$ both elements of which are strictly less than -1. Therefore C_e must be a negation. On the other hand, for each j it is the case that $\perp^* \leq \Phi(\phi)_j \sim_{\tau_j} \Phi'(\Delta_{\tau,\perp^*}(\phi))_j$ and hence $\perp^* \leq V(C_e)[\Phi(\phi)] \sim_1 V(C_e)[\Phi'(\Delta_{\tau,\perp^*}(\phi))]$. This implies $\perp^* \leq V(C_e)[\Phi(\phi)]$ and since this value must be among $\{\perp, \top\}$ (C_e being a negation, $\perp$ being $\forall$-simple) we have that $V(C_e)[\Phi(\phi)]$ equals $\top$. Thus also $V(B)[\Phi(\phi)] = \top$ and we have our contradiction.

III. Let $S = (H, \leq, \top)$ where $H = \{-1, -i, i, 1, 2\}$ and pick any valuation V on S such that $V(a^1 \vee b^1) = 1$ and that for each τ, the V-images of free variables of type τ comprise the set $M^\tau(H - \{1\})$. One notes that 1 is not of the form $\neg(r)$ for any $r \in H$, that 1 is $\forall$-simple, and that for any $r, s \in H$, if $r \rightarrow s = 1$ then $s = 1$ is the case. Assume formula A exists lacking both $\vee$ and $\forall$, yet for which $V(A) = 1$. Picking A of minimal degree we can infer that it has the form $\exists x^\tau B$. First, suppose $V(B)[\phi] = 1$ for some $\phi \in M^\tau$. An inductive argument[12] shows that some $\phi' \in M^\tau(H - \{1\})$ exists such that $1 \leq \phi \sim_\tau \phi'$. But then $1 \leq V(B)[\phi] \sim_1 V(B)[\phi']$ so $V(B)[\phi'] \geq 1$ and hence $V(B)[\phi'] = 1$

is the case. By construction there exists free variable a^τ such that $V(a^\tau) = \phi'$ and hence $V(B_{x^\tau}[a^\tau]) = V(B)[V(a^\tau)] = V(B)[\phi'] = 1$ holds. But $B_{x^\tau}[a^\tau]$ has lower degree than A so we conclude that $V(B)[\phi] = 1$ is impossible and in fact $V(B)[\phi] < 1$ is the case in general. However since $V(A) = 1$ there must exist $\phi, \psi \in M^\tau$ such that $V(B)[\phi] = -i$ and $V(B)[\psi] = i$. A further inductive argument shows that for any type σ and elements $\phi', \psi' \in M^\sigma$ there exists $\theta \in M^\sigma$ such that $-i \leq \theta \sim_\sigma \phi'$ and $i \leq \theta \sim_\sigma \psi'$. Define map $\Delta\colon H \to M^\tau$ where $\Delta(\perp) = \phi$, $\Delta(\top) = \Delta(1) = \psi$ and $\Delta(-i)$ and $\Delta(i)$ are so chosen that $-i \leq \Delta(-i) \sim_\tau \psi$, $-i \leq \Delta(i) \sim_\tau \phi$, $i \leq \Delta(-i) \sim_\tau \phi$ and $i \leq \Delta(i) \sim_\tau \psi$. One checks that for all $r, s \in H$ that $r \sim_1 s \leq \Delta(r) \sim_\tau \Delta(s)$.[13] Now define $\theta\colon H \to H$ by $\theta(k) = V(B)[\Delta(k)]$. Then $\theta \in M^{(1)1}$, $\theta(k) < 1$ for all $k \in H$ and $\theta(\top) = i$ and $\phi(\perp) = -i$. But then $i \leq i \leftrightarrow \top \leq \theta(i) \leftrightarrow i$, so $i \leq \theta(i) < 1$. We infer $\theta(i) = i$. This implies $-i \leq i \leftrightarrow \perp \leq \theta(i) \leftrightarrow \theta(\perp) = i \leftrightarrow (-i) = \perp$ which is a contradiction.

7 Extensions to arbitrary type theory The language is extended as follows: A second "initial" type 0 (the type of individuals) is added. For any two types τ and σ, $(\tau)\sigma$ is also a type. Appropriate variables for each type are introduced. Terms are built up as before except that (1) for arbitrary types τ, σ if A is of type $(\tau)\sigma$ and B of type τ then (AB) (read A *applied to* B) is of type σ and (2) if A is of type σ, if a^τ, x^τ are free and bound variables of type τ and if A' is formed from A by replacing a^τ everywhere with x^τ, then $\lambda x^\tau A'$ is a term of type $(\tau)\sigma$.

