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Stationary Logic and Its Friends—I

ALAN H. MEKLER and SAHARON SHELAH*

This is the first of two papers that deal with $L(aa)$ and related logics. Here we establish: every consistent $L_{\omega_1\omega}(Q)$ sentence has an $\underline{F}$-determinate model ($\underline{F}$ is a countable fragment of $L_{\omega_1\omega}(aa)$); and it is consistent that $L(Q)$ has the weak Beth property. A logic has the weak Beth property if it satisfies Beth's theorem where the hypothesis has been strengthened to require that implicit definitions guarantee existence as well as uniqueness. In [4] Friedman showed that Beth's theorem fails for $L(Q)$. He asked whether $L(Q)$ has the weak Beth property. (This is also problem 8 in [5].) Friedman has argued that people were interested in the weak Beth property and the usual theorems of Beth and Craig just happen to be true (for $L_{\omega\omega}$).

The two sections of this paper can be read independently. The methodological link between the sections is the use of forcing (set theoretic rather than model theoretic) to construct models. How does forcing help us? In the model theoretic proofs of the theorems of Beth and Craig, saturated models are used. Mainly one uses that these models have lots of automorphisms. Such models are harder (or impossible) to come by for other logics. Forcing can be viewed as giving Boolean-valued models. So we can use automorphisms which also move truth values. Sometimes by using the completeness theorem (or more generally absoluteness arguments) we can get rid of the forcing.

In the second paper we will use our methods to investigate the relation between $L(aa)$ and other logics. In particular we'll show it is consistent that $\Delta(L(Q)) \subseteq L(aa)$; Craig $(L(Q^{cf}_\omega), L(aa))$ holds (Q^{cf}_ω expresses "the cofinality of a linear order is ω"); and there is a compact Beth closed logic stronger than $L_{\omega\omega}$.[1] These results should be viewed against a background of counterexamples

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which seemed to indicate no "natural" $\aleph_0$-compact logic (other than $L_{\omega\omega}$) satisfied any variant of Craig's or Beth's Theorems. In particular there is an implicit $L(Q)$ Beth definition with no explicit $L(aa)$-definition (see [8] for this result and further references).[2]

1 Determinate models By $L_{\omega_1\omega}(Q)$, we mean the language $L_{\omega_1\omega}$ augmented by the quantifier Qx which expresses "for uncountably many x". The second-order quantifier $aa\ s$ expresses "for a closed unbounded set of countable subsets s". (That is, $A \models aa\ s\ \psi(s)$, if $\{X \subseteq A | X$ countable and $A \models \psi[X]\}$ contains a subset C closed under unions of countable chains such that any countable $Y \subseteq A$ is contained in some element of C.) Note that the Q quantifier is expressible in terms of the aa quantifier. One feature of $L(aa)$ which seems not to be shared by sublogics such as $L(Q)$ and L^{POS} is the ability to say a property holds on a stationary co-stationary set. ($L(aa)$ was introduced but not investigated in [9]. For information on $L(aa)$ see [1], but note [2].)

Let $\underline{F} \subseteq L_{\omega_1\omega}(aa)$. A model A is said to be *$\underline{F}$-determinate* if for all $\psi(\bar{s}, \bar{t}, \bar{x}) \in \underline{F}$, $A \models aa\ \bar{s}\ \forall \bar{x}(aa\ t\ \psi(\bar{s}, t, \bar{x}) \vee aa\ t \neg\psi(\bar{s}, \bar{t}, \bar{x}))$. (Roughly, the only $\underline{F}$-definable stationary sets are cubs.) This notion was introduced in [7] and studied in the finitary case in [3]. In [3], it is conjectured that every consistent $L(Q)$-theory has a finitely (i.e., $L_{\omega\omega}(aa)$-) determinate model.

Our main result is

1.1 Theorem *If $\underline{F}$ is a countable subset of $L_{\omega_1\omega}(aa)$, then every consistent $L_{\omega_1\omega}(Q)$-sentence has an $\underline{F}$-determinate model.*

We will show

1.2 Theorem *For every consistent sentence $\phi \in L_{\omega_1\omega}(Q)$, there is a notion of forcing $\mathbf{P}$ such that $V^{\mathbf{P}} \models$ there is an $L_{\omega_1\omega}(aa)$-determinate model of ϕ.*

The completeness theorem for $L_{\underline{A}}(aa)$ (where $\underline{A}$ is a countable admissable set) and the result above imply there is an $\underline{F}$-determinate model of ψ.

1.3 Corollary (of Theorem 1.1) *For any countable $\underline{F} \subseteq L_{\omega_1\omega}(aa)$ every relativized projective class (RPC) of $L_{\omega_1\omega}(Q)$ has an $\underline{F}$-determinate member. In particular every consistent $L^{POS}_{\omega_1\omega}$ sentence has an $\underline{F}$-determinate model.*

1.4 Corollary (of Theorem 1.1) *The $L(aa)$-elementary classes are not contained in the relativized projective classes of $L_{\omega_1\omega}(Q)$.*

This result should be contrasted with paragraph 5 of [1]. There a sentence ψ of $L(aa)$ which is not expressible in $L_{\infty\infty}$ is given. The proof there cannot be extended to projective classes. The referee has pointed out that a fairly direct omitting types argument shows the sentence $aa\ s \exists x \forall y(y < x \leftrightarrow s(y))$ is not *RPC* in $L_{\omega_1\omega}(Q)$.

Proof of Theorem 1.2: Fix A a model of ϕ. We can assume A has cardinality ω_1. We will use a quantifier $\bar{Q}\bar{x}$ (where $\bar{x}$ is a sequence of distinct variables) which expresses "for uncountably many disjoint tuples". Rather than deal with this quantifier directly we can assume A has an ω_1-like linear ordering $\leq$. Then in A and in any model $L(Q)$-equivalent to A, $\bar{Q}\bar{x}$ is first-order expressible.

