Notre Dame Journal of Formal Logic
Volume 25, Number 4, October 1984

Some Remarks on Equivalence in Infinitary and Stationary Logic

MATT KAUFMANN*

The logic $L(aa)$ is obtained by adding a quantifier aa ("almost all") ranging over countable sets (see [1]). If instead one extends first-order finitary logic by allowing arbitrary conjunctions and disjunctions and quantification $\exists\langle x_\alpha : \alpha < \lambda\rangle$ and $\forall\langle x_\alpha : \alpha < \lambda\rangle (\lambda < \kappa)$ and restricts to formulas with fewer than κ free variables, one obtains the infinitary logic $L_{\infty\kappa}$ (see, for example, [3]). The following theorem extends Section 5 of [1] and answers a question of Nadel.

Theorem 1 *For every cardinal κ, there are structures A and B such that A and B satisfy the same sentences of $L_{\infty\kappa}$ but not of $L(aa)$.*

Proof: Fix $\kappa > \omega$. It is routine to construct a chain $\langle A_\alpha : \alpha < \kappa^+\rangle$ of structures for the vocabulary $\{X, Y, P, \epsilon\}$, satisfying the following inductive hypotheses (1) through (4) below. Here X, Y, and P are unary relation symbols and ϵ is binary. We write $A_\alpha = (A_\alpha; X_\alpha, Y_\alpha, P_\alpha, \in)$, and we use the standard notation $[Z]^\omega$ for the set of countably infinite subsets of a set Z.

(1) $A_\alpha = X_\alpha \cup Y_\alpha$, $Y_\alpha = [X_\alpha]^\omega$, $P_\alpha \subseteq Y_\alpha$, and $\in$ is the membership relation on $X_\alpha \times Y_\alpha$. Also $\alpha < \beta$ implies $A_\alpha \subsetneq A_\beta$.

(2) $|X_\alpha| = 2^\kappa$.

(3) Suppose that $Z_0 \subseteq X_\alpha$ and $|Z_0| \leq \kappa$, and that $|Z| = \kappa$ and $(Z_0 \cup [Z_0]^\omega, P_\alpha \cap [Z_0]^\omega, \in) \subseteq (Z \cup [Z]^\omega, P, \in)$. Then for some $Z' \subseteq X_{\alpha+1}$, there is an isomorphism j from $(Z \cup [Z]^\omega, P, \in)$ onto $(Z' \cup [Z']^\omega, P_{\alpha+1} \cap [Z']^\omega, \in)$ such that j extends the identity function on $Z_0 \cup [Z_0]^\omega$.

(4) If α is a limit ordinal of cofinality ω, then for all $s \in \left[\bigcup_{\beta<\alpha} X_\beta\right]^\omega - \bigcup_{\beta<\alpha} [X_\beta]^\omega$, we have $s \in P_\alpha$.

*We thank Mark Nadel for raising questions that led to the theorems in this paper. We also thank Ken Kunen for bringing Lemma 3 to our attention.

Received September 19, 1983; revised November 16, 1983

Similarly define a sequence $\langle B_\alpha : \alpha < \kappa^+ \rangle$ satisfying (1), (2), and (3) above (with $A_\alpha = (A_\alpha; X_\alpha, Y_\alpha, P_\alpha, \in)$ changed to $B_\alpha = (B_\alpha; X_\alpha, Y_\alpha, P_\alpha, \in)$), along with (4) changed so that $s \notin P_\alpha$, i.e.,

(4′) If α is a limit ordinal of cofinality ω, then for all $s \in \left[\bigcup_{\beta<\alpha} X_\beta\right]^\omega - \bigcup_{\beta<\alpha} [X_\beta]^\omega$, we have $s \notin P_\alpha$.

Now let $A = \bigcup_{\alpha<\kappa^+} A_\alpha$ and $B = \bigcup_{\alpha<\kappa^+} B_\alpha$.