From the semantic structure $S = (H, \leq, h)$ the objects M^τ, $\sim_\tau$, $M^\tau(H')$, k^τ and $\Delta_{\tau,k}$ are all built up for the additional types in the manner suggested for $\tau = 1$ and $(1)1$. For $\tau = 0$ one puts $M^0 = M^0(H') = \{0\}$, $0 \sim_0 0 = \text{h}$ and $k^0 = \Delta_{0,k}(0) = 0$. Valuations are defined as before with the additional clause

$$(6)\ V(\lambda x^\tau A, \mathfrak{A})(\phi) = V(A_{x^\tau}[a^\tau], \mathfrak{A}^{\phi}_{a^\tau}),$$

where a^τ is an appropriate free variable not occurring in A. It is necessary to verify that $V(\lambda x^\tau A)$ so defined does indeed belong to the set $M^{(\tau)\sigma}$ (A being of type σ).[14] Also to be checked is that V respects λ-conversion: that is, $(\lambda x^\tau A)B$ and $A_{x^\tau}[B]$ are always V-equivalent.

The proofs for arbitrary type theory go through word for word except for one minor modification: The possible presence in A (the presumed counterexample) of subpseudo-terms of the form $(\lambda x^\tau B)C$ has to be dealt with. These are eliminated by the process of λ-conversion (replace $(\lambda x^\tau B)C$ with $B_{x^\tau}[C]$) which eventually transforms A into a "normal" term, one lacking any subpart of the form $(\lambda x^\tau B)C$. The presumed counterexample A is assumed to be normal at the outset, to have minimal degree as such, and the proofs proceed as before. What is missing is the knowledge that the λ-conversion process for arbitrary A does eventually stop and deliver up a term in normal form. This can be verified by adapting a variant of the notion of "rank" used in Schütte [4] and proving by his same line of reasoning that all terms have finite rank. Normal forms for arbitrary terms are then demonstrated by induction on rank.

8 Independence in intermediate logic Each of the ten independence proofs involves a valuation V whose valid schemata go far beyond intuitionistic logic. The case for $\neg$ deserves special mention. The proof that definition of $\neg$ requires

$\forall$ can be done with a two-element Heyting lattice while proof that definition of $\neg$ requires $\rightarrow$ is done with the next simplest Boolean lattice. Thus even for two-valued logic a definition of $\neg$ requires $\forall$ and for classical (Boolean) logic generally, a definition of $\neg$ also requires $\rightarrow$.[15]

Consider however the following schemata:

$$\neg(A) \vee \neg\neg(A)$$
$$\neg(\forall x^\tau A) \rightarrow \exists x^\tau \neg(A)$$
$$\forall x^\tau(A \vee B) \rightarrow (\forall x^\tau A) \vee B\ ,$$

where in the third, B is not to contain unbound occurrence of x^τ. These are in clear violation of intuitionistic philosophy and yet in every case except one, the three are V-valid.[16] It is just the first schema which fails validity and this only for the valuation V in proof III. Indeed if we add to higher-order intuitionistic logic the schema $\neg(A) \vee \neg\neg(A)$ then $\exists x^1[(\neg(x^1) \rightarrow A)\ \&\ (\neg\neg(x^1) \rightarrow B)]$ becomes a definition of $A \vee B$ not needing the symbol $\forall$. This explains why independence proof III is qualitatively different from the other nine.

9 An interpretation The Heyting lattice provides subclassical logic with perhaps its most natural semantic structure. There may be an analogy with classical logic for which the two-element lattice provides not only a semantics but also a starting point for interpretation, and so a second look at the general shape of a Heyting lattice might be called for. Indeed, nonclassical phenomena arise in a Heyting lattice which even the intuitionistic interpretation, in a sense to be clarified, avoids accepting at face value.

To illustrate, let us assume that some general Heyting lattice is given and that all formulas A, B, C, etc. have been mapped onto it by means of some valuation. For simplicity we will identify a formula A with its value in the lattice. What at once catches the eye is that the truth of a formula A (the case where $A = \top$) is not dependent on the truth or falsity of its subformulas. This is rather different from the classical situation where the meaning of each connective and quantifier can be spelled out by means of a truth table.[17] On a Heyting lattice one notes that $A \rightarrow B = \top$ may occur whether or not either of A or B take on the classical values $\top$ or $\bot$. It is also seen that $A \vee B = \top$ is possible even though neither A nor B may coincide with $\top$. In an analogous manner $\exists xA = \top$ is possible even though for no (relevant) ϕ is $A_x[\phi] = \top$ the case.[18]

The idea of a Heyting lattice comes from the pioneering work of intuitionists and it is of interest to note their attitude toward this behavior. The concern of intuitionism is proof and not classical truth, but by observing which formulas an intuitionist is willing to assert, an interpretation emerges for comparison. In this interpretation, implication (at least) is treated in a way befitting the Heyting lattice. The intuitionist is willing to assert $A \rightarrow B$ without specifically asserting or denying either A or B in the process. However, he will not assert $A \vee B$ without first asserting one of A or B and he refuses to assert $\exists xA$ without previously asserting some $A_x[\phi]$. Clearly his loyalties lean more to the classical than to the Heyting lattice.