Fix $\underline{A}$ a countable fragment of $L_{\omega_1\omega}(Q)$ so that $\phi \in \underline{A}$. For every formula $\psi \in \underline{A}$ there is a first-order (i.e., $L_{\omega_1\omega}$) formula $\hat{\psi}$ such that "$\leq$ is ω_1-like" $\rightarrow$ $(\psi \leftrightarrow \hat{\psi})$" is universally valid. We will assume if $\psi \in \underline{A}$, then $\hat{\psi} \in \underline{A}$.

Let $M(\nu) = \omega(1+\nu)$. If i is a successor ordinal or 0, a sequence $\bar{\alpha}_i$ will always denote ordinals in $M(i) - M(i-1)$. Let $\mathbf{P} = \{\psi[\bar{\alpha}_0, \bar{\alpha}_{i_1}, \ldots, \bar{\alpha}_{i_n}] | 0 < i_1 < \ldots < i_n, A \models \exists \bar{x}_0\, \bar{Q}\bar{x}_1, \ldots \bar{Q}\bar{x}_n\, \psi \wedge$ "all the free variables of ψ are pairwise not equal"$\}$. Order $\mathbf{P}$ by implication ($p \leq q$ if p is stronger than q, i.e., $\vdash p \rightarrow q$).

1.5 Lemma $\mathbf{P}$ *is equivalent to adding* ω_1 *Cohen reals.*

Proof: Define $\mathbf{P}_i = \{\psi(\bar{\alpha}) \in \mathbf{P} | \bar{\alpha} \subseteq M(i)\}$. Note $\mathbf{P}_0$ is countable and so is the same as adding a Cohen real. (We will explain below why $\mathbf{P}_0$ is nontrivial.)

First we must show: if $Y \subseteq \mathbf{P}_i$ is dense, then $X = \{q | \text{there is } p \in Y \text{ such that } q \leq p\}$ is dense in $\mathbf{P}$. Suppose $q = \psi[\bar{\alpha}_{k_1}, \ldots, \bar{\alpha}_{k_r}, \bar{\beta}_{k_{r+1}}, \ldots, \bar{\beta}_{k_n}] \in \mathbf{P}$, where $k_r \leq i < k_{r+1}$. If $p \leq \bar{Q}\bar{x}_{r+1} \ldots \bar{Q}\bar{x}_n \psi[\bar{\alpha}_{k_1}, \ldots, \bar{\alpha}_{k_r}, \bar{x}_{r+1}, \ldots, \bar{x}_n]$ and $p \in \mathbf{P}$, then $p \wedge q$ is a condition.

Next let $\tilde{Q}_i$ be the canonical $\mathbf{P}_i$-name for $\{p \in \mathbf{P}_{i+1} | p \wedge q$ is a condition for every $q \in G$ (the $\mathbf{P}_i$-generic set)$\}$. $\mathbf{P}_{i+1}$ can be construed as $\mathbf{P}_i * \tilde{Q}_i$. Also $V^{\mathbf{P}_i} \models \tilde{Q}_i$ is countable. For limit i, $\mathbf{P}_i$ is the direct limit of $\{\mathbf{P}_j | j < i\}$. Since $\mathbf{P}$ is the direct limit of the $\mathbf{P}_i$'s, to prove the lemma we need only show the $\tilde{Q}_i$ are nontrivial (i.e., $V^{\mathbf{P}_i} \models \tilde{Q}$ is nontrivial). The language of A has a linear order. For any ordering of a finite set of ordinals in $M(i+1) - M(i)$, there is a $\tilde{Q}_i$ condition which imposes this ordering. So forcing with $\tilde{Q}_i$ adds a $V^{\mathbf{P}_i}$ generic linear ordering of the ordinals in $M(i+1) - M(i)$.

Note that the lemma above shows $\mathbf{P}$ satisfies the ccc.

1.6 Lemma

(1) *For any* $p \in \mathbf{P}$ *and* $q = \psi[\bar{\alpha}_0, \ldots, \bar{\alpha}_{i_n}]$ *either* $p \wedge q$ or $p \wedge \neg q \in \mathbf{P}$.
(2) *If* G *is a* $\mathbf{P}$*-generic set, then it is the complete* $\underline{A}$ *diagram of a model* $\underline{A}$*-equivalent to* A.

Proof: (1) Without loss of generality we can assume p and q involve the same constants. Assume p is $\phi[\bar{\alpha}_0, \ldots \bar{\alpha}_{i_n}]$ and ϕ says the α's are pairwise different. An easy induction shows for all $\bar{a}_0, \ldots, \bar{a}_k$:

$$\text{if } A \models \bar{Q}\bar{x}_{k+1} \ldots \bar{Q}\bar{x}_n \phi[\bar{a}_0, \ldots \bar{a}_k, \bar{x}_{k+1}, \ldots, \bar{x}_n]$$

then either

$$A \models \bar{Q}\bar{x}_{k+1} \ldots \bar{Q}\bar{x}_n [\phi \wedge \psi[\bar{a}_0, \ldots, \bar{a}_k, \bar{x}_{k+1}, \ldots, \bar{x}_n]]$$

or

$$A \models \bar{Q}\bar{x}_{k+1} \ldots \bar{Q}\bar{x}_n [\phi \wedge \neg\psi[\bar{a}_0, \ldots, \bar{a}_k, \bar{x}_{k+1}, \ldots, \bar{x}_n]].$$

Take $\bar{a}_0$ so that $A \models \bar{Q}\bar{x}_1 \ldots \bar{Q}\bar{x}_n \phi[\bar{a}_0, \bar{x}_1, \ldots, \bar{x}_n]$. By applying this fact we can finish the proof.
(2) We work in $V[G]$. Since $\mathbf{P}$ is ccc, ω_1 (in V) remains the first uncountable ordinal. If $\alpha \neq \beta$ then $\{p | p \rightarrow \neg(\alpha = \beta)\}$ is dense. So for $\alpha \neq \beta$, $\neg(\alpha = \beta) \in G$. Let M be the model whose universe is ω_1 and whose atomic theory is given by

G. (We can assume $\underline{A}$ has no function symbols, so M is a structure of the appropriate similarity type.) By (1), G is complete (ψ or $\neg\psi \in G$). Sinc G is closed under conjunction, G is consistent. Also G contains the $\underline{A}$ theory of A. Note as well that $M \models$ "$\leq$ is ω_1-like", since for $\alpha \in M(i)$ and $\beta \notin M(i)$, $\beta \leq \alpha$ is not a condition.

