Notre Dame Journal of Formal Logic
Volume 26, Number 1, January 1985

Individuals and Points

BOWMAN L. CLARK

The concept of a point has been of perpetual interest to philosophers and mathematicians alike. Contemporary mathematicians and philosophers have approached the subject in three ways: One is to take as basic individuals, volumes [10], regions [18], lumps [8], or spheres [14], and to define points in terms of sets of nested individuals by way of a relation, contained in the interior of [10], nontangential part of [18], completely contained in [8], or concentric with [14]. Another technique is to utilize algebraic operations, those of a Boolean ring [13] or a distributive lattice [15], on a set of individuals, and to define points in terms of certain subsets of this set that meet certain conditions. Presumably a set of any of the above basic individuals would do, except spheres, provided one allowed for disconnected volumes, regions, pieces or lumps. A third technique has been to take spheres [5], intervals [9], events [7], or any of the above basic individuals would do, and to define points as the atomic parts of these individuals; that is, as individuals which have only themselves, excluding the null element, as parts. Although different programs within these three groups have differed in detail, they are sufficiently similar to justify this three-way classification, which we shall call the nesting definitions, the algebraic definitions, and the atomic definitions.

In a recent paper [4], I presented an axiomatized calculus of individuals based on a primitive two-place predicate, 'x is connected with y'. which was the relation utilized by Whitehead [18] for his theory of Extensive Connection in which he proposed a nesting definition for points. Whitehead's theory of Extensive Connection was his last formulation of what was to have been the basis of the fourth volume of *Principia Mathematica*, a volume on geometry to be written by Whitehead himself.[1] In my paper, with slight alteration, I used Whitehead's mereological definitions to construct a calculus of individuals with pseudo-Boolean operators, pseudo-Boolean because of the absence of the null element as in the traditional formulations of the calculus of individuals. With the presence of the predicate, 'x is a nontangential part of y', I was also able

Received November 16, 1983

to introduce the pseudo-topological operators, the interior of x and the closure of x, pseudo-topological because of the absence of the null individual and boundary elements. Just as the domain of the traditional calculus of individuals is a Boolean algebra with the null element removed, the calculus of individuals based on connection turned out to be a closure algebra with the null element and the boundary elements removed. It is the purpose of this paper to utilize this new calculus of individuals with its operators to define a point along the lines of the algebraic definitions. I shall present only as many of the definitions, axioms, and theorems of the calculus of individuals based on connection as are required for this purpose.

Following Whitehead we shall take the basic individuals of the system to be space-time regions, so that our points will be space-time points. We could just as well take the basic individuals to be spatial regions, in which case our points would be spatial points. Although the predicate, 'x is connected with y', is taken as primitive and undefined, heuristically we would like it to be the case that two spatio-temporal regions are connected if, and only if, they have a spatio-temporal point in common. The proof of a theorem to this effect will be taken to be a mark of the success of our definition of points. In what follows we shall assume classical first-order quantification theory with identity[2] and some form of set theory, although the use of set theory is minimal and, as I suggested in the earlier paper, can be eliminated for the calculus of individuals. The lower case letters, '$\ldots, x, y, z$', with or without subscripts, will be taken as individual variables ranging over spatio-temporal regions, and the upper case letters, '$\ldots, X, Y, Z$', will be taken as individual variables ranging over sets of spatio-temporal regions. Points once constructed will have the upper case letters, '$\ldots, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$', ranging over them.

Definitions for the traditional mereological predicates, 'x is part of y', 'x is a proper part of y', 'x overlaps y', and 'x is discrete from y', are as follows[3]:

D0.2 $Px, y =_{\text{def}} (z)(Cz, x \supset Cz, y)$
D0.3 $PPx, y =_{\text{def}} Px, y \cdot {\sim}Py, x$
D0.4 $Ox, y =_{\text{def}} (\exists z)(Pz, x \cdot Pz, y)$
D0.5 $DRx, y =_{\text{def}} {\sim}Ox, y.$

From these definitions and the following two mereological axioms, the traditional mereological theorems of the calculus of individuals are provable;

A0.1 $(x)[Cx, x \cdot (y)(Cx, y \supset Cy, x)]$
A0.2 $(x)(y)[(z)(Cz, x \equiv Cz, y) \supset x = y].$

The advantage of the present calculus of individuals is that we are able to introduce, in addition to the traditional mereological predicates, the predicates, 'x is externally connected to y', 'x is a tangential part of y', and 'x is a nontangential part of y'. These definitions are as follows:

D0.6 $ECx, y =_{\text{def}} Cx, y, \cdot {\sim}Ox, y$
D0.7 $TPx, y =_{\text{def}} Px, y \cdot (\exists z)(ECz, x \cdot ECz, y)$
D0.8 $NTPx, y =_{\text{def}} Px, y \cdot {\sim}(\exists z)(ECz, x \cdot ECz, y).$

It is the later relation which Whitehead used to nest his regions in order to construct his points. And it is this relation which we shall use to introduce our pseudo-topological operators.

In order to introduce the pseudo-Boolean and pseudo-topological operators, we need to introduce a definition for 'the fusion of a set of regions'. Since a fusion of a set of regions, $f'X$, will itself be a region, we may introduce the definition as follows:

D1.1 $x = f'X =_{\text{def}} (y)[Cy, x \equiv (\exists z)(z \in X \cdot Cy, z)]$.

