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Trees and Finite Satisfiability: Proof of a Conjecture of Burgess

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The method of trees, expounded in such books as Jeffrey's [2] and Smullyan's [3], provides a sound and complete positive algorithmic test for unsatisfiability (and hence, for validity as well).[1] On occasion, the method can also be used to demonstrate finite satisfiability, i.e., truth in some model with a finite domain. Applied to the sentence $\exists xFx$, for example, the method yields the one-branch tree:

$$\begin{array}{c} \exists xFx \\ | \\ Fa \end{array} ,$$

from which the argument of the usual completeness proof proves the existence of a model with a one-element domain in which the sentences $\exists xFx$ and Fa are both true. But the method does not invariably demonstrate the finite satisfiability of a finitely satisfiable sentence. $\forall x\exists yRxy$, for example, is true in every one-element model in which R is interpreted as the identity relation. Applied to this sentence, the method produces the infinite one-branch tree:

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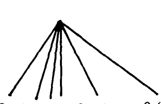
$$\forall x \exists y Rxy$$
$$|$$
$$\exists y Ray$$
$$|$$
$$Rab$$
$$|$$
$$\exists y Rby$$
$$|$$
$$Rbc$$
$$|$$
$$\exists y Rcy$$
$$|$$
$$Rcd$$
$$\vdots\ ;$$

the completeness proof then supplies a model for these sentences that has an infinite domain consisting of the constants a, b, c, d, . . . (or objects in one-one correspondence with these constants). That $\forall x \exists y Rxy$ has a finite model, however, is shown neither by the method itself nor by any of the standard proofs of the method's adequacy.

Like the set of (Gödel numbers of) unsatisfiable sentences, the set of finitely satisfiable sentences is recursively enumerable; indeed, as a theorem due to Trachtenbrot asserts, the two sets are recursively inseparable. Since the method of trees is simple, intuitive, and widely used, it would be pleasant and possibly even useful if there should turn out to be a simple and intuitive sound and complete modification that also shows the finite satisfiability of any finitely satisfiable sentence to which it is applied. Is there such a modification of the method of trees?

Consider the rule which we call *new EI* (it is due to John Burgess): Suppose that the sentence $\exists x\theta(x)$ is on an open branch B of a tree but that there is no constant c such that the sentence $\theta(c)$ is on B. Let $b_1, \ldots, b_k$ be the constants occurring in sentences on B and let a be some constant occurring in no sentence on B. Then the tree may be extended by affixing

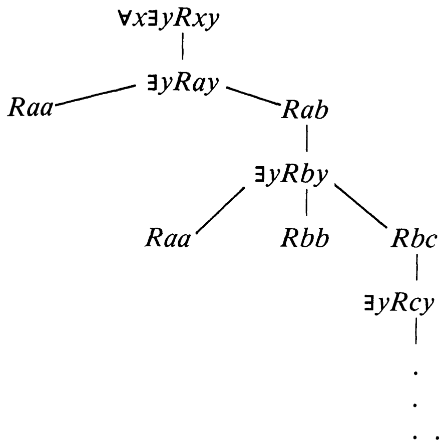


$$\theta(b_1) \ldots \theta(b_k)\ \theta(a)$$

to the bottom of B.

We call a branch *finished* (with respect to a method) if no more rules (of the method) apply to it.

Let us refer to the method obtained from the method of trees by substituting new *EI* for the usual rule *EI* for the existential quantifier as *the new method.* Applied to the finitely satisfiable $\forall x \exists y Rxy$, e.g., the new method yields:



The new method is clearly sound, and it is apparent from the completeness proof for the (old) method and the fact that new *EI* is a *finitely* branching rule that the new method is also complete for unsatisfiability. Moreover, it is evident from the proof that if some branch of a tree generated from a sentence ϕ by the new method is finite, finished, and open (like the leftmost branch in the tree above), then ϕ is finitely satisfiable.

Burgess has conjectured [1] that the new method shows the finite satisfiability of any finitely satisfiable sentence to which it is applied, and thus that the question posed some five paragraphs back has the answer, "Yes, the *new* method."

To prove Burgess's conjecture, it suffices to show that if ϕ is finitely satisfiable, then some branch of a tree generated from ϕ by the new method is finite, finished, and open. (Note that being finite, finished, and open is an effectively decidable property of branches, and thus that the new method provides sound and complete positive algorithmic tests for unsatisfiability and finite satisfiability both.)