We'll show by induction on the construction of formulas in $\underline{A}$ that $M \models \phi[\bar{\alpha}]$ iff $\phi(\bar{\alpha}) \in G$. Since $\leq^M$ is ω_1-like, it is enough to consider $\phi \in \underline{A} \cap L_{\omega_1\omega}$. The cases for atomic sentences and negation are easy. If $\Lambda\Phi[\bar{\alpha}] \in G$, then since G is complete and consistent, $\phi[\bar{\alpha}] \in G$, for all $\phi \in \Phi$. So $M \models \phi[\bar{\alpha}]$ for all $\phi \in \Phi$. Suppose $\neg\Lambda\Phi[\bar{\alpha}] \in G$. Then $\{p \in \mathbf{P} |$ there is $\phi \in \Phi$ so that $p \rightarrow \neg\phi[\bar{\alpha}]\}$ is dense below $\neg\Lambda\Phi[\bar{\alpha}]$. So for some $\phi \in \Phi$, $\neg\phi[\bar{\alpha}] \in G$. Hence $M \models \neg\Phi[\bar{\alpha}]$. If $M \models \exists x\phi(x, \bar{\alpha})$, then the induction hypothesis and the consistency of G imply $\exists x\phi(x, \bar{\alpha}) \in G$. Suppose $\exists x\phi(x, \bar{\alpha}) \in G$. As in (1), we can establish by induction: for all ψ, if $A \models \exists\bar{x}_0 \bar{Q}\bar{x}_1 \ldots \bar{Q}\bar{x}_n \exists y\psi(\bar{x}, y)$, then either for some i and $x' \in \bar{x}_i$ $A \models \exists\bar{x}_0\ \bar{Q}\bar{x}_1 \ldots \bar{Q}\bar{x}_n\psi(\bar{x}, x')$, or $\left(\text{letting } \hat{\psi}(\bar{x}, y) \text{ be } \psi(\bar{x}, y) \wedge \bigwedge_{x \in \bar{x}} x \neq y\right)$ $A \models \exists\bar{x}_0 y \bar{Q}\bar{x}_1 \ldots \bar{Q}\bar{x}_n \hat{\psi}(\bar{x}, \bar{y})$, or for some $k \geq 1$, $A \models \exists\bar{x}_0 \bar{Q}\bar{x}_1 \ldots \bar{Q}\bar{x}_k y \bar{Q}\bar{x}_{k+1} \ldots \bar{Q}\bar{x}_n \hat{\psi}(\bar{x}, y)$.

To see this, choose precise submodels $A_0 \prec A_1 \prec \ldots \prec A_n = A$ of A and $\bar{a}_i \in A_i - A_{i-1}$ for $i = 0, \ldots, n$ (where $A_{-1} = \emptyset$) such that $A \models \exists y\psi(\bar{a}, y)$. If $A \models \psi(\bar{a}, b)$ for b occurring in $\bar{a}$ then we have the first case. Otherwise $A \models \hat{\psi}(\bar{a}, b)$ for some b. If $b \in A_0$, then $A \models \exists\bar{x}_0 y \bar{Q}\bar{x}_1 \ldots \bar{Q}\bar{x}_n \hat{\psi}(\bar{x}, y)$. If $b \in A_k - A_{k-1}$ for $1 \leq i \leq n$, then $A \models \exists\bar{x}_0 \bar{Q}\bar{x}_1 \ldots \bar{Q}\bar{x}_k y \bar{Q}\bar{x}_{k+1} \ldots \bar{Q}\bar{x}_n \hat{\psi}(\bar{x}, y)$. Using this fact it is easy to show $\{p |$ for some γ, $p \rightarrow \phi(\gamma, \bar{\alpha})\}$ is dense below $\exists x\phi(x, \bar{\alpha})$.

To complete the proof of Theorem 1.2 we need more information about the forcing. Fix an ordinal i and suppose G_i is $\mathbf{P}_i$ generic. In $V[G_i]$ define $\mathbf{Q} = \{p \in \mathbf{P} | p \wedge q \in \mathbf{P}$ for all $q \in G_i\}$. (Note by Lemma 1.5, $\mathbf{P} \simeq \mathbf{P}_i * \tilde{\mathbf{Q}}$ where $\tilde{\mathbf{Q}}$ is the canonical name of $\mathbf{Q}$.) Suppose ξ, $\nu_1 < \ldots < \nu_n$, $\tau_1 < \ldots < \tau_n$ are countable ordinals such that:

(*) $\quad i < \nu_1, \tau_1$; and for all k if $\alpha < \nu_k(\tau_k)$ then $\alpha + \xi < \nu_k(\tau_k)$.