With this definition of 'the fusion of a set of regions', we can introduce the pseudo-Boolean operators, the sum of x and y, $x + y$, the complement of x, $-x$, the intersect of x and y, $x \wedge y$, as well as the universal individual, a^*, as follows:

D1.2 $x + y =_{\text{def}} f'\{z\colon Pz, x \vee Pz, y\}$
D1.3 $-x =_{\text{def}} f'\{y\colon {\sim}Cy, x\}$
D1.4 $a^* =_{\text{def}} f'\{y\colon Cy, y\}$
D1.5 $x \wedge y =_{\text{def}} f'\{z\colon Pz, x \cdot Pz, y\}$.

In addition to these definitions, we need the usual axiom asserting the existence of the fusion of any nonempty set:

A1.1 $(X)({\sim}X = \Lambda \supset (\exists x)x = f'X)$.

It is because of this axiom that our operators are pseudo-Boolean. There is no null element, since there is no fusion of the null set; and as a consequence, no $x \wedge y$ where $x \wedge y = 0$, or where ${\sim}Ox, y$, and no $-a^*$. This, of course, is standard for the traditional calculus of individuals, and is what distinguishes the domain of a calculus of individuals from a Boolean algebra.

It is, however, the presence of tangential and nontangential parts in the present calculus which makes it possible to introduce the pseudo-topological operators, the interior of x, ix, and the closure of x, cx. The interior of x is the fusion of all the nontangential parts of x, and may be introduced as follows:

D2.1 $ix =_{\text{def}} f'\{y\colon NTPy, x\}$.

The closure of x, cx, may be introduced as $-i-x$, or as follows:

D2.2 $cx =_{\text{def}} f'\{y\colon {\sim}Cy, i-x\}$,

in which case, '$cx = -i-x$' becomes provable as a theorem, where ${\sim}x = a^*$. We also need the following axiom with the presence of the pseudo-topological operators:

A2.1′ $(x)((\exists z)(NTPz, x \cdot {\sim}z = x) \cdot (y)\{(z)[(Cz, x \supset Oz, x) \cdot$
$(Cz, y \supset Oz, y)] \supset (z)(Cz, x \wedge y \supset Oz, x \wedge y)\})$.

The '$(\exists z)\, NTPz, x$' component in the above axiom[4] assures us that every individual has an interior; the '${\sim}z = x$' component that the calculus is nonatomic; and the remainder that the intersect of two open individuals is itself open. It should be remarked that it is the absence of a null element, along with the presence of external connectedness, that makes these operators pseudo-topological; there are no boundary elements. Usually x is a boundary element if, and only if, $ix = 0$. With A2.1 and the definitions, theorems concerning the traditional properties of these topological operators are provable, except those involving the nonexistent regions, theorems such as '$c0 = 0$'; and '$ix \wedge iy = i(x \wedge y)$' holds only if $x \wedge y$ exists.

With this basis at hand, we can now propose a definition of 'X is a point' as follows:

D3.1 $PT(X) =_{\text{def}} (x)(y)\{(x \in X \cdot y \in X) \supset [ECx,y \vee (Ox,y \cdot x \wedge y \in X)]\} \cdot (x)(y)[(x \in X \cdot Px,y) \supset y \in X] \cdot (x)(y)[x + y \in X \supset (x \in X \vee y \in X)] \cdot \sim X = \Lambda.$

Although the above definition bears a certain similarity to the algebraic definitions of a point, strictly speaking it is not, due to the presence of external connectedness. We may also introduce a definition for 'point X is incident in region x':

D3.2 $IN(X,x) =_{\text{def}} PT(X) \cdot x \in X.$

We also need the following axiom for the existence of points:

A3.1 $(x)(y)[Cx,y \supset (\exists X)(PT(X) \cdot x \in X \cdot y \in X)].$

From D3.1 the following theorems immediately follow:

T3.1 $(x)(y)(X)\{(PT(X) \cdot x \in X \cdot y \in X) \supset [ECx,y \vee (Ox,y \cdot x \wedge y \in X)]\}$
T3.2 $(x)(y)(X)[(PT(X) \cdot x \in X \cdot Px,y) \supset y \in X]$
T3.3 $(x)(y)(X)[(PT(X) \cdot x + y \in X) \supset (x \in X \vee y \in X)].$

With D3.1, D3.2, and A3.1 added to the calculus of individuals based on connection, the following also become provable theorems. Under each I have listed the definitions, axioms, and theorems from which they follow. T0.__, T1.__, and T2.__ refer to the theorems in [4].

T3.4 $(x)(y)(X)[(PT(X) \cdot x \in X \cdot y \in X) \supset Cx,y]$
(T3.1; T0.28)

T3.5 $(x)(y)(X)[(PT(X) \cdot x \in X) \supset x + y \in X]$
(T3.2; T1.17)

T3.6 $(x)(y)(X)\{PT(X) \supset [(x \in X \vee y \in X) \equiv x + y \in X]\}$
(T3.3; T3.5)

T3.7 $(X)(PT(X) \supset (\exists x)x \in X)$
(D3.1)

T3.8 $(X)(PT(X) \supset a^* \in X)$
(T3.6; T3.2)

T3.9 $(x)(X)[PT(X) \supset \sim(x \in X \cdot -x \in X)]$
(T3.3; T1.37)

T3.10 $(x)(X)[PT(X) \supset (x \in X \vee -x \in X)]$
(T3.8;T1.41; T1.32)

T3.11 $(x)(X)[PT(X) \supset (x \in X \equiv \sim -x \in X)]$
(T3.9; T3.10)

T3.12 $(x)(X)\{PT(X) \supset [(z)(z \in X \supset Cz,x) \equiv x \in X]\}$
(T1.37; T3.11; T3.2)

T3.13 $(x)(\exists X)\,IN(X,x)$
(A3.1; T0.1; D3.2)

