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Principal Congruences of Tetravalent Modal Algebras

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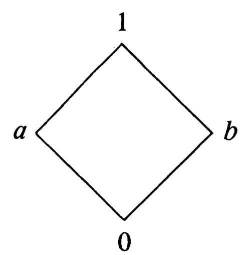
Abstract We show that tetravalent modal algebras form a discriminator variety. Consequently, we obtain a characterization for congruences and mainly for principal congruences.

1 Discriminator variety T and congruences We begin with the four-element algebra S_4 (there is no connection with the modal system S_4 of Lewis and Langford) of type $(2,2,1,1,0)$. Its operations are the two lattice operations $\wedge$, $\vee$ on



with the two unary operations:

x	0	a	b	1
$\sim x$	1	a	b	0
∇x	0	1	1	1

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and the nullary operation 1. For convenience, we also consider the term $\Delta x = \sim\nabla \sim x$.

This algebra is a De Morgan algebra (i.e., a bounded distributive lattice with a De Morgan negation $\sim$ verifying $\sim\sim x = x$ and $\sim(x \wedge y) = \sim x \vee \sim y$, having one more unary operation ∇ (the possibility) satisfying the identities $\sim x \vee \nabla x = 1$ and $x \wedge \sim x = \sim x \wedge \nabla x$. The variety $\underline{T}$ generated by the algebra S_4 (called Tetravalent Modal Algebras Variety) and some of its properties have been studied in [3].

Now from [2], we recall the notion of the Birula-Rasiowa transformation Φ, that is a mapping from the set of prime filters of a tetravalent modal algebra A, into itself, defined by $\Phi(P) = \mathbf{C} \sim P$, where $\mathbf{C}$ denotes set-theoretical complement and $\sim P = \{\sim x : x \in P\}$.

If π_0 is a family of prime filters in A, closed under Φ, and we set $a \equiv b$ $(mod.\ \pi_0)$ iff for each $P \in \pi_0$, $a \in P \Leftrightarrow b \in P$, then we have that $\equiv (mod.\ \pi_0)$ is a congruence relation on A and the kernel of the natural homomorphism h from A onto the quotient algebra $A/_{\equiv} = A/_{\pi_0}$ (i.e.: $\{x \in A : h(x) = 1\}$), is the set $N_h = \bigcap_{P \in \pi_0} P$. Moreover, it is not hard to prove that if N is a strong filter in A (i.e. a filter N verifying $x \in N \Rightarrow \Delta x \in N$), then the family $\pi[N]$ of all prime filters in A, which contain N, is closed under Φ and $N = \bigcap_{P \in \pi[N]} P$.

Now define

$$x \dagger y = [\Delta(x \wedge y) \vee \sim\Delta(x \vee y)] \wedge [\nabla(x \wedge y) \vee \sim\nabla(x \vee y)]$$

and

$$t(x,y,z) = [(x \dagger y) \wedge z] \vee [\sim(x \dagger y) \wedge x] \ .$$

Using some of the identities valid in any algebra $A \in \underline{T}$, the following properties for the operation $\dagger$ can be easily obtained:

(P_1) $x = y \Rightarrow x \dagger y = 1$
(P_2) $x \dagger y = y \dagger x$
(P_3) $x \dagger 1 = \Delta x$
(P_4) $\Delta(x \dagger y) = x \dagger y$
(P_5) $\nabla(x \dagger y) = x \dagger y$
(P_6) $\sim(x \dagger y)$ and $(x \dagger y)$ are Boolean complements.

It is not difficult to check that t is the ternary discriminator function on S_4, i.e., that on S_4:

$$t(x,y,z) = \begin{cases} z \text{ if } x = y \\ x \text{ if } x \neq y \end{cases} .$$

This means that the algebra S_4 is quasiprimal and thus the variety $\underline{T}$ is a discriminator variety. From this fact follow many important features of $\underline{T}$; for instance, by the famous theorem of McKenzie [4], $\underline{T}$ has a finite set of equational axioms (an explicit set of six axioms was given in [2]). Moreover, the subdirectly irreducible algebras in $\underline{T}$ are precisely S_4 and its subalgebras $S_3 = \{0, a, 1\}$ and

$S_2 = \{0,1\}$. These are the only directly indecomposable algebras in $\underline{T}$ and every finite algebra in $\underline{T}$ is uniquely representable as a product of copies of S_2, S_3, and S_4. Moreover, $\underline{T}$ is congruence-uniform, congruence-regular, arithmetical, and enjoys the congruence-extension property [5].