A one-to-one function f with domain $\alpha_1 < \ldots < \alpha_m$ is said to be *ξ-appropriate* with respect to $\bar{\nu}$, $\bar{\tau}$ if:

(0) $\bar{\nu} \subseteq \bar{\alpha}$ and for all k, $f(\nu_k) = \tau_k$
(1) For all k, $\alpha_k \in M(i)$ implies $f(\alpha_k) = \alpha_k$
(2) For all j and k_j, $\alpha_i \in M(\nu_k)$ iff $f(\alpha_j) \in M(\tau_k)$
(3) Suppose $\alpha_j \in M(\delta + 1) - M(\delta)$, $\alpha_{j+1} \in M(\delta' + 1) - M(\delta')$, $f(\alpha_j) \in M(\rho + 1) - M(\rho)$, and $f(\alpha_{j+1}) \in M(\rho' + 1) - M(\rho)$. Further assume $\delta' = \delta + \beta$ and $\rho' = \rho + \beta'$. Then $\beta \equiv \beta' \pmod{\xi}$, i.e., for some $\beta_0 < \xi$ and some μ, μ', $\beta = \xi \cdot \mu + \beta_0$ and $\beta' = \xi \cdot \mu' + \beta_0$. Further $\beta < \xi$ iff $\beta' < \xi$.

(We often omit saying "with respect to $\bar{\nu}$, $\bar{\tau}$".)

1.7 Lemma *We work in $V[G_i]$ and conditions are in $\mathbf{Q}$. For every $\phi(\bar{s}, \bar{x}) \in L_{\omega_1\omega}(aa)$ there is an ordinal $\xi_\phi > 0$ such that: if $\nu_1 < \ldots \nu_n$, $\tau_1 < \ldots < \tau_n$ and ξ_ϕ satisfy (*), f with domain $\bar{\alpha}$ is ξ_ϕ-appropriate and $\psi(\bar{\alpha}) \in \mathbf{Q}$, then*

$\psi(\bar\alpha) \Vdash \tilde{M} \vDash \phi[M(\bar\nu), \bar\alpha]$ *iff* $\psi(f(\bar\alpha)) \Vdash \tilde{M} \vDash \phi(M(\bar\tau), f(\bar\alpha))$. *Here* $\tilde{M}$ *is the canonical name of the model formed from G.* [*Note*: $\psi(f(\bar\alpha)) \in \mathbf{Q}$.] *Further,* $1 \Vdash \tilde{M} \vDash aa\ \bar{s}\ \forall\bar{x}(aa\ t\ \phi(\bar{s}, t, \bar{x}) \vee aa\ t \neg\phi(\bar{s}, t, \bar{x}))$.

Proof: We define ξ_ϕ and prove the lemma by induction on the construction of formulas.

Atomic There are two cases. First suppose ϕ is $s_k(x_j)$. Let $\xi_\phi = 1$. Now $\psi(\bar\alpha) \Vdash \tilde{M} \vDash M(\nu_k)(\alpha_j)$ iff $\alpha_j \in M(\nu_k)$ iff $f(\alpha_j) \in M(\tau_k)$ (since f is 1-appropriate) iff $\psi(\bar{f}(\alpha)) \Vdash \tilde{M} \vDash M(\tau_k)(f(\alpha_j))$.

Assume ϕ has no second-order variables. Let $\xi_\phi = 1$. $\psi(\bar\alpha) \Vdash \tilde{M} \vDash \phi(\bar\alpha)$ iff $\psi(\bar\alpha) \wedge \neg\phi(\bar\alpha) \notin \mathbf{Q}$ iff $\psi(f(\bar\alpha)) \wedge \neg\phi(f(\bar\alpha)) \notin \mathbf{Q}$.

Conjunction Suppose ϕ is $\Lambda\Phi$. Let $\xi_\phi = \sup\{\xi_\phi | \phi \in \Phi\}$. The induction hypothesis is verified since a conjunction is forced iff each conjunct is.

Negation Suppose ϕ is $\neg\theta(\bar{s}, \bar{x})$. Let $\xi_\phi = \xi_\theta\cdot\omega$. Suppose $\psi(\bar\alpha) \nVdash \tilde{M} \vDash \phi(M(\bar\nu), \bar\alpha)$. For some $\psi'(\bar\alpha, \bar\gamma) \leq \psi(\bar\alpha)$, $\psi'(\bar\alpha, \bar\gamma) \Vdash \tilde{M} \vDash \theta(M(\bar\nu), \bar\alpha)$. We can extend f to a ξ_θ-appropriate $\hat{f}$ whose domain is $\bar\alpha \cup \bar\gamma$. So $\psi'(\hat{f}(\bar\alpha), \hat{f}(\bar\gamma)) \Vdash \tilde{M} \vDash \theta(M(\bar\tau), f(\bar\alpha))$. So $\psi(f(\bar\alpha)) \nVdash \tilde{M} \vDash \phi(M(\bar\tau), f(\bar\alpha))$. Since f^{-1} is ξ_ϕ-appropriate, the other direction has also been proved.

Existential quantification Suppose ϕ is $\exists y\theta(\bar{s}, \bar{x}, y)$. Let $\xi_\phi = \xi_\theta\cdot\omega^2$. Suppose $\psi(\bar\alpha) \Vdash \tilde{M} \vDash \phi(M(\bar\nu), \bar\alpha)$. Consider $\psi'(f(\bar\alpha), \bar\gamma) \leq \psi(f(\bar\alpha))$. Choose $\hat{f}$ a $\xi_\theta\cdot\omega$ appropriate extension of f^{-1} with domain $f(\bar\alpha) \cup \bar\gamma$. Choose β and $\psi''(\bar\alpha, \hat{f}(\bar\gamma), \bar\delta, \beta) < \psi'(\bar\alpha, \hat{f}(\bar\gamma))$ so that $\psi''(\bar\alpha, \hat{f}(\bar\gamma), \bar\delta, \beta) \Vdash \tilde{M} \vDash \theta(M(\bar\nu), \bar\alpha, \beta)$. Choose g a ξ_θ-appropriate extension of $\hat{f}^{-1}$ with domain $\bar\alpha \cup \hat{f}(\bar\gamma) \cup \bar\delta \cup \{\beta\}$. So $\psi''(f(\bar\alpha), \bar\gamma, g(\bar\delta), g(\beta)) \Vdash \tilde{M} \vDash \exists y\theta(M(\bar\tau), f(\bar\alpha), y)$.

