Notre Dame Journal of Formal Logic
Volume 26, Number 2, April 1985

Recursively Saturated Models Generated by Indiscernibles

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The theorem which is proved here has its origins in a question raised by A. Macintyre: Is there a recursively saturated model of Peano Arithmetic which is generated by a set of indiscernibles? To give this question respectability, we understand PA to be formalized so as to include terms for all definable functions. Since recursively saturated models are in some sense large whereas models generated by indiscernibles are small, the positive answer to Macintyre's question obtained by Abramson and Knight [3] was unexpected. Their proof showed that every consistent extension of PA has a countable, recursively saturated model which is generated by a set of indiscernibles. The countability comes as no surprise, for by stretching and shrinking the indiscernibles generating a recursively saturated model, one can obtain indiscernible generators for a recursively saturated model of arbitrary infinite cardinality.

This answer to Macintyre's question suggests the following modification of his question: Is *every* countable, recursively saturated model of PA generated by a set of indiscernibles? We demonstrate here that this question also has a positive answer. It is natural to consider variations of this question with PA replaced by some other theory, such as an extension of *ZFC* which has definable Skolem functions (e.g., $ZFC + V = L$). The answer to the question for such theories is also positive. What is unusual is that not all Skolem functions need be definable, but in order to carry out the proof, the existence of what may be called a β-function, which is a binary function encoding all finite sequences, is required.

*Research was supported in part by NSF Grant MCS 79-05028.

Received January 15, 1982; revised July 7, 1982

Theorem *Suppose T is a theory in a recursive language which includes the binary function symbol β such that for each $n < \omega$ the sentence*

$$\forall x_0, \ldots, x_{n-1} \forall y_0, \ldots, y_{n-1} \exists x \left[\bigwedge_{i<j<n} x_i \neq x_j \rightarrow \bigwedge_{i<n} \beta(x_i, x) = y_i \right]$$

is a consequence of T. Then every countable, recursively saturated model of T is generated by a set of indiscernibles.

Notice that T need not be a Skolem theory. An example of a theory satisfying the hypothesis of the theorem is ZF if included in the formalization is a symbol β for function evaluation: if f is a function and x an element in its domain, then $\beta(x, f) = f(x)$.

For introductory information about recursive saturation see Barwise-Schlipf [2]. Two important properties that result from a structure $\mathfrak{B}$ being both countable and recursively saturated are the following:

(i) $\mathfrak{B}$ is resplendent. Thus, whenever T_0 is a recursive theory consistent with $Th(\mathfrak{B})$, then $\mathfrak{B}$ has an expansion to a recursively saturated model of T_0.

(ii) $\mathfrak{B}$ is homogeneous. Consequently, if $\mathfrak{B}' \equiv \mathfrak{B}$ is countable, recursively saturated and realizes the same types as $\mathfrak{B}$, then $\mathfrak{B}' \cong \mathfrak{B}$.

To begin the proof of the theorem, let $\mathfrak{A}$ be a countable, recursively saturated model of T. We can assume that $<$ is a binary relation symbol of the language, that $<$ linearly orders A, and that for each $n < \omega$ the sentence

$$(*) \qquad \forall x_0, \ldots, x_{n-1} \forall y_0, \ldots, y_{n-1} \forall z \exists x > z \left[\bigwedge_{i<j<n} x_i \neq x_j \rightarrow \bigwedge_{i<n} \beta(x_i, x) = y_i \right]$$

is true in $\mathfrak{A}$. To see why, notice that if $<$ linearly orders A with order type ω then the above sentences (*) hold in $(\mathfrak{A}, <)$, so by (i) we can also assume that $(\mathfrak{A}, <)$ is recursively saturated.

If we were working just with PA we would have available the standard integers for coding purposes. To make up for their unavailability in our general setting, we employ the following lemma.