The class of countable subsets of $|A|$ satisfying the hypothesis of (4) (for some $\alpha < \kappa^+$) is clearly closed unbounded in the class of countable subsets of $|A|$, so $A \models aas\,P(s \cap X)$. More precisely, $A \models aas\,\exists y[\forall x(x \epsilon y \leftrightarrow s(x) \wedge X(x)) \wedge P(y)]$. Similarly, (4′) implies that $B \models aas\,\neg P(s \cap X)$, hence $B \models \neg aas\,P(s \cap X)$. Therefore A and B do not satisfy the same sentences of $L(aa)$.

On the other hand, consider the set of all partial isomorphisms f from (a subset of) $|A|$ to $|B|$ satisfying the following conditions: $|\text{domain}(f) \cap X^A| \leq \kappa$, $[\text{domain}(f) \cap X^A]^\omega = \text{domain}(f) \cap Y^A$, and $[\text{range}(f) \cap X^B]^\omega = \text{range}(f) \cap Y^B$. It is easy to see, using (3), that this set is an appropriate back-and-forth system for $L_{\infty\kappa^+}$ (cf. [2], or simply check by induction on formulas that every such function preserves truth, and $\varnothing$ is such a function). Therefore A and B satisfy the same sentences of $L_{\infty\kappa^+}$. □

Theorem 1 produces structures of power 2^κ which are $L_{\infty\kappa^+}$-equivalent but not isomorphic. We now consider whether $L_{\infty\kappa^+}(aa)$-equivalent models of power κ^+ must be isomorphic. (Here $L_{\infty\kappa^+}(aa)$ is formed using the formation rules of $L_{\infty\kappa^+}$ and of $L(aa)$, so 'aa' still ranges over countable sets.) This can fail for Abelian groups if $\kappa = \omega$. This follows from the following immediate consequence of [4], 1.5(4) and 5.2, where we write aa^κ to denote the interpretation of 'aa' using the cub filter on $P_\kappa(\kappa)$ instead of $P_{\omega_1}(\kappa)$: (Assuming $V = L$) For every regular cardinal κ which is not weakly compact, there are 2^κ different Abelian groups of power κ which are pairwise $L_{\infty\kappa}(aa^\kappa)$-equivalent.

Here we consider only the interpretation of 'aa' over countable sets, and we treat only successor κ, leaving open the limit case. A well-known construction easily yields $L_{\infty\omega_1}(aa)$-equivalent linear orders of power ω_1 which are not isomorphic; this is Proposition 1. Theorems 2 and 3 extend this result to cardinals greater than ω_1 under certain hypotheses.

Proposition 1 *There is a family of 2^{ω_1} pairwise nonisomorphic linear orders of power ω_1 which are all $L_{\infty\omega_1}(aa)$-equivalent.*

Proof: For all $X \subseteq \omega_1$, let $\langle L^X, <^X \rangle$ be the ordered sum $\sum_{\alpha<\omega_1} A_\alpha$, where A_α is $(\mathrm{Q}, <)$ if $0 < \alpha \in X$ and A_α is $1 + (\mathrm{Q}, <)$ otherwise. If the symmetric difference $(X - Y) \cup (Y - X)$ is stationary then $\langle L^X, <^X \rangle$ and $\langle L^Y, <^Y \rangle$ are not isomorphic. Now it is well-known that there is a family $\{X^Y : Y \subseteq \omega_1\}$ of 2^{ω_1} *stationary subsets of* ω_1, each with stationary complement, whose pairwise symmetric differences are stationary. For example, let $\{S_\alpha : \alpha < \omega_1\}$ be disjoint stationary sets by Ulam's Theorem (see, e.g., [6], II.6.11, 6.12), and for $\varnothing \neq Y \subsetneq \omega_1$ let $X^Y = \bigcup\{S_\alpha : \alpha \in Y\}$. It remains to prove that if X, Y, $\omega_1 - X$, and $\omega_1 - Y$ are stationary, then $\langle L^X, <^X\rangle$ and $\langle L^Y, <^Y\rangle$ are $L_{\infty\omega_1}(aa)$-equivalent. In fact, we show that for any isomorphism f from an initial segment I of L^X onto an initial segment J of L^Y, such that I and J either both have or both do not have suprema (in $<^X$ and $<^Y$, respectively), we have $\langle L^X, <^X\rangle \models \phi(domf)$ iff $\langle L^Y, <^Y\rangle \models \phi(rnf)$, all $\phi \in L_{\infty\omega_1}(aa)$; that is, $(L^X, <^X) \models \phi[s]$ iff $(L^Y, <^Y) \models \phi[f \circ s]$ for every assignment s of the free variables of ϕ into the domain of f. (The conclusion follows by setting $f = \varnothing$.) The proof is a standard induction on complexity of ϕ; we focus on the 'aa' step. (The '$\exists$' step follows as in the proof of (*) in Theorem 2.) Suppose $\langle L^X, <^X\rangle \models aas\,\phi(s, domf)$. Then since X and $\omega_1 - X$ are stationary, $\langle L^X, <^X\rangle \models \phi(s, domf)$ for some initial segment s which has a supremum, and for one with no supremum. It follows from the inductive hypothesis that $\langle L^Y, <^Y\rangle \models \phi(s, rnf)$ for every sufficiently large initial segment s; hence $\langle L^Y, <^Y\rangle \models aas\,\phi(s, rnf)$. The reverse direction is similar. To be more careful we should replace $\phi(s, \dots)$ by a formula ϕ' in the above argument, in which each occurrence of $s(x)$ is replaced by $\bigvee_{n<\omega} x = y_n$, where the variables y_i do not occur in ϕ and are interpreted by the elements of s. We omit the details. □