T3.14 $(x)(y)[Cx,y \equiv (\exists X)(PT(X)\cdot IN(X,x)\cdot IN(X,y))]$
(T3.2; A3.1; D3.2)

T3.15 $(x)(y)[Ox,y \equiv (\exists X)(PT(X)\cdot IN(X,x)\cdot IN(X,y)\cdot{\sim}ECx,y)]$
(T3.1; A3.1; T0.19; T0.27; D3.2)

T3.16 $(x)(y)[ECx,y \equiv (\exists X)(PT(X)\cdot IN(X,x)\cdot IN(X,y)\cdot{\sim}Ox,y)]$
(T3.1; A3.1; T0.26; T0.27; D3.2)

T3.17 $(x)(y)\{Px,y \equiv (X)[(PT(X)\cdot IN(X,x)) \supset IN(X,y)\}$
(T3.2; A3.1; T3.11; T1.36; T1.32; T1.25; D3.2)

T3.18 $(x)(X)[(PT(X)\cdot IN(X,ix)) \supset IN(X,x)]$
(T3.2; T2.4; D3.2)

T3.19 $(x)(X)[(PT(X)\cdot IN(X,x)) \supset ((\exists z)\,z = -x \supset IN(X,cx))]$
(T3.2; T2.41; D3.2)

T3.20 $(x)(X)\{[PT(X)\cdot IN(X,x)\cdot{\sim}(\exists z)(z\in X\cdot ECz,x)] \supset IN(X,ix)\}$
(T.31; T2.21; T3.12; D3.2)

T3.21 $(x)(X)\{[PT(X)\cdot IN(X,x)\cdot(\exists z)(z\in X\cdot ECz,x)] \supset {\sim}IN(X,ix)\}$
(T0.27; T2.21; T3.12; D3.2)

Needless to say, T3.14, T3.15, T3.16, and T3.17 are critical theorems in the above list. They conform to our heuristic interpretation of these four mereological relations and justify our definition of a point. T3.18–T3.21 are perhaps equally critical, for they relate points to our pseudo-topological operators. In T3.20 and T3.21 we can see the distinction between interior points and boundary points emerging. In fact, T3.21 shows us that the boundary elements which were eliminated at the level of regions begin to emerge at the level of points. Any point incident in region x, but not incident in ix, will be a boundary point of x. And the null region eliminated at the level of regions will emerge at the level of sets of points as the null set. This we shall see shortly.

Perhaps a word should be said concerning these missing traditional elements at the level of regions and the proofs of the above theorems. In the earlier paper [4], we had to restrict Universal Instantiation in the presence of these nonexistent regions. This complicated the proofs and theorems involving $x \wedge y$ and $-x$. We do not find quite the same complication in the above proofs and theorems. In the sets which constitute our points, if $x \wedge y$ does not exist, then x is externally connected to y. In the case of $-x$, some theorems, for example T3.9, T3.10, and T3.17, do have to be proven on the condition that $(\exists z)\,z = -x$ and on the condition that ${\sim}(\exists z)\,z = -x$, or equivalently, $x = a^*$. To instantiate any of these theorems, however, in terms of $x \wedge y$ or $-x$, does still require that $(\exists z)\,z = x \wedge y$ or $(\exists z)\,z = -x$.

Our next task is to explore the relation between regions and sets of points incident in particular regions. For this purpose it will be helpful to introduce another kind of variable to range over sets of sets of regions. Along with '$\ldots, X, Y, Z$', ranging over sets of regions, we shall let '$\ldots, \dot{X}, \dot{Y}, \dot{Z}$' range over sets of sets of regions, and consequently, sets of points. Also, it will be helpful to introduce '$\dot{V}$' for the set of all points, '$\dashv\dot{X}$' for the complement of $\dot{X}$ restricted to the set of all points, and '$P(x)$' for the set of all the points incident in the region x. Their definitions are as follows:

D3.3 $\dot{V} =_{\text{def}} \{X: PT(X)\}$
D3.4 $\lrcorner\dot{X} =_{\text{def}} \dot{V} \cap -\dot{X}$
D3.5 $P(x) =_{\text{def}} \{X: PT(X) \cdot x \in X\}$.

With these definitions at hand, the following become theorems of the system:

T3.22 $\lrcorner\dot{V} = \Lambda$
(D3.3; D3.4)
T3.23 $\dot{V} = P(a^*)$
(D3.3; D3.5)
T3.24 $(x)\, P(x) \subseteq P(a^*)$
(T3.17; T1.25; D3.5)
T3.25 $\lrcorner P(a^*) = \Lambda$
(T3.22; T3.23)
T3.26 $(x) \lrcorner P(x) = P(-x)$
(T3.11; D3.3; D3.5; D3.4)
T3.27 $(x)(y)\, P(x \wedge y) \subseteq P(x) \cap P(y)$
(T1.50; T3.2; T1.48; D3.5)
T3.28 $(x)(y)(\sim ECx, y \supset P(x) \cap P(y) = P(x \wedge y))$
(T3.1; D3.5; T3.27)
T3.29 $(x)(y)\, P(ix) \cap P(iy) = P(ix \wedge iy)$
(T3.28; T2.25; D3.5)
T3.30 $(x)(y)\, P(x) \cup P(y) = P(x + y)$
(T3.6; D3.5).