Lemma *There are distinct $a_0, a_1, a_2, \ldots \in A$ such that whenever $1 \leqslant n < \omega$, then $\beta(a_n, a_0) = a_{n+1}$.*

Proof: We are looking for elements $a_0, a_1 \in A$ which satisfy a certain recursive set of sentences, so by recursive saturation it suffices to show that for each $n < \omega$ there are distinct $a_0, a_1, a_2, \ldots, a_{n+2} \in A$ such that $\beta(a_{i+1}, a_0) = a_{i+2}$ for $i \leqslant n$. For any distinct $a_1, a_2, \ldots, a_{n+2}$ there are by (*) infinitely many x for which $\beta(a_{i+1}, x) = a_{i+2}$ for $i \leqslant n$. Choose a_0 to be such an x not in $\{a_1, a_2, \ldots, a_{n+2}\}$.

Fix a_0 and a_1 (and consequently $a_2, a_3, a_4, \ldots$) as in the lemma.

We now fix some notation. For any ordered set X (usually $X \subseteq \omega$ or $X \subseteq A$) and $n < \omega$ let $[X]^n$ be the set of increasing n-tuples from X. Let $[X]^{\geqslant n} = \bigcup_{n \leqslant i < \omega} [X]^i$.

For $n \geqslant 2$, set $\beta(x_0, x_1, \ldots, x_n) = \beta(\beta(x_0, x_1, \ldots, x_{n-1}), x_n)$.

We will make use of a pairing function which maps $\omega \times \omega$ onto the set of odd natural numbers; for example, let $\langle i, s\rangle = (i+s)(i+s+1) + (2i+1)$.

Let $\langle \gamma_j^i(\bar{x}, x): i, j < \omega\rangle$ be an effective listing of all formulas such that $\gamma_j^i(\bar{x}, x)$ is $(n_i + 1)$-ary, and if

$$\Gamma_i(\bar{x}, x) = \{\gamma_j^i(\bar{x}, x): j < \omega\},$$

then all recursive types are included among $\{\Gamma_i(\bar{x}, x): i < \omega\}$.

Our first main goal will be to obtain a subset $X \subseteq A$ which has properties (0)-(5) stated below.

(0) $(\mathfrak{A}, X)$ is recursively saturated.
(1) $X \neq \phi$ and X has no last element.
(2) If $(b, b') \in [X]^2$, then $\beta(b, b') = a_0$ and $\beta(a_0, b) = a_1$.
(3) For any $i, j < \omega$ and $(b_0, b_1, \ldots, b_{n_i}) \in [X]^{n_i+1}$,

$$\exists x \bigwedge_{k \leqslant j} \gamma_k^i(b_0, b_1, \ldots, b_{n_{i-1}}, x) \rightarrow \bigwedge_{k \leqslant j} \gamma_k^i(b_0, b_1, \ldots, b_{n_{i-1}}, \beta(a_{2i+2}, b_0, b_1, \ldots, b_{n_i})).$$

Before stating the other two properties that X must have, we make the following

Observation: If X satisfies just (1)-(3) above, then X generates a recursively saturated, elementary substructure of $\mathfrak{A}$. Moreover, if $Y \subseteq X$ is nonempty with no last element, then Y also generates a recursively saturated, elementary substructure of $\mathfrak{A}$. (Thus, showing the existence of X satisfying (1)-(3) will suffice for affirmatively answering Macintyre's question.)

(4) Whenever $i, s < \omega$ and $(b_0, b_1, \ldots, b_s) \in [X]^{s+1}$, then

$$\beta(a_{\langle i,s\rangle}, b_0, b_1, \ldots, b_s) \in \{a_0, a_1\}.$$

For the last property that X must have, we will need a notion of freeness.

Definition For any $r < \omega$ a subset $Y \subseteq A$ is *r-free* iff whenever $N < \omega$, $r \leqslant s < \omega$, $f: N \times [s]^{\geqslant r} \rightarrow 2$ and $(c_0, c_1, \ldots, c_{s-1}) \in [Y]^s$, then there is $c_s > c_{s-1}$, with $c_s \in Y$, such that: if $i < N$, $r \leqslant m < s$ and $(j_0, j_1, \ldots, j_{m-1}) \in [s]^m$, then

$$\beta(a_{\langle i,m\rangle}, c_{j_0}, c_{j_1}, \ldots, c_{j_{m-1}}, c_s) = a_0 \text{ iff } f(i, j_0, \ldots, j_{m-1}) = 0.$$

The last property is

(5) X is 0-free.