For the next theorems we need some properties of the cub filter from Kueker [5]. (The first part is Corollary 2.2 of [5]; the second part follows easily and is also well-known.)

Lemma 1 *Suppose that D is cub in $P_{\omega_1}(B)$ and $A \subseteq B$.*
(i) *There is a countable family $\mathfrak{F}$ of finitary functions on B such that for every countable $s \subseteq B$ which is closed under the functions in $\mathfrak{F}$, we have $s \in D$.*
(ii) *There is a cub $C \subseteq P_{\omega_1}(A)$ such that $C \subseteq \{s \cap A : s \in D\}$.* □

The following well-known characterization of saturated dense linear orders is also useful. For this purpose it is useful to define the *coinitiality* $ci(I)$ of a proper initial segment I of $(L, <)$ to be the cofinality of $\langle L - I, <^{-1} \restriction (L - I)\rangle$. We understand the cofinality to be 0 if there is a greatest element; similarly for coinitiality.

Lemma 2 *For any regular cardinal λ, a dense linear order $(L, <)$ is λ-saturated iff $cf(L) \geq \lambda$ or L has a greatest element, $ci(L) \geq \lambda$ or L has a least element, and for every initial segment I of $(L, <)$ with $\varnothing \subsetneq I \subsetneq L$, $cf(I) \geq \lambda$ or $ci(I) \geq \lambda$.* □

The following theorem generalizes Proposition 1 above using a similar familiar construction of nonisomorphic orders via disjoint stationary sets. We will use the notation $|A^S| = A^S$, $|A^S(\alpha)| = A^S(\alpha)$, and so on.

Theorem 2 *Suppose $\kappa^{<\kappa} = \kappa$. Then there is a family of 2^{κ^+} pairwise nonisomorphic dense linear orders of power κ^+ which are all $L_{\infty\kappa^+}(aa)$-equivalent.*

Proof: The case $\kappa = \omega$ is just Proposition 1, so suppose $\kappa > \omega$. Since $\kappa^{<\kappa} = \kappa$, there is a saturated model $(L, <)$ of $Th(\mathrm{Q}, <)$, of power κ. For each $S \subseteq \kappa^+$ define A^S to be the ordered sum of linear orders $A^S_\alpha(\alpha < \kappa^+)$, where $A^S_\alpha = 1 + (L, <)$ if $\alpha \in S$ and $cf(\alpha) = \kappa$, and $A^S_\alpha = (L, <)$ otherwise. As in the proof of Proposition 1, it suffices to show that A^S is $L_{\infty\kappa^+}(aa)$-equivalent to $A^{S'}$ for all S, $S' \subseteq \kappa$. This is a special case of the following claim.