Congruence-regularity of $\underline{T}$ will allow us to characterize any congruence of an algebra $A \in \underline{T}$, by means of a family of prime filters in A. Thus, we have:

1.1 Theorem *Let $A \in \underline{T}$ and $\alpha \in Con(A)$. Then we have:*

$$\alpha = \equiv(mod.\ \pi[[1]_\alpha]) \ .$$

Proof: If $A \in \underline{T}$ and $\alpha \in Con(A)$, it is well known that the class $[1]_\alpha$ is the kernel of the natural homomorphism $h: A \to A/\alpha$. By [1], $[1]_\alpha$ is a proper strong filter in A. Therefore $\pi[[1]_\alpha]$ is a family of prime filters in A, closed under Φ, such that:

(a) $[1]_\alpha = \bigcap_{P \in \pi[[1]_\alpha]} P.$

On the other hand, we have:

(b) $[1]_{\equiv(mod.\ \pi[[1]_\alpha])} = \bigcap_{P \in \pi[[1]_\alpha]} P.$

From (a), (b) and congruence-regularity of $\underline{T}$, it follows $\alpha = \equiv(mod.\ \pi[[1]_\alpha])$.

2 Principal congruences The main use of the quasiprimality of S_4 will be an analysis of the principal congruence structure on algebras of $\underline{T}$. We know [5] that for any algebra $A \in \underline{T}$ and any $a, b, c, d \in A$, we have:

$$(c,d) \in \theta(a,b) \text{ iff } t(a,b,c) = t(a,b,d) \ .$$

Using our explicit formula for t, we can now deduce our main result about principal congruences:

2.1 Theorem *Let $A \in \underline{T}$, a, $b \in A$. Then:*

$$\theta(a,b) = \theta(a \dagger b, 1) \ .$$

Proof: For the statement, it is sufficient that we have $(a \dagger b, 1) \in \theta(a,b)$ and $(a,b) \in \theta(a \dagger b, 1)$. Then, we must prove:

(I) $t(a,b,a \dagger b) = t(a,b,1)$
(II) $t(a \dagger b, 1, a) = t(a \dagger b, 1, b)$.

We have:

$$\begin{aligned} t(a,b,a \dagger b) &= [(a \dagger b) \wedge (a \dagger b)] \vee [\sim(a \dagger b) \wedge a] \\ &= (a \dagger b) \vee [\sim(a \dagger b) \wedge a] \\ &= [(a \dagger b) \wedge 1] \vee [\sim(a \dagger b) \wedge a] \\ &= t(a,b,1) \ . \end{aligned}$$

Thus, we get (I).

Now we have:

(a) $t(a \dagger b, 1, a) = [[(a \dagger b) \dagger 1] \wedge a] \vee [\sim[(a \dagger b) \dagger 1] \wedge (a \dagger b)]$
$= [\Delta(a \dagger b) \wedge a] \vee [\sim\Delta(a \dagger b) \wedge (a \dagger b)]$
$= (a \dagger b) \wedge a.$

Similarly we get:

(b) $t(a \dagger b, 1, b) = (a \dagger b) \wedge b.$

Let us prove:

(c) $(a \dagger b) \wedge a = (a \dagger b) \wedge b.$

Obviously we have $t(a,b,a) = t(a,b,b)$, since $(a,b) \in \theta(a,b)$. Thus:

$$\begin{aligned} t(a,b,a) &= [(a \dagger b) \wedge a] \vee [\sim(a \dagger b) \wedge a] \\ &= [(a \dagger b) \vee \sim(a \dagger b)] \wedge a \\ &= 1 \wedge a = a = t(a,b,b) \\ &= [(a \dagger b) \wedge b] \vee [\sim(a \dagger b) \wedge a] \ . \end{aligned}$$

This condition implies $(a \dagger b) \wedge b \leq a$ and we get $(a \dagger b) \wedge b \leq (a \dagger b) \wedge a$. The other inequality is proved similarly and we have (c). Thus, condition (II) holds, which completes the proof.

As a consequence of this theorem, we obtain a description of the class $[1]_{\theta(a,b)}$:

2.2 Corollary *Let* $A \in \underline{T}$, $a, b \in A$. *Then*:

$$[1]_{\theta(a,b)} = F[a \dagger b] \ ,$$

being $F[a \dagger b]$ *the principal filter, in* A, *generated by* $a \dagger b$.

Proof: If $x \in A$ and $x \in [1]_{\theta(a,b)}$, then, by the previous result, $(x,1) \in \theta(a \dagger b, 1)$. From this, we get the following equivalences:

$$\begin{aligned} & t(a \dagger b, 1, x) = t(a \dagger b, 1, 1) \\ & \Leftrightarrow (a \dagger b) \wedge x = (a \dagger b) \wedge 1 \\ & \Leftrightarrow (a \dagger b) \wedge x = (a \dagger b) \Leftrightarrow a \dagger b \leq x \Leftrightarrow x \in F[a \dagger b] \ . \end{aligned}$$

Therefore

$$[1]_{\theta(a,b)} = F[a \dagger b] \ .$$

Finally, using Theorem 1.1, there follows a characterization of the principal congruence $\theta(a,b)$ by means of an explicit family of prime filters:

2.3 Corollary *Let* $A \in \underline{T}$, $a, b \in A$. *Then*:

$$\theta(a,b) = \equiv (mod.\ \pi[F[a \dagger b]])$$

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