Lemma *There is $X \subseteq A$ possessing properties (0)-(5) above.*

Proof: Since conditions (1)-(5) just require that $(\mathfrak{A}, a_0, a_1, X)$ satisfy some recursive set of first-order sentences, it will suffice by (i) to get such an X and then appeal to the resplendency of $\mathfrak{A}$ to guarantee that (0) also holds.

Let $g: \omega \times [\omega]^{\geqslant 0} \rightarrow 2$ be a recursive function with the following property: whenever $N < \omega$, $s < \omega$, $f: N \times [s]^{\geqslant 0} \rightarrow 2$, and $(k_0, k_1, \ldots, k_{s-1}) \in [\omega]^s$, then there is some $k_s > k_{s-1}$ such that if $i < N$, $m \leqslant s$ and $(j_0, j_1, \ldots, j_{m-1}) \in [s]^m$, then

$$g(i, k_{j_0}, k_{j_1}, \ldots, k_{j_{m-1}}, k_s) = f(i, j_0, j_1, \ldots, j_{m-1}).$$

It is very easy to construct such a function g. (We will use this g again in the proof.)

We are now going to find elements $b_0, b_1, b_2, b_3, \ldots \in A$ with the following properties:

(6) $b_0 < b_1 < b_2 < \ldots$.

(7) If $i < j < \omega$, then $\beta(b_i, b_j) = a_0$ and $\beta(a_0, b_i) = a_1$.

(8) For any $i, j < \omega$ and $(m_0, m_1, \ldots, m_{n_i}) \in [\omega]^{n_i+1}$,

$$\exists x \bigwedge_{k \leqslant j} \gamma_k^i(b_{m_0}, \ldots, b_{m_{n_i-1}}, x) \to \bigwedge_{k \leqslant j} \gamma_k^i(b_{m_0}, \ldots, b_{m_{n_i-1}}, \beta(a_{2i+2}, b_{m_0}, \ldots, b_{m_{n_i}})).$$

(9) Whenever $i, s < \omega$ and $(m_0, m_1, \ldots, m_s) \in [\omega]^{s+1}$, then

$$\beta(a_{\langle i,s\rangle}, b_{m_0}, b_{m_1}, \ldots, b_{m_s}) = a_0 \text{ iff } g(i, m_0, m_1, \ldots, m_s) = 0.$$

At this point notice that if $b_0, b_1, b_2, \ldots$ satisfy (6)-(9), then $X = \{b_0, b_1, b_2, \ldots\}$ satisfies (1)-(5). To get the b_m's we require of them one more property.

(10) No two of the following are equal:

$$\begin{array}{ll} a_n & (n < \omega), \\ b_i & (i < \omega), \\ \beta(a_{2i+2}, b_{m_0}, \ldots, b_{m_k}) & (i < \omega,\ k < n_i,\ (m_0, \ldots, m_k) \in [\omega]^{k+1}), \\ \beta(a_{\langle i,s\rangle}, b_{m_0}, \ldots, b_{m_k}) & (i < \omega,\ k < s,\ (m_0, \ldots, m_k) \in [\omega]^{k+1}). \end{array}$$

Now, if we have $b_0, b_1, \ldots, b_m$ which satisfy all of those parts of (6)-(10) which refer only to $b_0, b_1, \ldots, b_m$, then b_{m+1} can be found by appealing to (*) and the recursive saturation of $\mathfrak{A}$.

Fix X as in the lemma.

The next step in the proof is to take the set X and, by carefully thinning it, construct a subset $Y \subseteq X$ which will be a set of indiscernibles generating a recursively saturated elementary substructure of $\mathfrak{A}$ which is isomorphic to $\mathfrak{A}$. This thinning process will produce sets which are more and more indiscernible in a sense made precise in the following definition.