Can the phenomena arising in a Heyting lattice be taken at face value? A bit of reflection may convince one that we regularly experience truths in our world in a most unclassical manner. Standing on the edge of a high cliff confronting the view which drops away at our feet we can be struck with the truth of "I would fall to great injury if I were to take another step forward." This forms a nonclassical experience of implication in that it is felt without first rejecting as false a premise (I step forward) or affirming as true a conclusion (I fall). For disjunction one can remember the afternoon we realize we have mislaid a garden tool which has got to be on the front or back lawn where we last used it. Although it does not turn up after considerable search we are unshaken in our certainty that it lies either out front of the house or out back. We are unclassically in the grip of the truth of $A \vee B$ and not that of either A or B. With fair frequency, existence is also viewed in this nonclassical manner as, for example, when we see a face and experience the strong sense of recognition and yet are completely unable to make an identification.

Based on these considerations, we suggest the following interpretation for subclassical logic:

$\neg(A) \equiv$ anything would be true if A were true
$A \vee B \equiv$ it is as if A is true or B is true
$A \,\&\, B \equiv$ both A and B are true
$A \rightarrow B \equiv B$ would be true if A were true
$\exists x A \equiv$ it is as if for some x, A is true of x
$\forall x A \equiv$ for all x, A is true of x.

The interpretations for & and $\forall$ are self-explanatory. The interpretation for $\neg$ is based on the lattice equivalence of $\neg(A)$ with $A \rightarrow \perp$. The novel features of the interpretation are contained in the treatment of $\rightarrow$, $\vee$, and $\exists$.

Since the examples given above for motivating the interpretation were drawn from personal subjective experience it might seem that the use of this interpretation is restricted to that domain. This is not so. One is quite free to treat the interpretation as involving an externally existing reality. Of course the world so viewed no longer fits the classical mold. Truths seem to form a part of the world in analogy with physical matter: the apple (a thing) and the danger (an implication) can both be perceived directly. The environment at times seems to hold truths (disjunctions) which are potent and yet not fully formed in the classical sense. In addition one encounters quasi-existing realities (truths involving existential quantification) for which the classical view has no name.

NOTES

1. The pattern's appeal is definitely related to this choice of primitives. Other choices do not reproduce its tidiness.

2. For reasons of symmetry we differ from Prawitz [3] by substituting $\neg$ for $\perp$.

3. One can refer to Umezawa [5] for comparable results. His focus is a proof-theoretic restriction of Prawitz's definability rather than the role of $\rightarrow$ and $\forall$.

4. Note that the $\sim_\tau$ maps are structure on the semantic system itself and not in the formal language. The logics studied in this paper do not include equality.

5. Also called a pseudo-Boolean algebra by some authors.

6. The notation is chosen to be compatible with arbitrary type theory.

7. Such a term is said to be formed by *application*.

8. Boldfaced $\boldsymbol{\phi}$, $\boldsymbol{\tau}$, $\boldsymbol{a}^{\boldsymbol{\tau}}$, etc. denote finite sequences of appropriate objects, their j-th element being written ϕ_j, τ_j, $a_j^{\tau_j}$, etc.

9. In arbitrary type theory this is simply $\Delta_{\tau,k} \in M^{(\tau)\tau}(\{\perp, k\})$.

10. We are temporarily considering elements $\boldsymbol{\phi}$ as functions on the set $\{0, \ldots, n\}$.

11. Recall that i^{τ} is the sole element of $M^{\tau}(\{i\})$ and similarly for $\top^{\tau_j}$ which is used later in this proof.

12. The induction is taken over types and is really only needed for the generalization to arbitrary type theory. In the present context, verification for types 1 and (1)1 is performed directly. Further reference to "inductive argument" elsewhere in this proof is to be read in the same light.

13. In arbitrary type theory this is simply $\Delta \in M^{(1)\tau}$.

14. This amounts to showing the sets M^{τ} form a "general model" in the sense of Henkin [1].

15. These remarks are of course relative to our chosen list of connectives and quantifiers.

16. That the second schema is V-valid in proof III requires argument similar to that developed within that proof.

17. Infinitary truth tables in the case of quantifiers.

18. Of course the classical situation and the Heyting lattice do exhibit *some* similar behavior. For both $A \;\&\; B = \top$ is the case only when each of $A = \top$ and $B = \top$ hold; $\forall x A = \top$ is the case only when for all (relevant) ϕ, $A_x[\phi] = \top$ happens to be true; $\neg(A) = \top$ holds only if $A = \perp$ is the case. Beyond this, similarities cease.

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