The above theorems indicate something of the relationship between the algebraic operators on the set of all space-time regions and the set theoretic operators on the subsets of the set of all points. T3.27, T3.28, and T3.29 are of particular interest. T3.27 and T3.28 are due to the fact that there is no boundary element (or region) where ECx, y; there is no region to correspond to the set of boundary points which the two externally connected regions have in common; whereas T3.29 shows that the interiors of two connected regions do have a common region to correspond to the set of their common points. This, as we shall see, gives us the desired result that the interior of a set of boundary points is identical to the null set. In order to compare the pseudo-topological interior and closure operators on the set of all regions with the topology of the subsets of the set of all points, we must introduce an interior operator, I, on the subsets of $\dot{V}$, which associates with each set of points the set of all its interior points. It may be defined as follows:

D3.6 $I\dot{X} = \dot{Y} =_{\text{def}} (\exists x)(\exists y)(\dot{X} = P(x) \cap P(y) \cdot \dot{Y} = P(ix) \cap P(iy) \vee$ $[\dot{Y} = \Lambda \cdot \sim(\exists x)(\exists y)(\dot{X} = P(x) \cap P(y) \cdot \dot{Y} = P(ix) \cap P(iy))]$.

This closure operator, C, can be introduced in the conventional way as:

D3.7 $C\dot{X} =_{\text{def}} \lrcorner I \lrcorner \dot{X}$.

The reason for introducing the interior operator symbol, I, in terms of $\cap$ is to simplify the proofs where $\sim(\exists z)\, z = x \wedge y$. It immediately follows from this definition that the interior of the set of points incident in region x is identical to the set of points incident in the interior of the region x:

T3.31 $(x)\,IP(x) = P(ix)$.

Some other theorems which follow in the system with these definitions added are:

T3.32 $I\dot{V} = \dot{V}$
(T3.31; T2.28; T3.23)

T3.33 $(x)(y)\,I(P(x) \cap P(y)) = IP(x) \cap IP(y)$
(D3.6; T3.31)

T3.34 $(x)\,IP(x) \subseteq P(x)$
(T3.17; D3.2; T2.4; D3.5; T3.31)

T3.35 $(x)\,IIP(x) = IP(x)$
(T3.31; T2.27; T3.31)

T3.36 $I\Lambda = \Lambda$
(T3.7; D3.6)

T3.37 $(x)(y)(ECx, y \supset I(P(x) \cap P(y)) = \Lambda)$
(T0.27; T2.19; T2.23; T3.14; D3.2; D3.5)

T3.38 $CP(a^*) = P(a^*)$
(T3.24; T3.36; T3.25; D3.7)

T3.39 $(x)\,CP(x) = P(cx)$
(D3.7; T3.26; T3.31; T3.26; T2.37; T3.38)

T3.40 $C\Lambda = \Lambda$
(T3.22; T3.32; T3.22)

T3.41 $(x)\,P(x) \subseteq CP(x)$
(T3.17; T2.41; D3.5; T3.24; T3.38; T1.32)

T3.42 $(x)(y)\,C(P(x) \cup P(y)) = CP(x) \cup CP(y)$
(T3.33; T3.26; D3.7; T1.32)

T3.43 $(x)\,CCP(x) = CP(x)$
(T3.35; T3.26; D3.7; T3.35; T3.22; T3.23; T1.32).

T3.31 and T3.39 show the relationship between the interior (closure) of a set of points incident in a region and the interior (closure) of the region. The important theorems in the above list, however, are T3.32, T3.33, T3.34, and T3.35 for the interior operator and T3.40, T3.41, T3.42, and T3.43 for the closure operator on the sets of points incident in regions. They express the traditional topological properties of sets of points, and they, unlike those of the algebraic operators on regions, are existentially unconditioned, due to the presence of the null set and sets of boundary points at the level of sets of points. It is T3.37 which characterizes the sets of boundary points. It is T3.37 which characterizes the sets of boundary points. Again, however, T3.39, T3.41, T3.42, and T3.43 require that the theorem be proven both for the case in which $(\exists z)\, z = -x$ and the case in which $\sim(\exists z)\, z = -x$, or $x = a^*$, in order to remove any existential conditions on the theorems. Thus the absence of a null region and boundary elements at the level of regions appears to have no serious consequences beyond the complication of the proofs of the theorems.

The following four theorems express the relationship between the regions and their mereological relations and the sets of points incident in regions and their topological operators. The theorems confirm our original heuristic interpretation of these relations.

T3.44 $(x)(y)(Cx, y \equiv {\sim}P(x) \cap P(y) = \Lambda)$
(T3.14; D3.2)

T3.45 $(x)(y)(Ox, y \equiv {\sim}IP(x) \cap IP(y) = \Lambda)$
(T3.44; T2.23; T2.19; T3.31)

T3.46 $(x)(y)[ECx, y \equiv ({\sim}P(x) \cap P(y) = \Lambda \cdot IP(x) \cap IP(y) = \Lambda)]$
(D0.6; T3.44; T3.45)

T3.47 $(x)(y)(Px, y \equiv P(x) \subseteq P(y))$
(T3.7; D3.2).

Further investigation of the topological properties of the sets of points incident in regions, although important, is largely of mathematical interest. Let us turn to a question of more interest to a philosopher or constructionist, the question of the alternative techniques for defining, or constructing, points. Our D3.1, as suggested above, has a close parallel in the algebraic definitions; nonetheless, the basis used, with slight modification and extension, was one proposed by Whitehead [18] for a nesting definition. In his earlier work [16],[17], he based his construction of physical space-time and its geometry on the mereological relation x is a part of y; or conversely, y extends over x. Nicod [10] had pointed out that the individuals, so nested, could converge toward boundaries as well as interiors of individuals without discrimination, and he suggested an alternative nesting relation: x is in the interior of y. Also, de Laguna [6], attempting to overcome this difficulty, suggested a primitive relation: x can be connected with y, in terms of which one can construct the relation: x is a nontangential part of y, apparently Nicod's undefined relation. As a consequence, Whitehead [18] chose de Laguna's primitive relation as a basis for his last attempt to formulate his final theory of points.