Definition A subset $Y \subseteq A$ is *n-indiscernible* if whenever $\phi(x_0, x_1, \bar{y})$ is an $(n+2)$-ary formula and $(b), b_1, \ldots, b_{n-1}), (c_0, c_1, \ldots, c_{n-1}) \in [Y]^n$, then $(\mathfrak{A}, X) \models \phi(a_0, a_1, \bar{b}) \leftrightarrow \phi(a_0, a_1, \bar{c})$.

Let $B_0, B_1, B_2, \ldots$ be a list of all those subsets of ω which are recursive in some type realized in $\mathfrak{A}$.

We will form a decreasing sequence $X \supseteq X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$ such that for each $r < \omega$:

(11) $(\mathfrak{A}, X, X_r)$ is recursively saturated.

(12) $X_r \neq \phi$ and has no last element.

(13) X_r is r-indiscernible.

(14) X_r is r-free.

(15) (If $r > 0$ then) whenever $i < \omega$ and $(b_0, b_1, \ldots, b_{r-1}) \in [X_r]^r$, then $\beta(a_{\langle i,r-1\rangle}, b_0, b_1, \ldots, b_{r-1}) = a_0$ iff $i \in B_{r-1}$.

Assuming that such a sequence has been found, we can complete the proof of the theorem as follows. Let $Y = \{y_r: r < \omega\} \subseteq X$ be a sequence such that for any $r < \omega$, the r-tuple $(y_0, y_1, \ldots, y_{r-1})$ realizes the same type in $(\mathfrak{A}, a_0, a_1)$ as does some (or, equivalently: every) r-tuple in the set $[X_r]^r$. The set Y is easily constructed by utilizing the homogeneity (see (ii)) of $(\mathfrak{A}, X)$. Clearly such a Y is a set of indiscernibles with no last element. By the observation following (3), the set Y generates a recursively saturated, elementary substructure $\mathfrak{B}$ of $\mathfrak{A}$. It follows from (2) that $a_0, a_1 \in B$ (and thus also $a_2, a_3, a_4, \ldots \in B$). By condition (15), each B_r is recursive in the type of $(a_0, a_1, y_0, y_1, \ldots, y_r)$, so that each type realized in $\mathfrak{A}$ is recursive in some type realized in $\mathfrak{B}$. Thus, $\mathfrak{A}$ and $\mathfrak{B}$ are countable, recursively saturated structures which are elementarily equivalent and realize the same types; therefore, as noted in (ii), they are isomorphic.

To construct the sequence $\langle X_r: r < \omega\rangle$, first notice that we can set $X_0 = X$. Now, proceeding by induction, suppose that X_r has already been constructed and that we wish to construct X_{r+1}. To do so we first construct an intermediate subset $Z \subseteq X_r$ such that:

(16) $(\mathfrak{A}, X, Z)$ is recursively saturated;
(17) $Z \neq \phi$ and has no last element;
(18) Z is $(r + 1)$-free;
(19) whenever $i < \omega$ and $(b_0, b_1, \ldots, b_r) \in [Z]^{r+1}$, then

$$\beta(a_{\langle i,r\rangle}, b_0, b_1, \ldots, b_r) = a_0 \text{ iff } i \in B_r.$$

If we can get $Z \subseteq X_r$ satisfying only (17)-(19), then by the resplendency of $(\mathfrak{A}, X, X_r)$, we will be able to get such a Z which satisfies (16) as well. We will proceed similarly to the manner we used to obtain X, utilizing the function $g: \omega \times [\omega]^{\geqslant 0} \to 2$ obtained then.