Second-order quantification Suppose ϕ is $aa\,t\,\theta(\bar{s}, t, \bar{x})$. Let $\xi_\phi = \xi_\theta\cdot\omega$. Take $\nu_{n+1} > \nu_n, \bar\alpha$ so that for all $\beta < \nu_{n+1}$, $\beta + \xi_\theta < \nu_{n+1}$. Suppose $\psi'(\bar\alpha, \bar\gamma, \bar\delta)$ decides whether $M \vDash \theta(M(\bar\nu), M(\nu_{n+1}), \bar\alpha)$, where $\bar\gamma \in M(\nu_{n+1})$ and $\bar\delta \notin M(\nu_{n+1})$. For convenience, assume $\psi'(\bar\alpha, \bar\gamma, \bar\delta) \Vdash \tilde{M} \vDash \theta(M(\bar\nu), M(\nu_{n+1}), \bar\alpha)$. We claim $\psi'(\bar\alpha, \bar\gamma, \bar\delta) \Vdash \tilde{M} \vDash aa\,t\,\theta(M(\bar\nu), t, \bar\alpha)$. Consider $\nu' > \nu_n, \bar\alpha, \bar\gamma$ such that for all $\beta < \nu'$, $\beta + \xi_\phi < \nu'$ (such ν' form a cub). Further suppose $p < \psi'(\bar\alpha, \bar\gamma, \bar\delta)$. We can choose $\bar\delta'$ so that $p \wedge \psi'(\bar\alpha, \bar\gamma, \bar\delta') \in \mathbf{Q}$ and there is a function g ξ_θ-appropriate with respect to $\nu_1, \ldots, \nu_{n+1}$ and $\nu_1, \ldots, \nu_n, \nu'$ with domain $\bar\alpha \cup \bar\gamma \cup \bar\delta \cup \{\nu_{n+1}\}$ such that g is the identity on $\bar\alpha \cup \bar\gamma$ and $g(\bar\delta) = \bar\delta'$. By induction $p \wedge \psi'(\bar\alpha, \bar\gamma, \bar\delta') \Vdash \tilde{M} \vDash \theta(M(\bar\nu), M(\nu'), \bar\alpha)$. So $\psi'(\bar\alpha, \bar\gamma, \bar\delta) \Vdash \tilde{M} \vDash aa\,t\,\theta(M(\bar\nu), t, \bar\alpha)$. Similarly if $\psi'(\bar\alpha, \bar\gamma, \bar\delta) \Vdash \tilde{M} \vDash \neg\theta(M(\bar\nu), M(\nu_{n+1}), \bar\alpha)$, then $\psi'(\bar\alpha, \bar\gamma, \bar\delta) \Vdash \tilde{M} \vdash aa\,t\neg\theta(M(\bar\nu), M(\nu_{n+1}), \bar\alpha)$. Note we have shown $1 \Vdash \tilde{M} \vDash aa\bar{s}\forall x(aa\,t\,\theta(\bar{s}, t, \bar{x}) \vee aa\,t\neg\theta(\bar{s}, t, \bar{x}))$. Choose $\tau_{n+1} > \tau_n, f(\bar\alpha)$ so that for all $\beta < \tau_{n+1}$, $\beta + \xi_\phi < \tau_{n+1}$. If $\psi(\alpha) \nVdash \tilde{M} \vDash aa\,t\,\theta(M(\bar\nu), t, \bar\alpha)$. Then for some $\psi'(\bar\alpha, \bar\gamma, \bar\delta)$ as above $\psi'(\bar\alpha, \bar\gamma, \bar\delta) \Vdash \tilde{M} \vDash \neg\theta(M(\bar\nu), M(\nu_{n+1}), \bar\alpha)$. Choose $\hat{f}$ a ξ_θ-appropriate extension of f with domain $\bar\alpha \cup \bar\gamma \cup \bar\delta \cup \{\nu_{n+1}\}$ so that $\hat{f}(\nu_{n+1}) = \tau_{n+1}$. By induction $\psi(\hat{f}(\bar\alpha), \hat{f}(\bar\gamma), \hat{f}(\bar\delta)) \Vdash \tilde{M} \vDash \neg\theta(M(\bar\tau), M(\tau_{n+1}), f(\bar\alpha))$. So by the above argument $\psi(f(\bar\alpha)) \nVdash \tilde{M} \vDash \phi(M(\bar\tau), f(\alpha))$.

We can now complete the proof of Theorem 1.2. Suppose G is $\mathbf{P}$-generic and $\psi(\bar{s}, \bar{t}, \bar{x}) \in L_{\omega_1\omega}(aa)$ (in $V[G]$). Choose i so that $\psi \in V[G_i]$ (where $G_i = G \cap P_i$). But Lemma 1.7 implies $V[G] \vDash M \vDash aa\bar{s}\forall\bar{x}[aa\,t\,\psi(\bar{s}, t, \bar{x}) \vee aa\,t\neg\psi(\bar{s}, t, \bar{x})]$.

It is possible to strengthen Theorem 1.2.

1.8 Theorem *It is relatively consistent with ZFC that every consistent $\phi \in L_{\omega_1\omega}(Q)$ has an $L_{\omega_1\omega}(aa)$ determinate model.*

Proof: Let $\mathbf{P}$ be the forcing which adds ω_1 Cohen reals. Suppose G is $\mathbf{P}$-generic and $\phi \in V[G]$ is a consistent $L_{\omega_1\omega}(Q)$ sentence. We can construe $V[G]$ as $V[G'][H]$ where $\phi \in V[G']$ and H is $\mathbf{Q}$-generic for some $\mathbf{Q} \in V[G']$ which adds ω_1 Cohen reals. By Lemma 1.5, $V[G'][H] \models \phi$ has an $L_{\omega_1\omega}(aa)$-determinate model.

