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The Rule of Procedure *Re* in Łukasiewicz's Many-Valued Propositional Calculi

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A modified form of modus ponens is used to give new formalizations of Łukasiewicz's finite and infinite-valued propositional calculi. In the infinite-valued case, we establish the independence of the axiom schemes.

Let C and N be the primitive implication and negation functors respectively of Łukasiewicz (see [4]). All the propositional calculi we consider here have 1 as the only designated truth-value. The usual primitive rule of procedure in formalizations of propositional calculi with C and N, or with C as the only primitive functor(s) is modus ponens (with respect to C). We consider here an alternative rule of procedure which occurred as a derived rule in [7], p. 101, and which has been considered in [1] and [2] as a primitive rule of procedure for the two-valued propositional calculus. This rule of procedure is as follows.

Re Let P, Q and R be formulas and let the result of replacing one occurrence of the subformula CPQ in R by Q be S. Then, if P and R are correct formulas, S is a correct formula.

Clearly, modus ponens is a special case of *Re*. Reductions in the number of axiom schemes of a similar nature to those we obtain here have been described in the two-valued case in [11].

We shall give several formalizations of the $\aleph_0$-valued and m-valued propositional calculi with C and N as the only primitive functors and *Re* as the only primitive rule of procedure. We also give several formalizations of the $\aleph_0$-valued and m-valued propositional calculi with C as the only primitive functor and *Re* as the only primitive rule of procedure. For each formalization we shall establish weak deductive completeness (i.e., the provability of all generalized tautologies), and for the formalizations of the $\aleph_0$-valued propositional calculi we establish the independence of the axiom schemes and primitive rule of procedure.

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We first establish some lemmas. We shall use a slightly nonstandard form of hypothetical deduction. The formalization considered in the six lemmas has no axioms and *Re* is the only primitive rule of procedure. The expressions to the left and right of the "yields" sign are formula schemes so that each hypothetical deduction has infinitely many assumption formulas and infinitely many conclusions. (Some lemmas therefore have two interpretations, depending on the number of primitive functors.) Each numbered formula scheme is preceded by a line of proof. For example, (1.1, *P*/1.1, 1.1, *Re* * 1.2) means that by taking the formula scheme 1.1 as an instance of *P* in 1.1, and then by 1.1 and one or more applications of *Re* we obtain 1.2. As usual, $APQ =_{df} CCPQQ$.

Lemma 1 *CCPQCCQRCPR* ⊦ *CPP*, *CCPCQRCQCPR*.

Proof: 1.1 *CCPQCCQRCPR*.
(1.1, *P*/1.1, 1.1, *Re* * 1.2)
1.2 *CQCCQRR*.
(1.2, *Q*/1.1, *R/P*, 1.1, *Re* * 1.3)
1.3 *CPP*.
(1.1, *P/Q*, *Q/CCQRR*, *R/CPR*, 1.2, *Re* * 1.4)
1.4 *CCCCQRRCPRCQCPR*.
(1.1, *Q/CQR* * 1.5)
1.5 *CCPCQRCCCQRRCPR*.
(1.1, *P/CPCQR*, *Q/CCCQRRCPR*, *R/CQCPR*, 1.5, 1.4, *Re* * 1.6)
1.6 *CCPCQRCQCPR*.

Lemma 2 *CCPCQRCQCPR*, *CQCPP* ⊦ *CPCQP*.

Proof: 2.1 *CQCPP*.
2.2 *CCPCQRCQCPR*.
(2.2, *P/Q*, *Q/P*, *R/P*, 2.1, *Re* * 2.3)
2.3 *CPCQP*.

Lemma 3 *CSCCPQCCQRCPR* ⊦ *CQCPP*, *CCPQCCQRCPR*.

Proof: 3.1 *CSCCPQCCQRCPR*.
(3.1, *S*/3.1, 3.1, *Re* * 3.2)
3.2 *CCPQCCQRCPR*.
(3.2, Lemma 1 * 3.3)
3.3 *CPP*.
(3.1, *S/Q*, *Q/P*, *R/P*, 3.3, *Re* * 3.4)
3.4 *CQCPP*.

Lemma 4 *CPP*, *CRCAPQAQP* ⊦ *CQCPP*, *CAPQAQP*.

Proof: 4.1 *CRCAPQAQP*.
(4.1, *R*/4.1, 4.1, *Re* * 4.2)
4.2 *CAPQAQP*.
(4.1, *R/Q*, *Q/P*, *CPP*, definition of *A*, *Re* * 4.3)
4.3 *CQCPP*.

Lemma 5 *CPP*, *CRACPQCQP* ⊢ *CQCPP*, *ACPQCQP*.

Proof: 5.1 *CRACPQCQP*.
(5.1, *R*/5.1, 5.1, *Re* * 5.2)
5.2 *ACPQCQP*.
(5.1, *R/Q*, *Q/P*, *CPP*, *P/CPP*, definition of *A*, *Re* * 5.3)
5.3 *CQCPP*.

Lemma 6 *CPP*, *CRCCNQNPCPQ* ⊢ *CQCPP*, *CCNQNPCPQ*.

Proof: 6.1 *CRCCNQNPCPQ*.
(6.1, *R*/6.1, 6.1, *Re* * 6.2)
6.2 *CCNQNPCPQ*.
(*CPP*, *P/NP* * 6.3)
6.3 *CNPNP*.
(6.1, *R/Q*, *Q/P*, 6.3, *Re* * 6.4)
6.4 *CQCPP*.