We are going to find elements $c_0, c_1, c_2, \ldots \in X_r$ with the following properties:

(20) $c_0 < c_1 < c_2 < \ldots.$
(21) Whenever $i < \omega$, $r < s < \omega$ and $(m_0, m_1, \ldots, m_s) \in [\omega]^{s+1}$, then

$$\beta(a_{\langle i,s\rangle}, c_{m_0}, c_{m_1}, \ldots, c_{m_s}) = a_0 \text{ iff } g(i, m_0, m_1, \ldots, m_s) = 0.$$

(22) Whenever $i < \omega$ and $(m_0, m_1, \ldots, m_r) \in [\omega]^{r+1}$, then

$$\beta(a_{\langle i,r\rangle}, c_{m_0}, c_{m_1}, \ldots, c_{m_r}) = a_0 \text{ iff } i \in B_r.$$

Since X_r is r-free we can find $c_0, c_1, c_2, \ldots$. Let $Z = \{c_0, c_1, c_2, \ldots\}$; then Z satisfies (17)-(19), so we can assume that Z satisfies (16) also.

For the final step of the proof we will find $X_{r+1} \subseteq Z$ which is nonempty, has no last element, and is $(r + 1)$-indiscernible and $(r + 1)$-free. (Property (15) X_{r+1} will inherit from Z and (11) can be handled as before.) Here we use the key combinatorial fact of this proof: the Nešetřil-Rödl generalization of Ramsey's Theorem (see [4]). Actually, we need only that special instance of this theorem for which an independently discovered proof can be found in Appendix B of Abramson-Harrington [1]. Curiously, the use made in [1] of this combinatorial theorem is for the construction of large models of PA which have only very short sets of indiscernibles.

For each $N < \omega$ we can find $(d_0, d_1, \ldots, d_{N-1}) \in [Z]^N$ with the following two properties:

(23) $\{d_0, d_1, \ldots, d_{N-1}\}$ is $(r+1)$-indiscernible.

(24) Whenever $i < N$, $r < s < N$ and $(m_0, m_1, \ldots, m_s) \in [N]^{s+1}$, then

$$\beta(a_{\langle i,s\rangle}, d_{m_0}, d_{m_1}, \ldots, d_{m_s}) = a_0 \text{ iff } g(i, m_0, m_1, \ldots, m_s) = 0.$$

The existence of $d_0, d_1, \ldots, d_{N-1}$ satisfying (23) and (24) follows easily from the Nešetřil-Rödl Theorem, the $(r+1)$-freeness of Z, and the recursive saturation of $(\mathfrak{A}, X, Z)$. Now using the resplendency of $(\mathfrak{A}, X, Z)$, there are $d_0, d_1, d_2, \ldots \in Z$ such that (23) and (24) are satisfied for each $N < \omega$. Then the set $X_{r+1} = \{d_0, d_1, d_2, \ldots\}$ has the required properties, thus completing the proof of the theorem.

It turns out that the Nešetřil-Rödl Theorem is rather heavy machinery for what is actually needed. A much weaker generalization of Ramsey's Theorem will suffice; the proof of this generalization is a nearly direct transcription of one of the standard proofs of Ramsey's Theorem. (The reason we can say that the theorem is much weaker than the Nešetřil-Rödl Theorem is that this theorem has an "infinite version", whereas the Nešetřil-Rödl Theorem does not (see Scholium 2.1 of [1]).)

To state the needed theorem, we borrow some terminology from the preceding proof. Consider a structure $\mathfrak{A} = (A, <, R_0, \ldots, R_N)$, where $<$ linearly orders A and for each $i \leqslant N$ there is n_i such that $R_i \subseteq [A]^{n_i}$. We will say that $\mathfrak{A}$ is *r-free* iff whenever $X \in [A]^{\geqslant 0}$ and $F_i\colon [X]^{n_i-1} \to 2$ for each $i \leqslant N$ for which $n_i > r$, then there is $c \in A$ such that the following: if $i \leqslant N$ and $n_i > r$ and $\langle x_0, \ldots, x_{n_i-2}\rangle \in [X]^{n_i-1}$, then $\langle x_0, \ldots, x_{n_i-2}, c\rangle \in R_i$ iff $F_i(x_0, \ldots, x_{n_i-2}) = 0$.