1.9 Theorem *It is relatively consistent with ZFC that some consistent $\phi \in L_{\omega_1\omega}(Q)$ has no $L_{\omega_1\omega}(aa)$ determinate model.*

Proof: Assume $MA + \neg CH$. Let ϕ say: (i) the model is the disjoint union of P and Q; (ii) P is the natural numbers $\left(\text{i.e., } \left(P(x) \leftrightarrow \bigvee_{n<\omega} x = c_n\right) \wedge \bigwedge_{n \neq m} c_n \neq c_m\right)$; E is a relation between P and Q; extensionality, $Q(x) \wedge Q(y) \rightarrow (x = y \leftrightarrow \forall z(zEx \leftrightarrow zEy))$; and $<$ is an $\aleph_1$-like ordering of Q. Let M be a model of ϕ. We can assume $Q^M = \{r_i | i < \omega_1\}$ is a set of reals.

Choose $S \subseteq \omega_1$ a stationary co-stationary set. Let $Y = \{\langle r_i, r_j\rangle | i < j$ and $j \in S\}$ and let $X = (Q^M)^2 - Y$. It is well known [cf. [6], Lemma 44.1, p. 564] that there is an F_σ set of pairs of reals A such that $A \supseteq Y$ and $A \cap X = \emptyset$. So there is a formula $\psi(x, y) \in L_{\omega_1\omega}$ such that $M \models \psi[a, b] \Leftrightarrow \langle a, b\rangle \in A$. But we have really coded S inside the model. Namely $\exists x \forall y(Q(y) \rightarrow (s(y) \leftrightarrow \psi(y, x)))$ holds for a stationary co-stationary set.

It might be thought the class of sentences considered is too narrow. One might consider sentences of the form $aa\bar{s}\psi(\bar{s})$ where ψ has no second-order quantifiers. (Under the assumption of finite determinacy every finitary sentence is equivalent to such a sentence.)

1.10 Counterexample There is a consistent sentence of the form $aa\bar{s}\psi(\bar{s})$ where ψ has no second-order quantifiers which has no finitely determinate model.

Proof: Let θ say of a model M: L is ω_1-like; A is a countable initial segment; $aa\, s$ "sup s exists"; and $f: M \times A \rightarrow M$ is such that if b is not a successor then $f(b, \_\_)$ enumerates an increasing cofinal sequence. We shall pretend the model is $\langle \omega_1, <, \omega, f\rangle$ (the general case is no harder, just messier). Let $B_{\alpha n} = \{\delta | f(\delta, n) = \alpha\}$. For some n there are $\alpha \neq \beta$ $B_{\alpha n}$ and $B_{\beta n}$ are both stationary (and disjoint) (cf. [6], Lemma 7.6, p. 59).

2 Weak Beth for $L(Q)$ In this section we will show it is consistent that $L(Q)$ has the weak Beth property. Let P be a new relation symbol. A formula $\psi[P]$ is said to be a Beth definition of P if $\models \psi[P] \wedge \psi[P'] \rightarrow P = P'$. A Beth definition $\psi[P]$ is a weak Beth definition if for any structure M there is some relation R such that $M \models \psi[R]$. A logic L has the weak Beth property if every weak Beth definition is equivalent to an explicit definition.

Fix an $L(Q)$ Beth definition $\psi[P]$ which is not equivalent to an explicit definition. For notational convenience we'll assume P is unary. First we describe a notion of forcing $\mathbf{P}$ so that $V^{\mathbf{P}} \models$ "$\psi[P]$ is not a weak Beth definition". (This result establishes an "absolute" weak Beth property.)

Since $L(Q)$ is $\aleph_0$-compact we can choose A an $L \cup \{P\}$ structure such that $A \models \psi[P]$ and there are $a, b \in A$ such that $A \models P(a) \wedge \neg P(b)$ and $(A', a) \equiv_Q (A', b)$. [Here A' is the reduct of A to L and $\equiv_Q$ denotes $L(Q)$-equivalence.] There is no harm in assuming for every $\phi \in L(Q)$ there is a relation R_ϕ such that $A \models \phi(\bar{x}) \leftrightarrow R_\phi(\bar{x})$. (So the first-order structure of A contains all the $L(Q)$ information.)

2.1 Proposition *There is $(A^*, a, b) \equiv_Q (A, a, b)$ such that $A^*|L$ is ω-homogeneous and only countably many types (over the empty set) are realized in $A^*|L$. Further if $\bar{c}$ and $\bar{d}$ realize the same type and for some e $\{\bar{x}|\bar{c}x$ realizes the same type as $\bar{c}e\}$ is uncountable then $\{x|\bar{d}x$ realizes the same type as $\bar{c}e\}$ is uncountable.*