(*) Suppose that f maps a proper initial segment I of A^S isomorphically onto a proper initial segment I' of $A^{S'}$, and that $cf(I) = cf(I') = ci(I) = ci(I') = \kappa$. Then for all $\phi \in L_{\infty\kappa^+}(aa)$, $A^S \models \phi(domf)$ iff $A^{S'} \models \phi(rnf)$.

Noting that the atomic case of (*) is trivial, we proceed by induction on ϕ. The propositional steps are clear, so suppose ϕ is $\exists\bar{x}\psi$ and $A^S \models \psi(\bar{a}, domf)$ where f, I, I' are as in (*). Choose $J \supsetneq I$ and $J' \supsetneq I'$ such that $\bar{a} \subseteq J$ and $cf(J) = cf(J') = ci(J) = ci(J') = \kappa$. Now $J - I$ and $J' - I'$ are saturated by Lemma 2. Hence they are isomorphic, and if g is an isomorphism then the inductive hypothesis applied to $f \cup g$ to show $A^{S'} \models \psi((f \cup g)(\bar{a}), rnf)$. Hence $A^{S'} \models \exists\bar{x}\psi(\bar{x}, rnf)$. The reverse direction is similar.

Finally, suppose $A^S \models aas\,\psi(s, domf)$. It is clear that the set of all $s \in P_{\omega_1}(A^S)$ such that the set $[s \cap A^S - I]$ is a dense linear order without endpoints is cub in $P_{\omega_1}(A^S)$. Hence there is a cub family of such s, such that $A^S \models \psi(s, domf)$. Now by Lemma 1(ii) we may choose a cub $C \subseteq P_{\omega_1}(I)$ such that for all $s \in C$, there is $t \in P_{\omega_1}(A^S - I)$ such that t is a dense linear order without endpoints and $A^S \models \psi(s \cup t, domf)$. Let $C' = \{f[s] \cup t\colon s \in C, t \in P_{\omega_1}(A^{S'} - I'), t \text{ a dense linear order without endpoints}\}$. Clearly C' is cub. The proof that $A^{S'} \models \psi(u, rnf)$ for all $u \in C'$ is similar to the argument in the '$\exists$' case, since one may choose 'g' to extend any isomorphism between the appropriate countable dense linear orders without endpoints.

We do not know if the condition $\kappa^{<\kappa} = \kappa$ can in general be weakened to $2^{<\kappa} = \kappa$ in Theorem 2. However, our final theorem asserts that this is the case if $V = L$. In fact, what we need is the following consequence of Jensen's principle $\square_\kappa$.[1]

Lemma 3 *Assume $V = L$. Then for every infinite cardinal κ and every regular cardinal $\mu \leq \kappa$, there is a stationary subset E of $\{\alpha < \kappa^+ : cf(\alpha) = \mu\}$ such that for all $\beta < \kappa^+$, $E \cap \beta$ is not stationary in β.* □

Theorem 3 *Assume $V = L$. Then for every infinite cardinal κ there is a family of 2^{κ^+} linear orders of power κ^+ which are pairwise $L_{\infty\kappa^+}(aa)$-equivalent and nonisomorphic.*

Proof: By GCH in L, we may assume that κ is singular (or else Theorem 2 applies). Then there is a (unique) special model A of $Th(\mathrm{Q}, <)$ of power κ. By Lemma 3 there is a stationary subset $E \subseteq \{\alpha < \kappa^+ : cf(\alpha) = cf(\kappa)\}$ such that $E \cap \beta$ is not stationary in β, all $\beta < \kappa^+$. These facts are exactly what we use about L in this proof.

For any $S \subseteq \kappa^+$ define $A^S_\alpha = 1 + A$ if $\alpha \in S$, and $A^S_\alpha = A$ if $\alpha \notin S$, and let A^S be the ordered sum $\Sigma\{A^S_\alpha\colon \alpha < \kappa^+\}$. By partitioning E into κ^+ disjoint stationary subsets (as before, see, e.g., [6], II.6.11), we see (as in the proof of Proposition 1) that it suffices to prove that A^S and $A^{S'}$ are $L_{\infty\kappa^+}(aa)$-equivalent whenever S and S' are stationary subsets of E such that $E - S$ and $E - S'$ are stationary.