We have then the Nicod-de Laguna-Whitehead nesting relation in our calculus of individuals based on connection; namely, x is a nontangential part of y (D0.8). What is also of interest is that the nesting relation of Menger [8], x is completely contained in y, is also available in our present system. Menger informally characterizes his relation as holding between the interiors of lumps and renders it as, 'the closure of x is contained in y'. With our pseudo-topological operators and our definition of 'x is open' (D2.4), we can define his relation as follows:

$$CCx, y =_{\text{def}} OPx \cdot OPy \cdot Pcx, y.$$

The presence of both these nesting relations in the present system makes it possible to compare the two relations as well as the two resulting definitions of a point. Likewise, it makes possible a comparison of these two nesting definitions of a point with our modified algebraic definition of a point given above.

Certainly, the modified algebraic definition of a point in this paper, as well as the nesting definitions mentioned above, have a distinct advantage over nesting definitions such as Tarski's [14] nested spheres, for the former require that the individuals have no particular shape. But our modified algebraic definition has an apparent advantage over all the above nesting definitions in that it does not require that the individuals be continuous, or connected in the topological sense. With the introduction of the sum of two regions, $z = x + y$, it allows for separated and scattered regions as individuals. If, however, one desires nested continuous, or connected, individuals for the construction of points, as Nicod, de Laguna, Whitehead, and apparently, Menger, the presence of the pseudo-topological operators on regions makes it possible to define such individuals. We need first a definition for 'x is separated from y' as follows[5]:

D2.6 $SPx, y =_{\text{def}} \sim Ccx, y \cdot \sim Cx, cy.$

Then we can define 'x is a connected individual' along the usual lines as follows:

D2.7 $CONx =_{\text{def}} \sim (\exists z)(\exists y)(z + y = x \cdot SPz, y).$

In short, x is connected if, and only if, x cannot be divided into two exhaustive parts which are separated. There is also no reason to think, once we have an advanced enough geometry, that Tarski's spheres could not be defined in the present system. Thus the extended calculus of individuals based on connection with its modified algebraic treatment of points is a far more general and powerful system than any of those alluded to in the above remarks. The only advantage we would claim at present for our modified algebraic definition of a point is the straightforward way in which the pseudo-topology of the regions is mirrored in the topology of the subsets of the set of all points.

A word also should be said about the atomic definitions of points and our present proposal. Since the system presented here is nonatomic (A2.1′), we cannot treat points as atoms in this extended version of the calculus of individuals based on connection. On the other hand, if we made the basic calculus atomic, we would lose most of the topological advantages of the present treatment. There are good reasons not to sacrifice the present advantages for an atomic definition of points. As Mortensen and Nerlich [9] point out, an atomic definition of points is also philosophically questionable, because of what they call "the epistemological priority of intervals over points: Separation and intervals between points are somehow visible in a way that points are not" (p. 217). Although the notion of "epistemological priority" is not a clear one, it is the unobservability of points in contrast to the observability of events, volumes, regions, lumps, spheres, etc. that has been the impetus for constructing points as sets of the latter, beginning with Whitehead's early attempts to those of the present day. As Menger [8] puts it, "a topology of lumps seems to be closer to the physicist's concept of space than is the point set theoretical concept. For naturally all the physicist can measure and observe are pieces of space, and the individual points are merely given as the result of approximations" (p. 85). To treat points as atomic parts of lumps puts points on the same level as the lumps. A point is simply a lump which has no lump, except itself, as a part.

In the beginning of the present paper, we allowed our lower case variables to range over spatio-temporal regions. The interesting question arises: Can the temporal ordering of regions be mirrored in the ordering of points somewhat analogous to the way in which we have seen the topological properties mirrored? In order to examine this possibility, let us add to our calculus of individuals another two-place primitive predicate, 'Bx, y', to be taken as a rendering of 'x is wholly before y'. We immediately have two obvious axioms:

A4.1 $(x)\{\sim Bx,x\cdot(y)(z)[(Bx,y\cdot By,z)\supset Bx,z]\}$
A4.2 $(x)(y)(Bx,y\supset\{\sim Cx,y\cdot(z)(w)[(Pz,x\cdot Pw,y)\supset Bz,w]\})$.

A4.1 tells us that wholly before is irreflexive and transitive, and A4.2 relates the new primitive relation to the mereological relations in such a way as to characterize the relation as *wholly* before, rather than *partially* before. Our heuristic interpretation of 'Bx,y' which shall guide our construction is that all the points of x are before all the points of y. With this new primitive, we can now formulate definitions for 'x is after y', 'x is contemporaneous with y', 'x is partially contemporaneous with y', 'x is partially before y', and 'x is partially after y' as follows:

D4.1 $Ax,y =_{\mathrm{def}} By,x$
D4.2 $COx,y =_{\mathrm{def}} (z)[Pz,x\supset\sim(Bz,y\vee Az,y)]\cdot(z)[Pz,y\supset\sim(Bz,x\vee Az,x)]$
D4.3 $PCx,y =_{\mathrm{def}} (\exists z)(\exists w)(Pz,x\cdot Pw,y\cdot COz,w)$
D4.4 $PBx,y =_{\mathrm{def}} (\exists z)(Pz,x\cdot Bz,y)$
D4.5 $PAx,y =_{\mathrm{def}} (\exists z)(Pz,x\cdot Az,y)$.