Proof: Choose a listing $\{\phi_{n,m}|n, m < \omega\}$ of the formulas of $L(Q)$ (without P) so that for each $n\{\phi_{n,m}|m < \omega\}$ enumerates the formulas with n free variables. We will make further demands of this listing. Let B be the two sorted structure $\langle A, U, a, b\rangle \cup \langle \omega, <\rangle$ together with relations $R_n \leq \omega \times A^n \times A^n (n < \omega)$ defined so that for all $n, i < \omega$, $B \models R_n(i, \bar{x}, \bar{y}) \leftrightarrow \bigwedge_{j \leq i} \phi_{nj}(\bar{x}) \leftrightarrow \phi_{nj}(\bar{y})$. The listing can be chosen so that: for all $i, n < \omega$ $B \models R_n(i, \bar{x}, \bar{y}) \rightarrow \forall w \exists z R_{n+1}(i, \bar{x}, w, \bar{y}, z)$ and $B \models R_n(i, \bar{x}, \bar{y}) \rightarrow \forall w[QzR_{n+1}(i, \bar{x}, w, \bar{x}, z) \rightarrow QzR_{n+1}(i, \bar{x}, w, \bar{y}, z)]$. We choose U a countable subset of A so that for all n, $B \models \forall i \forall \bar{x} \exists \bar{y}\left(\bigwedge_{y \in \bar{y}} U(y) \wedge R_n(i, \bar{x}, \bar{y})\right)$. Let $B^* = \langle A^*, U^*, a, b\rangle \cup \langle \omega^*, <\rangle$ be a model $\equiv_Q B$ with ω^* nonstandard. Then it is easy to see A^* is the desired model.

Fix C a countable homogeneous elementary (first-order) substructure of $A^*|L$ such that every type realized in A^* is also realized in C. (Note that our assumption about the R_ϕ's implies two sequences realize the same first-order type iff they realize the same $L(Q)$-type.) Of course C is unique up to isomorphism. Define $\mathbf{P} = \{M|M$ is an L structure on some limit $\alpha < \omega_1$ and $M \simeq C\}$ together with the partial ordering defined by: $M < M'$ if $M \succ M'$ and for all types p and $\bar{\beta}$ there is $\gamma \in M - M'$ such that $M \models p(\bar{\beta}, \gamma)$ iff there is some $\bar{c} \in C$ realizing the same type as $\bar{\beta}$ such that $A^* \models Qxp(\bar{c}, x)$. Note: by the choice of A^* the "some $\bar{c}$" above can be replaced by "all $\bar{c}$".

It is easy to check the following.

2.2 Proposition *For any α, $\{M|\alpha \in M\}$ is dense (in $\mathbf{P}$). Also, $\mathbf{P}$ is closed under unions of countable chains.* (To prove the second assertion use the fact that the union of a countable elementary chain of countable homogeneous structures all realizing the same types is isomorphic to every member of the chain.)

Assume that G is a $\mathbf{P}$-generic set. So taking $\bigcup_{M \in G} M$ we have a structure on ω_1, which we will call B. We now work in $V[G]$.

2.3 Proposition *For all $\phi \in L(Q)$ and $\bar{\beta} < \omega_1$, $B \models \phi[\bar{\beta}]$ iff there is $M \in G$ so that $M \models R_\phi[\bar{\beta}]$.*

Proof: This is an easy induction.

2.4 Proposition *B is $L_{\infty\omega}(Q)$-equivalent to $A^*|L$.*

Proof: We show by induction on formulas ϕ that if $\bar{\beta}$ and $\bar{c}$ realize the same type then $B \models \phi[\bar{\beta}]$ iff $A * \models \phi[\bar{c}]$. The only interesting case occurs when we encounter the Q quantifier. Suppose $B \models Qx\phi[x, \bar{\beta}]$. Since only countably many types are realized in B, there is some type p such that $B \models Qxp(x, \bar{\beta}) \wedge (p(x, \bar{\beta}) \rightarrow \phi(x, \bar{\beta}))$. (The induction hypothesis is used to see $B \models p(x, \bar{\beta}) \rightarrow \phi(x, \bar{\beta})$.) By the choice of A^* and P, since $\bar{\beta}$ and $\bar{c}$ realize the same type $A^* \models Qxp(x, \bar{c})$. Hence $A^* \models Qx\phi(x, \bar{c})$. The other direction is similar.

2.5 Theorem *There is no $U \subseteq \omega_1$ so that $B \models \psi[U]$.*

Proof: Suppose such a U exists.

Claim There are α, $\beta \in \omega_1$ such that α and β realize the same type and $\alpha \in U$ and $\beta \notin U$.

Proof (*of Claim*): Otherwise U is $L_{\infty\omega}(Q)$-definable in B. Since A^* is $L_{\infty\omega}(Q)$-equivalent to B, this definition works in A^*. But either a and b (the elements of A^*) are both in the set so defined or both outside. So A^* must have two subsets which satisfy the Beth definition.

Choose a name $\tilde{U}$ for U and pick $M \in G$ so that $M \Vdash \tilde{B} \models \psi[\tilde{U}] \wedge \check{\alpha} \in \tilde{U} \wedge \check{\beta} \notin \tilde{U}$. We can assume α, $\beta \in M$. (So M also forces α and β realize the same type.) Here $\tilde{B}$ is the canonical name for B (i.e., derived from the canonical name for G.) Let $\mathbf{P}' = \{M' | M' \leq M\}$. We can replace $\tilde{U}$ by a $\mathbf{P}'$ name $\tilde{U}'$ such that: $1 \Vdash^{\mathbf{P}'} \tilde{B}' \models \psi[\tilde{U}'] \wedge \alpha \in \tilde{U}' \wedge \beta \notin \tilde{U}'$. (Here $\tilde{B}'$ is the obvious $\mathbf{P}'$ name for B.) Further we can assume each element of $\tilde{U}'$ has the form $\langle \gamma, p \rangle$ where $\gamma \in \omega_1$ and $p \in \mathbf{P}'$.

Let h be an automorphism of M such that $h(\alpha) = \beta$. Extend h to a permutation of ω_1 by letting h be the identity function outside the universe of M. Since for all limit ordinals $\nu \geq$ the universe of $Mh \restriction \nu$ is a permutation of ν, h induces an automorphism $\hat{h}$ of $\mathbf{P}'$. Let $\tilde{W} = \{\langle h(\nu), \hat{h}(p)\rangle | \langle \nu, p \rangle \in \tilde{U}'\}$. We will show $1 \Vdash \tilde{M} \models \psi(\tilde{W})$. So $1 \Vdash \tilde{B}' \models \psi(\tilde{W}) \wedge \psi(\tilde{U}) \wedge \tilde{W} \neq \tilde{U}$. But being a Beth definition is absolute.