Let us say that a dense linear order $(L, <)$, without a last element, is *short λ-saturated* if it satisfies the criteria of Lemma 2 except that we make no restriction on the cofinality of L. Now fix a strictly increasing sequence $\langle \kappa_i : i < cf(\kappa) \rangle$

with sup κ. A dense linear order B of power κ is *short special* if it is the union of an increasing chain $\langle B_i : i<cf(\kappa)\rangle$ of elementary submodels such that B_i is short κ_i^+-saturated for all $i<cf(\kappa)$. We claim:

(1) Suppose $S \subseteq E$. For all $\alpha<\beta<\kappa^+$ let $A(\alpha,\beta)$ be the ordered sum $\Sigma\{A_\gamma^S : \alpha \le \gamma < \beta\}$. Then $A(\alpha,\beta)$ is short special.

We prove (1) by induction on β. Suppose (1) holds for all $\beta'<\beta$, and fix $\alpha<\beta$. Notice that if $\beta=1$ then $A(0,\beta)=A_0^S$ is short special because it is special. Next, suppose $\beta=\gamma+1$ where $\alpha<\gamma$ (as the case $\beta=\alpha+1$ follows just as the case $\beta=1$). Then $A(\alpha,\beta)$ is the ordered sum $A(\alpha,\gamma)+A_\gamma^S$. If $\gamma\notin S$ then since $A(\alpha,\gamma)$ is short special and A_γ^S is special with no least element, it is easy using Lemma 2 to see that $A(\alpha,\beta)$ is special. If $\gamma\in S$ then $\gamma\in E$ so $cf(\gamma)=cf(\kappa)$. Therefore $A(\alpha,\beta)$ is special, because it is the ordered sum of two special models by the following claim:

(2) If B is a short special dense linear order and $cf(B)=cf(\kappa)$, then B is special.

To prove (2), suppose B is the increasing union of submodels $B_i (i<cf(\kappa))$ where each B_i is short κ_i^+-saturated. By reindexing if necessary, we may choose an increasing sequence $\langle a_i : i<cf(\kappa)\rangle$ cofinal in B such that $a_i\in B_i$ for all $i<cf(\kappa)$. Then if B_i' is the submodel obtained by restricting B_i to $\{x\in B_i : x<a_i\}$, then B_i' is κ_i^+-saturated and still $B=\bigcup\{B_i' : i<cf(\kappa)\}$, so B is special. This concludes the proof of (2), and hence of the successor step of (1).

Finally we prove (1) for β a limit, $\alpha<\beta$ fixed. Since $S\subseteq E$, we may choose a continuous strictly increasing sequence $\langle \beta_i : i<cf(\beta)\rangle$ from $\beta - S$ which has supremum β, such that $\alpha=\beta_0$. For each $i<cf(\beta)$ let $A[i]=A(\beta_i,\beta_{i+1})$. Then each model $A[i]$ is short special by the inductive hypothesis, and $A(\alpha,\beta)=\Sigma\{A[i] : i<cf(\beta)\}$. For all $i<cf(\beta)$ we write $A[i]$ as an increasing union of elementary submodels $A[i]_j\,(j<cf(\kappa))$ such that $A[i]_j$ is κ_j^+-saturated. Now let $B_j=\Sigma\{A[i]_j : i<cf(\beta)\}$. Since $\beta_i\notin S$ for all $i<cf(\beta)$, no model $A[i]_j$ has a least element (except perhaps for $A[0]_j$). It follows easily that B_j is short κ_j^+-saturated for all $j<cf(\beta)$. Since $A(\alpha,\beta)$ is the increasing union of the models $B_j\,(j<cf(\kappa))$, it follows that $A(\alpha,\beta)$ is short special, so the proof of (1) is complete.

Notice that since there is a unique special model of power κ, (1) and (2) together immediately imply

(3) if $\alpha<\beta$, $\alpha'<\beta'$, $\alpha\in S$ iff $\alpha'\in S'$, and $cf(\beta)=cf(\beta')=cf(\kappa)$, then $A^S(\alpha,\beta)\cong A^{S'}(\alpha',\beta')$.