Accordingly, suppose that M is a model with domain D, D contains exactly n members, $0 < n < \omega$, and $M \vDash \phi$. We may assume that M assigns denotations to no constants not occurring in ϕ. Let us call a model N *good* if for some m, $0 \leq m \leq n$, there are m distinct constants $a_0, \ldots, a_{m-1}$ not occurring in ϕ, m distinct elements $e_0, \ldots, e_{m-1}$ of D, and $N = M^{a_0 \ldots a_{m-1}}_{e_0 \ldots e_{m-1}}$. Thus M is good ($m = 0$).

A *branch B* of a tree is said to be *true* in N if all sentences on B are true in N; B will be called *good* if it is true in some good model.

The sole branch of the tree generated from ϕ by 0 applications of the rules is good, as M is good and $M \vDash \phi$.

Lemma *If B is a good branch and a rule R (= a propositional calculus or quantifier-flipping rule, UI (universal instantiation), an identity rule, or new EI) is applied to some sentence (or sentences in the case of the identity rule "= for =") on B, then some extension of B that results from the application of R is good.*

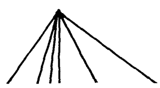
Proof: Suppose that N is good and B is true in N. Since any reduct of a good model to the language of B is still good, we may assume that N assigns denotations to no constants not occurring in some sentence on B.

If R is a propositional calculus or quantifier-flipping rule, then the argument of the usual soundness proof shows that some extension of B is true in N.

Similarly, if R is UI, applied to $\forall x\,\theta(x)$, then the argument of the usual soundness proof suffices, except in the one case in which there are no constants in sentences on B, and a constant a is introduced by UI. But in this case $N = M$ by our assumption. Since $n \geq 1$, M_e^a is good, where e is any element of D. And $M_e^a \models \theta(a)$. Thus the extension of B with $\theta(a)$ at the bottom is good.

If R is an identity rule, then again the argument of the usual soundness proof works.

Suppose that R is new EI, applied to $\exists x\theta(x)$, and



$$\theta(b_1)\ldots\theta(b_k)\,\theta(a)$$

is affixed to the bottom of B, where $b_1, \ldots, b_k$ are the constants occurring in sentences on B and a is a constant not in any sentence on B. If $N \models \theta(b_j)$ for some j, $1 \leq j \leq k$, then we are done, as the extension of B with $\theta(b_j)$ on the bottom is true in N. But if not, then, since $N \models \exists x\theta(x)$, $N_e^a \models \theta(a)$ for some e in D; $e \neq N(b_1), \ldots, e \neq N(b_k)$. Since N is good, for some m, $0 \leq m \leq n$, some constants $a_0, \ldots, a_{m-1}$, and elements $e_0, \ldots, e_{m-1}$ of D, $N = M_{e_0 \ldots e_{m-1}}^{a_0 \ldots a_{m-1}}$ and, by our assumption, $a_0, \ldots, a_{m-1}$ all occur in sentences on B. Thus $e \neq e_0 = N(a_0), \ldots, e \neq e_{m-1} = N(a_{m-1})$; $m < n$; N_e^a is good and we are done, as the extension of B with $\theta(a)$ at the bottom is true in N_e^a.

It is easy to see that there is a natural number r such that no good branch B contains more than r sentences: Let q be the number of constants in ϕ. All sentences on B are "subsentences" of ϕ, or negations thereof, obtained by repeatedly taking propositional components, taking instances of quantified sentences with respect to a set of constants containing $\leq q + n$ members, and substituting some among these $q + n$ constants for others in these subsentences.

Thus if a tree is generated from ϕ by the new method in the standard manner (apply the propositional calculus and quantifier-flipping rules as many times as possible, then apply new EI as many times as possible, then apply UI as many times as possible, then apply the identity rules as many times as possible, then loop and apply the propositional calculus...), the tree will always contain at least one good branch. At each application of a rule, any good branch will either be extended or not; if it is extended, then one of its extensions is good and contains at least one more sentence. Since a good branch cannot contain more than r sentences, eventually the tree will contain a good branch B to which no more rules can be applied. The branch B will be finite and finished, and open, as no branch true in any model (whether good or not) is closed.

NOTE

1. Sentences are here assumed to contain no function symbols.

REFERENCES

[1] Burgess, John, Personal communication, Fall 1982.

[2] Jeffrey, R. C., *Formal Logic: Its Scope and Limits,* 2nd Ed., McGraw-Hill, New York, 1981.

[3] Smullyan, Raymond M., *First-Order Logic,* Springer-Verlag, New York, 1968.

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