With these additional axioms and definitions, the following become theorems of our extended calculus:

T4.1 $(x)\sim Bx,x$
(A4.1)
T4.2 $(x)(y)[(Bx,y\cdot By,z)\supset Bx,z]$
(A4.1)
T4.3 $(x)(y)(Bx,y\supset\sim By,x)$
(T4.2; T4.1)
T4.4 $(x)(y)(Bx,y\equiv\{\sim Cx,y\cdot(z)(w)[(Pz,x\cdot Pw,y)\supset Bz,w]\})$
(A4.2; T0.5)
T4.5 $(x)(y)(z)[(Px,y\cdot Bz,y)\supset Bz,x]$
(A4.2; T0.5)
T4.6 $(x)(y)(z)[(Px,y\cdot By,z)\supset Bx,z]$
(A4.2; T0.5)
T4.7 $(x)(y)(Bx,y\supset\sim Px,y)$
(A4.2; T0.11)
T4.8 $(x)(y)[(z)(Pz,x\supset Bz,y)\equiv Bx,y]$
(T0.5; T4.6)
T4.9 $(x)\sim Ax,x$
(T4.1; D4.1)
T4.10 $(x)(y)(z)[(Ax,y\cdot Ay,z)\supset Ax,z]$
(T4.2; D4.1)
T4.11 $(x)(y)(Ax,y\supset\sim Ay,x)$
(T4.3; D4.1)
T4.12 $(x)(y)(Ax,y\equiv\{\sim Cx,y\cdot(z)(w)[(Pz,x\cdot Px,y)\supset Az,w]\})$
(T4.4; D4.1; T0.2)
T4.13 $(x)(y)(Ax,y\supset\sim Px,y)$
(T4.12; T0.11)
T4.14 $(x)(y)(z)[(Px,y\cdot Ay,z)\supset Ax,z]$
(T4.5; D4.1)

T4.15 $(x)(y)(z)[(Px,y \cdot Az,y) \supset Az,x]$
(T4.6; D4.1)

T4.16 $(x)(y)[(z)(Pz,x \supset Az,y) \equiv Ax,y]$
(T0.5; T4.14)

T4.17 $(x)\, COx,x$
(T4.7; T4.13; D4.2)

T4.18 $(x)(y)(COx,y \equiv COy,x)$
(D4.2)

T4.19 $(x)(y)(Px,y \supset PCx,y)$
(T4.17; T0.5)

T4.20 $(x)\, PCx,x$
(T4.19; T0.5)

T4.21 $(x)(y)(PCx,y \equiv PCy,x)$
(D4.3)

T4.22 $(x)(y)(COx,y \supset PCx,y)$
(T0.5; D4.3)

T4.23 $(x)(y)[\sim COx,y \equiv (PBx,y \vee PAx,y)]$
(T4.18; D4.4; D4.5)

T4.24 $(x)(y)(z)[Bx+y,z \supset (Bx,z \cdot By,z)]$
(T4.6; T1.17; T1.15)

T4.25 $(x)(y)(z)[Bz,x+y \supset (Bz,x \cdot Bz,y)]$
(T4.5; T1.17; T1.15)

T4.26 $(x)(y)(z)\{(\exists w)\, w = x \wedge z \supset [Bx,y \supset Bx \wedge z,y]\}$
(T4.6; T1.50)

T4.27 $(x)(y)(z)\{(\exists w)\, w = y \wedge z \supset [Bx,y \supset Bx,y \wedge z]\}$
(T4.5; T1.50)

T4.28 $(x)(\sim Bx,a^* \cdot \sim Ax,a^*)$
(T4.7; T4.13; T1.25)

T4.29 $(x)\, PCx,a^*$
(T4.19; T1.25)

T4.30 $(x)(y)[Bx,y \supset (Bx,iy \cdot Bix,y \cdot Bix,iy)]$
(T4.6; T4.5; T4.4; T2.4).

T4.3–T4.8 give us some indication of how the temporal ordering relation, x is before y, functions with reference to the relation, x is a part of y. T4.9–T4.16 are due to the fact that the relation, x is after y, is the converse of the relation, x is before y. T4.24–T4.29 indicate something of the way the temporal ordering relation orders the pseudo-algebraic elements. And T4.30, along with A4.3, indicate the way in which it orders the interiors of regions.

With our primitive temporal ordering relation holding between regions, it is now possible to define a temporal ordering relation holding between points as follows:

D5.1 $B(X, Y) =_{\text{def}} PT(X) \cdot PT(Y) \cdot (\exists x)(\exists y)(x \in X \cdot y \in Y \cdot Bx,y)$.

There is no ambiguity between 'Bx,y' and '$B(X, Y)$' since the former contains individual variables ranging over regions and the latter variables ranging over sets of regions. With this definition we can define 'point X is after point Y' and 'point X is contemporary with point Y' in the usual way:

D5.2 $A(X, Y) =_{\text{def}} B(Y, X)$
D5.3 $C(X, Y) =_{\text{def}} PT(X) \cdot PT(Y) \cdot \sim B(X, Y) \cdot \sim A(X, Y)$.

Along with these new definitions, we need two new axioms:

A5.1 $(x)(y)(\sim Bx, y \supset (\exists X)(\exists Y)\{PT(X) \cdot PT(Y) \cdot x \in X \cdot y \in Y \cdot (z)(w)$ $[(z \in X \cdot w \in Y) \supset \sim Bz, w]\})$.
A5.2 $(x)(y)(X)(Y)\{(PT(X) \cdot PT(Y) \cdot x \in X \cdot y \in Y \cdot Bx, Y) \supset (z)(w)$ $[(z \in X \cdot w \in Y) \supset (\exists u)(\exists v)(Pu, z \cdot u \in X \cdot Pv, w \cdot v \in Y \cdot Bu, v)]\}$.