This yields the '$\exists$' step in the proof of the following claim, as in the proof of Proposition 1.

(4) For $\alpha<\kappa^+$ let $A^S(\alpha)$ denote $A^S(0,\alpha)$ (and similarly for S') and suppose $cf(\alpha)=cf(\alpha')=cf(\kappa)$ where $\alpha\in S$ iff $\alpha'\in S'$. Also suppose $f: A^S(\alpha)\cong A^{S'}(\alpha')$. Then for all $\phi\in L_{\infty\kappa^+}(aa)$, $A^S\models\phi(dom f)$ iff $A^{S'}\models\phi(rn f)$.

Having commented on the '$\exists$' step of the proof (by induction on ϕ) of (4), and noting that the atomic and propositional steps are obvious, let us treat the case

$\phi = aas\,\psi$. Moreover, even this step follows as in the proof of Theorem 2 if $cf(\kappa) > \omega$, since in that case we have $(A, s) \cong (B, t)$ for all special $A, B \models Th(\mathrm{Q}, <)$ of power κ and all countable $s, t \models Th(\mathrm{Q}, <)$ which are contained in A and B (respectively). (This follows just as the proof of uniqueness of special models.) So let us assume $cf(\kappa) = \omega$ and $A^S \models aas\,\psi(s, dom f)$, where $f: A^S(\alpha) \cong A^{S'}(\alpha')$. Also let us assume $\alpha, \alpha' \notin S$; the case $\alpha, \alpha' \in S$ is similar. By Lemma 1 there is a countable family $\mathfrak{F}$ of finitary functions on A^S such that for every countable $s \subseteq A$ which is closed under the functions of $\mathfrak{F}$, $A^S \models \psi(s, dom f)$. Since S is stationary, we may choose $\gamma \in S$ such that $A^S(\gamma)$ is closed under the functions of $\mathfrak{F}$, and such that γ is the limit of an ω-sequence $\alpha = \gamma_0 < \gamma_1 < \ldots < \gamma_n < \ldots$ of elements of $\kappa^+ - S$ of cofinality ω. For each n, (3) shows that there is an isomorphism f_n from $A^S(\alpha)$ onto $A^S(\gamma_n, \gamma_{n+1})$. Let C consist of all countable sets $s \subseteq A^S(\alpha)$ such that the set $s \cup \bigcup_{n<\omega} f_n[s]$ is closed under the functions in $\mathfrak{F}$. Clearly C is cub in $P_{\omega_1}(A^S(\alpha))$. Now whenever $\alpha' \leq \delta < \eta < \kappa^+$ with $cf(\delta) = cf(\eta) = \omega$ and $\delta, \eta \notin S'$, fix an isomorphism $f_{\delta,\eta}$ from $A^{S'}(\alpha')$ onto $A^{S'}(\delta, \eta)$, by (3). Let D be the family of countable subsets of $A^{S'}$ which have the following properties.

(I) $f^{-1}[s \cap A^{S'}(\alpha')] \in C$.

(II) For all $a, b \in s$ and $\alpha \leq \delta < \eta < \kappa^+$ with $cf(\delta) = cf(\eta) = \omega$ and $\delta, \eta \notin S$, if $a \in A^{S'}(\delta, \delta+1)$ and $b \in A^{S'}(\eta, \eta+1)$, then $f_{\delta,\eta}[s \cap A^{S'}(\alpha')] = s \cap A^{S'}(\delta, \eta)$ and $f_{\alpha,\delta}[s \cap A^{S'}(\alpha')] = s \cap A^{S'}(\alpha', \delta)$.

(III) For all $a \in s$ there is $\delta \in \kappa^+ - S$ of cofinality ω such that $a \in A^{S'}(\delta)$ and $s \cap A^{S'}(\delta, \delta+1) \neq \varnothing$.