A complete formalization of the $\aleph_0$-valued propositional calculus with *C* and *N* as the only primitive functors and modus ponens as the only primitive rule of procedure is given by the following axiom schemes (see [10], [6], [3]).

AX1 *CPCQP*
Ax2 *CCPQCCQRCPR*
Ax3 *CAPQAQP*
AX4 *CCNQNPCPQ*

The completeness of each of the following three sets of axiom schemes, with *Re* as the only primitive rule of procedure, follows by Lemmas 3, 1, and 2 in the first case, Lemmas 1, 4, and 2 in the second case, and Lemmas 1, 6, and 2 in the third case.

CN1.1 *CSCCPQCCQRCPR*
CN1.2 *CAPQAQP*
CN1.3 *CCNQNPCPQ*

CN2.1 *CCPQCCQRCPR*
CN2.2 *CRCAPQAQP*
CN2.3 *CCNQNPCPQ*

CN3.1 *CCPQCCQRCPR*
CN3.2 *CAPQAQP*
CN3.3 *CRCCNQNPCPQ*.

A complete formalization of the corresponding *m*-valued propositional calculus with modus ponens as the only primitive rule of procedure is given by the axiom schemes Ax1–Ax4 and *M*(*P*) defined in Section 14 of [10]. Each of the above formalizations with *Re* as the only primitive rule of procedure may be extended to formalizations of the corresponding *m*-valued calculi by taking the additional axiom scheme *M*(*P*). The plausibility of the above formalizations is established by the usual method. That *Re* is plausible follows from the observation that if P always takes the truth-value 1 then $CPQ =_T Q$ and hence $S =_T R$.

A complete formalization of the $\aleph_0$-valued propositional calculus with C as the only primitive functor and modus ponens as the only primitive rule of procedure is given by the following axiom schemes (see [8]).

A1 *CPCQP*
A2 *CCPQCCQRCPR*
A3 *CAPQAQP*
A4 *ACPQCQP.*

The completeness of each of the following sets of axiom schemes, with *Re* as the only primitive rule of procedure, follows by Lemmas 3, 1, and 2 in the first case, Lemmas 1, 4, and 2 in the second case, and Lemmas 1, 5, and 2 in the third case.

C1.1 *CSCCPQCCQRCPR*
C1.2 *CAPQAQP*
C1.3 *ACPQCQP*

C2.1 *CCPQCCQRCPR*
C2.2 *CRCAPQAQP*
C2.3 *ACPQCQP*

C3.1 *CCPQCCQRCPR*
C3.2 *CAPQAQP*
C3.3 *CRACPQCQP.*

A complete formalization of the corresponding m-valued propositional calculus with modus ponens as the only primitive rule of procedure is given by *A1–A4* and the additional axiom scheme $A(CP)^{m-1}QP$ (see [9]). Each of the above formalizations of the propositional calculus with C as the only primitive functor and *Re* as the only primitive rule of procedure may be extended to complete formalizations of the corresponding m-valued calculus by taking the additional axiom scheme $A(CP)^{m-1}QP$. The plausibility of each of the formalizations given above is established in the usual manner.

We now establish the independence of the axiom schemes and rule of procedure in each of the formalizations of the $\aleph_0$-valued propositional calculi given above. The independence of the rule of procedure *Re* follows from the observation that the generalized tautology *Cpp* is shorter than any of the axioms.

The axiom schemes are shown independent by the usual method, though somewhat indirectly. For the axiom schemes of the $\aleph_0$-valued propositional calculi with C and N as the only primitive functors we establish the independence of the formula schemes *Ax2*, *Ax3*, and *Ax4* in such a manner that the result extends to the corresponding result in each of the new formalizations given above. Similarly, for the $\aleph_0$-valued propositional calculi with C as the only primitive functor we establish the independence of *A2*, *A3*, and *A4*. The matrices given below also show that no further reduction in the number of axiom schemes can be achieved simply by replacing an axiom scheme T by a formula scheme such as $CP_1 \ldots CP_nT$, where $P_1, \ldots, P_n$ denote formula schemes. The matrices for *Ax3*, *A3*, and *A4* were found by computer program, although that for *A4* was given in [5] for a similar purpose. Above each matrix is the axiom scheme or schemes for which it establishes independence and below each matrix is a note of the assignment of values which demonstrates independence.

Ax2, A2

CPQ (P \ Q)	1	2	3	4	*NP*
*1	1	2	3	4	4
2	1	1	1	2	2
3	1	1	1	1	2
4	1	1	1	1	1

CCPQCCQRCPR
21232134224

Ax3

CPQ (P \ Q)	1	2	3	4	*NP*
*1	1	2	3	4	4
2	1	1	3	3	3
3	1	1	1	2	2
4	1	1	1	1	1

CAPQAQP
2123232

Ax4

CPQ (P \ Q)	1	2	*NP*
*1	1	2	1
2	1	1	2

CCNQNPCPQ
212211212

A3

CPQ (P \ Q)	1	2	3
*1	1	2	3
2	1	1	1
3	1	2	1

CAPQAQP
3132323

A4

CPQ (P \ Q)	1	2	3	4
*1	1	2	3	4
2	1	1	2	2
3	1	1	1	2
4	1	1	2	1

ACQPCPQ
2234243

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