These two axioms together perform functions analogous to those of A2.1 and A3.1. In the case of A2.1, the first half of the main conjunct assured us of the existence of needed individuals, namely, the interiors of regions, and the second half is needed because of the absence of the boundary elements in external connectedness. In the case of A3.1, we are assured of the existence of needed individuals, namely, points, and again are compensated for the absence of the boundary elements in external connectedness. If, for example, it were not for the missing boundary elements, A3.1 could be written: '$(x)(\exists X)(PT(X) \cdot x \in X)$'. Analogously, A5.1 assures us of the existence of needed elements, namely, a pair of points such that no region of the first will be wholly before any region of the second. A5.2 compensates for the missing boundary elements in external connectedness. If these elements were not missing, then A5.2 would be provable.

With D5.1–D5.3, A5.1, and A5.2 added to our system, the following become theorems of the system:

T5.1 $(X)(Y)[(PT(X) \cdot PT(Y)) \supset (B(X, Y) \vee C(X, Y) \vee A(X, Y))]$
(D5.2; D5.3)
T5.2 $(X)(Y)(B(X, Y) \equiv \{PT(X) \cdot PT(Y) \cdot (x)(y)[(x \in X \cdot y \in Y) \supset$ $(\exists z)(\exists w)(Pz, x \cdot z \in X \cdot Pw, y \cdot w \in Y \cdot Bz, w)]\})$
(A5.2; D5.1; T3.7)
T5.3 $(x)(y)\{Bx, y \equiv (X)(Y)[(PT(X) \cdot PT(Y) \cdot x \in X \cdot y \in Y) \supset$ $B(X, Y)]\}$
(D5.1; A5.1)
T5.4 $(X) \sim B(X, X)$
(T3.14; D3.2; A4.2; D5.1)
T5.5 $(X)(Y)(Z)[(B(X, Y) \cdot B(Y, Z)) \supset B(X, Z)]$
(T4.2; T4.5; T5.2; T3.7)
T5.6 $(X)(Y)(B(X, Y) \supset \sim B(Y, X))$
(T5.5; T5.4).

The important theorems here are, of course, T5.3 and T5.4–T5.6. T5.3 is analogous to T3.14–T3.17 in that just as the latter theorems indicated that our definition of points conforms to our heuristic interpretation of the mereological relations, T5.3 indicates that our temporal ordering of points conforms to our heuristic interpretation of 'Bx, y': x is wholly before y if, and only if, every point of x is before every point of y. And T5.4–T5.6 tell us that our temporal ordering relation on points has the conventional properties of being irreflexive, transitive, and asymmetrical.

In characterizing the primitive relation, x is wholly before y, we did not characterize it in such a way that the relation, point X is contemporaneous with point Y, turns out to be transitive. If we take 'Bx, y' to be a rendering of 'x is wholly in the causal past of y', then at the level of points we can construct the Minkowski cones in our space-time topology. '$\dot{X}$ = the causal past of Y', '$\dot{X}$ = the causal future of Y', and '$\dot{X}$ = the causal contemporaries of Y' can be defined as follows:

D5.4 $\dot{X} = CP'Y =_{\text{def}} \dot{X} = \{X : B(X, Y)\}$
D5.5 $\dot{X} = CF'Y =_{\text{def}} \dot{X} = \{X : A(X, Y)\}$
D5.6 $\dot{X} = CO'Y =_{\text{def}} \dot{X} = \{X : C(X, Y)\}$.

Sets analogous to Carnap's [1] world lines can be defined in the system as follows:

D5.7 $WL(\dot{X}) =_{\text{def}} {\sim}\dot{X} = \Lambda \cdot \dot{X} \subseteq \dot{V} \cdot (X)[X \in \dot{X} \supset ((\exists Y)(Y \in \dot{X} \cdot B(Y, X)) \cdot (\exists Z)(Z \in \dot{X} \cdot B(X, Z)))] \cdot (X)(Y)[(X \in \dot{X} \cdot Y \in \dot{X}) \supset (B(X, Y) \vee B(Y, X) \vee X = Y)] \cdot (X)(Y)[(X \in \dot{X} \cdot Y \in \dot{X} \cdot B(X, Y)) \supset (\exists Z)(Z \in \dot{X} \cdot B(X, Z) \cdot B(Z, Y))]$.

A world line, then, is a nonempty set of space-time points, ordered by the temporal relation in such a way that it has no initial member and no final member, is connected and dense. A world line is a causal temporal path through the space-time points. A Carnapian simultaneous space can then be defined as follows:

D5.8 $SS(\dot{X}) =_{\text{def}} (X)(Y)[X \in \dot{X} \cdot Y \in \dot{X}) \supset C(X, Y)] \cdot (\dot{Y})[WL(\dot{Y}) \supset (\exists X)(X \in \dot{Y} \cdot X \in \dot{X}]$.

A simultaneous space is a set of points any two of which are contemporary and it is complete in that it includes one point from each world line. There can be alternative simultaneous spaces for any one point, since the relation, X is contemporaneous with Y, is not transitive. These definitions are sufficient to indicate the possibilities of the constructed space-time topology.