It is easy to see that D is cub in $P_{\omega_1}(A^{S'})$. We will prove:

(5) For all $s \in D$, there exists $t \in C$ and $g \supseteq f$ such that $g\left[t \cup \bigcup_{n<\omega} f_n[t]\right] = s$ and g maps $A^S(\gamma)$ isomorphically onto $A^{S'}(\delta)$ where $\delta = sup(s) =_{def} sup\{\eta : (\exists a \in s)\ a \in A^{S'}(\eta, \eta+1)\}$.

By the inductive hypothesis, since $A^S \models \psi\left(t \cup \bigcup_{n<\omega} f_n[t], dom f\right)$ for all $t \in C$, then (5) implies

(6) For all $s \in D$, if $sup(s) \in S'$ then $A^{S'} \models \psi(s, rn f)$.

A nearly identical argument (starting with $\gamma \in \kappa^+ - S$ of cofinality ω rather than $\gamma \in S$) gives a cub $D_1 \subseteq P_{\omega_1}(A^{S'})$ such that

(7) For all $s \in D_1$, if $sup(s) \notin S'$ then $A^{S'} \models \psi(s, rn f)$.

From (6) and (7) one obtains $A^{S'} \models \psi(s, rn f)$ for all $s \in D \cap D_1$, and hence $A^{S'} \models aas\,\psi(s, rn f)$. The reverse direction of the 'aa' step is of course similar.

It remains then only to prove (5). Fix $s \in D$, and choose an ω-sequence $\alpha = \delta_0 < \delta_1 < \ldots < \delta_n < \ldots$ with supremum $\delta = sup(s)$, such that for all $n > 0$, $s \cap A(\delta_n, \delta_n + 1) \neq \varnothing$ and $\delta_n \in \kappa^+ - S'$ has cofinality ω. (Such a sequence exists by III.) Now let $g = f \cup \bigcup_{n<\omega} (f_{\delta_n, \delta_{n+1}} \circ f \circ f_n^{-1})$; so g maps $A^S(\gamma)$ isomorphically onto $A^{S'}(\delta)$. Let $t = f^{-1}[s \cap A^{S'}(\alpha')]$. Then $t \in C$ by (I), and

$$g\left[t \cup \bigcup_{n<\omega} f_n[t]\right] = f[t] \cup \bigcup_{n<\omega} f_{\delta_n,\delta_{n+1}} \circ f[t] = f[t] \cup \bigcup_{n<\omega} f_{\delta_n,\delta_{n+1}}[s \cap A^{S'}(\alpha')] =$$

$[s \cap A^{S'}(\alpha')] \cup \bigcup_{n<\omega} [s \cap A^{S'}(\delta_n, \delta_{n+1})]$ (the latter is by (II)) = s, as desired.□

NOTE

1. Ken Kunen brought Lemma 3 (and its proof) to our attention in the case $\mu > \omega$. A proof can be found for $\kappa = \omega$ in Baumgartner [2], p. 221, who attributes the result to "Magidor (and possibly others)". The proof for arbitrary κ is essentially the same; just begin by restricting at the start to $\{\alpha < \kappa^+ : cf(\alpha) = \mu\}$, μ as below.

REFERENCES

[1] Barwise, J., M. Kaufmann, and M. Makkai, "Stationary logic," *Annals of Mathematical Logic*, vol. 13 (1978), pp. 171-224.

[2] Baumgartner, J. E., "A new class of order types," *Annals of Mathematical Logic*, vol. 9 (1976), pp. 187-222.

[3] Dickmann, M., *Large Infinitary Languages*, North-Holland, Amsterdam, 1975.

[4] Eklof, P. and A. Mekler, "Infinitary stationary logic and Abelian groups," *Fundamenta Mathematica*, vol. CXII (1981), pp. 1-15.

[5] Kueker, D., "Countable approximations and Löwenheim-Skolem theorems," *Annals of Mathematical Logic*, vol. 11 (1977), pp. 57-103.

[6] Kunen, K., *Set Theory, Studies in Logic and the Foundations of Mathematics*, Vol. 102, North-Holland, Amsterdam, 1980.

Department of Mathematics
Purdue University
*West Lafayette, Indiana 47907**

*Present address: Austin Research Center, Burroughs Corporation, Austin, Texas, 78727.