For notational convenience all forcing will be relative to $\mathbf{P}'$ and we will omit putting $\check{}$ over names. Also let X_ν denote the ordinal ν when it is substituted for a second-order variable.

Claim Suppose $\varphi(X, \bar{x}, \bar{s})$ is an $L(aa)$ formula. For all $p \in \mathbf{P}'$, $\nu_1, \ldots, \nu_n$, $\tau_1, \ldots, \tau_m$ (where each τ_i is a limit ordinal $\geq$ universe of M and $< \omega_1$), $p \Vdash \tilde{B}' \models \varphi(\tilde{U}, \bar{\nu}, \bar{X}_\tau)$ if $\hat{h}(p) \Vdash \tilde{B}' \models \varphi(\tilde{W}, h(\bar{\nu}), \bar{X}_\tau)$.

Proof (*of Claim*): We proceed by induction on the construction of formulas. The definitions of $\tilde{W}$ and h have been chosen so that the atomic case is clear. Conjunction and negation are easy.

Suppose $p \Vdash \tilde{B}' \models \exists x \varphi(\tilde{U}, \bar{\nu}, x, \bar{X}_\tau)$. For $r < \hat{h}(p)$, choose $q < \hat{h}^{-1}(r)$ and γ such that $q \Vdash \tilde{B} \models \varphi(\tilde{U}, \bar{\nu}, \gamma, X_\tau)$. So $\hat{h}(q) \Vdash \tilde{B}' \models \varphi(\tilde{W}, h(\bar{\nu}), h(\gamma), \bar{X}_i)$. Hence $\hat{h}(p) \Vdash \tilde{B}' \models \exists x \varphi(\tilde{W}, h(\bar{\nu}), x, X_\tau)$. (The other direction is the same.) Now suppose $p \Vdash \tilde{B}' \models aas\ \varphi(\tilde{U}, \bar{\nu}, \bar{X}_\tau, s)$ and take G a $\mathbf{P}'$-generic set containing $h(p)$. Consider $\hat{h}^{-1}(G)$ and C a cub in $V[\hat{h}^{-1}(G)] (= V[G])$ such that for all $X_\gamma \in C$ $B \models \varphi(U, \bar{\nu}, \bar{X}_\tau, X_\gamma)$. (We can assume any element of C is X_γ for some limit ordinal $\gamma \geq$ the universe of M.) For any $X_\gamma \in C$ there is $q \in \hat{h}^{-1}(G)$ so that $q \Vdash \tilde{B} \models \varphi(\tilde{U}, \bar{v}, \bar{X}_\tau, X_\gamma)$. So $\hat{h}(q) \Vdash \tilde{B}' \models \varphi(\tilde{W}, h(\bar{v}), X_\tau, X_\gamma)$. Hence $V[G] \models$ for all $X_\gamma \in C$, $B \models \varphi(W, h(\bar{v}), \bar{X}_\tau, X_\gamma)$. Since G was arbitrary, $\hat{h}(p) \Vdash \tilde{B} \models aas\, \varphi(\tilde{W}, h(\bar{v}), \bar{X}_\tau, s)$.

We have proved:

2.6 Theorem *If $\psi[R]$ is a Beth definition which is not equivalent to an explicit definition then there is a notion of forcing* **P** *such that $V^{\mathbf{P}} \models$ "$\psi[R]$ is not a weak Beth definition".*

2.7 Remark: The forcing model constructed above has the property that any relation implicitly defined by an $L(aa)$-formula is defined by a disjunction of types.

2.8 Theorem *It is consistent, assuming the consistency of ZF, that every weak Beth definition is equivalent to an explicit definition.*

Proof: We can assume $V \models CH$. Let **Q** be the conditions which add a Cohen generic subset of ω_1 (i.e., force with functions from countable α's to 2). Suppose H is **Q**-generic. Suppose $\psi[R]$ is a Beth definition which is not equivalent to an explicit definition. In V find **P** as before. Note both V and $V[H]$ satisfy $1 \Vdash^{\mathbf{P}} \psi[R]$ is not a weak Beth definition. These are two possibilities; either **P** has (in V) a cofinal subset which is a complete ω_1-branching tree of height ω_1 *or* there is $M_0 \in \mathbf{P}$ such that $\{M | M \leq M_0\}$ is linearly ordered. In the first case there is an embedding from a cofinal subset of **P** onto a cofinal subset of **Q**. So there is $G \in V[H]$ which is **P**-generic (over V) such that $V[G] = V[H]$. So $V[H] \models \psi[R]$ is not a weak Beth definition. In the second case let $G = \{M | M$ is comparable with $M_0\}$. So G is **P**-generic (over $V[H]$) and $V[H][G] = V[H] \models \psi[R]$ is not a weak Beth definition.

Open Questions: (1) Is "$L(Q)$ has the weak Beth property" provable in *ZFC?* (2) Has every $\psi \in L(Q)$ a model M with countably many automorphism classes? (a and b are automorphically equivalent if there is $f \in Aut(M)$ so that $f(a) = b$.)

NOTES

1. Only a weak form of $\Delta(L(Q)) \subseteq L(aa)$ appears in the second paper. The other results mentioned will appear elsewhere.
2. The results of these papers are primarily due to Shelah. Mekler worked out the details of the proofs, contributed a few remarks, and wrote the paper.

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A. Mekler
Department of Mathematics
Simon Fraser University
Burnaby, British Columbia
Canada V5A 1S6

S. Shelah
Department of Mathematics
The Hebrew University
Jerusalem, Israel