Theorem *Suppose $\mathfrak{A} = (A, <, R_0, \ldots, R_N)$ is r-free and that $G\colon [A]^r \to s \in \omega$. Then there is $B \subseteq A$ such that B is homogeneous for G and $\mathfrak{A}|B$ is r-free.*

Proof: We can assume without loss of generality that each $n_i \geqslant r+1$. For if some $n_i \leqslant r$, then that R_i can be disregarded by appropriately modifying G so that if $G(a_0, \ldots, a_{r-1}) = G(b_0, \ldots, b_{r-1})$, then $\mathfrak{A}|\{a_0, \ldots, a_{r-1}\} \cong \mathfrak{A}|\{b_0, \ldots, b_{r-1}\}$. Also, we can assume that $A = \omega$. With these assumptions we will find a homogeneous B such that $\mathfrak{A}|B \cong \mathfrak{A}$.

The proof proceeds by induction on r.

$r = 1$. We can assume $s = 2$. Suppose there is no such B for which $G(b) = 0$ for each $b \in B$. Let $\langle a_0, a_1, \ldots, a_{p-1}\rangle \in [A]^p$ so that $G(a_i) = 0$ for $i < p$, $\mathfrak{A}|\{a_0, \ldots, a_{p-1}\} \cong \mathfrak{A}|p$, and for no $a_p \in A$ with $G(a_p) = 0$ is $\mathfrak{A}|\{a_0, \ldots, a_p\} \cong \mathfrak{A}|(p+1)$. Now construct $b_0 < b_1 < b_2 < \ldots$ such that $\mathfrak{A}|\{b_0, \ldots, b_{i-1}\} \cong \mathfrak{A}|i$ and $\mathfrak{A}|\{a_0, \ldots, a_{p-1}, b_i\} \cong \mathfrak{A}|(p+1)$ for each $i < \omega$. Then $G(b_i) = 1$ for each $i < \omega$, so that $B = \{b_0, b_1, b_2, \ldots\}$ is as required.

$r = t + 1$. Inductively define $a_0 < a_1 < a_2 < \ldots$ and $\omega = X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$ so that for each $i < \omega$,

$a_i = \min X_i < a_{i+1}$;

$\mathfrak{A}|(\{a_0, a_1, \ldots, a_{i-1}\} \cup X_i) \cong \mathfrak{A}$; and

whenever $x, y \in X_i$ and $\langle j_0, j_1, \ldots, j_{t-1}\rangle \in [i]^t$, then

$$G(a_{j_0}, a_{j_1}, \ldots, a_{j_{t-1}}, x) = G(a_{j_0}, a_{j_1}, \ldots, a_{j_{t-1}}, y).$$

Let $X = \{a_0, a_1, a_2, \ldots\}$, so that $\mathfrak{A}|X \cong \mathfrak{A}$. Let $H\colon [X]^t \to 2$ be such that if

$\langle x_0, x_1, \ldots, x_{t-1}\rangle \in [X]^t$ then for every (or some) $x \in X$ with $x > x_{t-1}$, $H(x_0, x_1, \ldots, x_{t-1}) = G(x_0, x_1, \ldots, x_{t-1}, x)$. Apply inductive hypothesis to $\mathfrak{A}|X$ and H to get B, which will be as required.

We close with a remark about the order type of the generating set of indiscernibles. The proof yielded a set of indiscernibles Y of order type ω which generated a structure isomorphic to $\mathfrak{A}$. If Y_0 is any countable ordered set of indiscernibles with no last element whose n-types are the same as those of Y, then the structure $\mathfrak{A}_0$ generated by Y_0 must also be countable, recursively saturated, and realize the same types as $\mathfrak{A}$, so that by (ii) $\mathfrak{A}_0 \equiv \mathfrak{A}$. Thus, not only is $\mathfrak{A}$ generated by a set of indiscernibles, but the order type of these indiscernibles can be that of any countable linearly ordered set with no last element, assuming that $<$ is part of the language and satisfies (*), as is the case with PA.

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