If it is the case, and it certainly appears to be the case, that we do not observe points, then they must be constructed in terms of something that we do observe. And if it is the case, as it appears to be, that as our ability to observe and measure improves, we observe smaller and smaller events, lumps, or regions, which approach points, then the calculus of individuals based on connection gives us a basis for constructing space-time points for science on the basis of these smaller and smaller individuals which approach points. Such a construction of points was the basis of Whitehead's [18] and Russell's [11] dream for constructing the physical world of natural science. Such a construction could also give us a more adequate construction of points for a program similar to Carnap's *Aufbau* [2]. After abstracting qualities from elementary experiences, Carnap utilizes ordered quadruples of real numbers as spatio-temporal points, and then assigns qualities to points such that q, a quality, is at $\langle t, x, y, z \rangle$, where t, x, y, and z are real numbers. But as Quine [11] has pointed out, such a treatment is hardly compatible with Carnap's phenomenological starting point. This suggestion for reconstructing the *Aufbau* and the above use of Carnap's definitions of world lines and instantaneous spaces must not be taken, however, as maintaining that the points as constructed here would have all the properties required for such a construction. No doubt additional axioms would be required as we saw, for example, in introducing A3.1, A5.1, and A5.2. Otherwise world lines and instantaneous spaces may not exist. It should also be pointed out that in order to utilize a construction of points such as the present one, Carnap's elementary experiences, like Whitehead's percipient events [16],[17] or actual occasions [18], must themselves be taken as regions, divisible into subregions and exemplifying the mereological relations of our calculus of individuals. In which case, then, 'q is at $\langle t, x, y, z \rangle$' could be replaced by a topological mapping of the qualified subregions of an elementary experience into the regions of a^*. This would, however, depend upon our first overcoming the companionship difficulty and the difficulty of imperfect community. And a solution becomes possible, if elementary experiences are taken as divisible regions.[6] Whether such a constructed system would be called phenomenal would depend upon how the term 'phenomenal' was used. In short, it would depend upon whether or not we take contemporaneous and past regions to be given to a momentary experience. Some philosophers would maintain this to be the case; others would not. This, of course, is not the place to reconstruct the *Aufbau*. I mention this only to suggest the possible usefulness of the above treatment of points.

NOTES

1. See [12], "Preface". Carnap also mentions this in [2].

2. Universal Instantiation has to be limited here because of the absence of certain elements; namely, the null individual and the boundary elements. It must be limited to elements which exist when these elements are introduced by '$f'X$'. See [4].

3. The numbering here is the numbering in [4].

4. It should be pointed out that there is a serious typographical error in A2.1 in [4]. It should read: '$(x)((\exists z)NTPz,x \cdot (y)\{(z)[(Cz,x \supset Oz,x) \cdot (Cz,y \supset Oz,y)] \supset (z)(Cz,x \wedge y \supset Oz,x \wedge y)\})$.' The only difference between A2.1 in [4] and the present A2.1$'$ is that '$\sim z = x$' has been added to make the pseudo-algebra nonatomic for the definition of points. The question of atomaticity versus nonatomiticity was left open in [4].

5. I have introduced this numbering here to conform with [4]. Thee definitions should be a part of the topological part of the calculus of individuals based on connection.

6. I outlined such a solution in [3] and developed it further along slightly different lines in "Qualia, Extension and Abstraction" (forthcoming).

REFERENCES

[1] Carnap, R., *Symbolic Logic and Its Applications*, trans. W. H. Meyer and John Wilkinson, Dover Publications, New York, 1958.

[2] Carnap, R., *The Logical Structure of the World and Pseudo-Problems in Philosophy*, trans. Rolf George, University of California Press, Los Angeles, 1969.

[3] Clarke, B., "Goodman on quality classes in the *Aufbau*," *The Southern Journal of Philosophy*, vol. 1 (1963), pp. 15–19.

[4] Clarke, B., "A calculus of individuals based on 'Connection,' " *Notre Dame Journal of Formal Logic*, vol. 22, no. 3 (1981), pp. 204–218.

[5] Huntington, E. V., "Postulates for abstract geometry," *Mathematische Annalen*, vol. 73 (1913), pp. 522–559.

[6] de Laguna, T., "Point, line and surface as sets of solids," *The Journal of Philosophy*, vol. 19 (1922), vol. 449–461.

[7] Martin, R. M., *Events, Reference and Logical Form*, Catholic University of America Press, Washington, D.C., 1978.

[8] Menger, K., "Topology without points," *Rice Institute Pamphlets*, 27 (1940), pp. 80–107.

[9] Mortensen, C. and G. Nerlich, "Physical topology," *Journal of Philosophical Logic*, vol. 7 (1978), pp. 209–223.

[10] Nicod, J., *Foundations of Geometry and Induction*, Harcourt, Brace and Co., New York, 1930; originally published as *La geometric dans le mode sensible*, Alcan, Paris, 1924.

[11] Quine, W. V., *From a Logical Point of View*, Harvard University Press, Cambridge, Massachusetts, 1961.

[12] Russell, B., *Our Knowledge of the External World*, G. Allen and Unwin, London, 1952.

[13] Stone, M. H., "Applications of the Boolean rings to topology," *Annals of Mathematics, Second Edition*, vol. 41 (1937), pp. 375–481.

[14] Tarski, A., "Foundations of the geometry of solids," in *Logic*, *Semantics*, *Metamathematics*, trans. J. H. Woodger, Oxford University Press, Oxford, 1956.

[15] Wallman, H., "Lattices and topological spaces," *Annals of Mathematics*, *Second Series*, vol. 39 (1938), pp. 112–126.

[16] Whitehead, A. N., *An Enquiry Concerning the Principles of Natural Knowledge*, Cambridge University Press, Cambridge, 1919.

[17] Whitehead, A. N., *The Concept of Nature*, Cambridge University Press, Cambridge, 1920.

[18] Whitehead, A. N., *Process and Reality*, The Macmillan Co., New York, 1929.

Department of Philosophy and Religion
University of Georgia
Athens, Georgia 30602