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# Epistemic Semantics for Classical and Intuitionistic Logic

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*1* In this paper I propose to explain how several nonstandard semantic systems for the propositional logic can be seen as employing essentially the same principles for defining the notion of logical entailment and the more basic idea of assignment of truth-values to molecular sentences. The systems are Kleene's three-valued matrix for the strong connectives, Łukasiewicz's three-valued matrix, supervaluations, and Beth's semantics for intuitionistic logic. Traditionally these systems have been viewed as dividing into two quite different approaches to semantic theory depending on the philosophical interpretation of the truth-values which they employ. Classical bivalent semantics, supervaluations, Łukasiewicz's three-valued logic, and frequently Kleene's strong connectives are presented as varieties of what Dummett calls realism, the view that truth-values and their informal readings are properly analyzed in terms of a correspondence theory of truth in which sentences are understood to describe a real world. Intuitionism and the strong connectives as Kleene originally interprets them represent varieties of antirealism or epistemic semantics.[1] Intuitionistic semantics is bivalent; an assignment of $T$ represents the epistemic fact that a sentence is provable and one of $F$ the fact that it is not provable. Kleene represents essentially the same constructivist notion of acceptability, but he uses the narrower concept of effectively decidable–in his words, "decidable by the algorithms"–and elects to represent the possibilities in terms of three values: a sentence is assigned $T$ if it is effectively decidable, $F$ if its negation is, and $N$ if neither it nor its negation is. (In the presentation below Beth's bivalent semantics are recast similarly into a trivially equivalent three-valued format.)

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The main results of the paper are two. First, it is shown that supervaluations, the strong connectives, and Łukasiewicz's three-valued semantics are all special cases of a general method of assigning values to whole expressions in terms of bivalent idealizations of values assigned to their parts. The theories so developed prove equivalent to the more usual matrix formulations of Kleene and Łukasiewicz and to the non-truth-functional projection in terms of partial models in the case of supervaluations.[2]

| | $\sim$ | $\&$ | $T$ | $F$ | $N$ | $\vee$ | $T$ | $F$ | $N$ | $\rightarrow$ | $T$ | $F$ | $N$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | $F$ | | $T$ | $F$ | $N$ | | $T$ | $T$ | $T$ | | $T$ | $F$ | $N$ |
| $F$ | $T$ | | $F$ | $F$ | $F$ | | $T$ | $F$ | $N$ | | $T$ | $T$ | $T$ |
| $N$ | $N$ | | $N$ | $F$ | $N$ | | $T$ | $N$ | $N$ | | $T$ | $N$ | $N$ |

Kleene's Strong Matrix

| | $\sim$ | $\&$ | $T$ | $F$ | $N$ | $\vee$ | $T$ | $F$ | $N$ | $\rightarrow$ | $T$ | $F$ | $N$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | $F$ | | $T$ | $F$ | $N$ | | $T$ | $T$ | $T$ | | $T$ | $F$ | $N$ |
| $F$ | $T$ | | $F$ | $F$ | $F$ | | $T$ | $F$ | $N$ | | $T$ | $T$ | $T$ |
| $N$ | $N$ | | $N$ | $F$ | $N$ | | $T$ | $N$ | $N$ | | $T$ | $N$ | $T$ |

Łukasiewicz's Three-Valued Matrix

| | $\sim$ | $\&$ | $T$ | $F$ | $N$ | $\vee$ | $T$ | $F$ | $N$ | $\rightarrow$ | $T$ | $F$ | $N$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | $F$ | | $T$ | $F$ | $N$ | | $T$ | $T$ | $T$ | | $T$ | $F$ | $N$ |
| $F$ | $T$ | | $F$ | $F$ | $F$ | | $T$ | $F$ | $N$ | | $T$ | $T$ | $T$ |
| $N$ | $N$ | | $N$ | $F$ | $\overbrace{N, F}$ | | $T$ | $N$ | $\overbrace{N, T}$ | | $T$ | $N$ | $\overbrace{N, T}$ |

The Non-Truth-Functional Projection of Supervaluations

The differences that the theories do possess have important consequences for the resulting logical entailment relations.[3] But what is interesting is that these differences appear to have little to do with the informal interpretation of the truth-values but arise rather from the autonomous issue of how much internal structure of sentential parts is relevant to the formation of bivalent idealizations.

Secondly, it is shown that Beth's intuitionistic semantics is an instance of a supervaluational language as most generally defined, and that the classical supervaluational language is in turn a special case of a Beth language.[4] Again the traditionally realistic classical superlanguage, the traditionally epistemic Beth language, and the traditionally realistic general superlanguage have different logics, but they do so not because of the intended interpretations of the truth-values. The differences may be traced rather to details in the projection of truth-values from parts to wholes that have no apparent relation to the meaning of the truth-values themselves.

That the specifically logical part of the semantic theory from different traditions of interpretation may be viewed as variations within a core of common principles suggests that issues in logical theory should be viewed as independent of those about the interpretation of truth-values. In particular it appears to be simplistic to characterize intuitionistic logic as epistemic or classical logic as realistic. The debate between realism and antirealism does not appear to turn on issues of logic alone.

*2* By a *language* we shall mean any pair $\langle Sen, Val \rangle$ in which *Sen* is the set of sentences of the propositional logic constructed in the usual way from negation, conjunction, disjunction, and the conditional, and *Val* is some set of functions whose domain is a subset of *Sen*. Members of *Val* are called the *acceptable valuations* of the language, and a valuation is called *total* if its domain coincides with *Sen* and is *partial* otherwise. A language is called *bivalent* if all its valuations range over $\{0, 1\}$, and it is called *normal* if whenever it assigns 0 or 1 to all the immediate parts of an expression it assigns one of these values to the whole in conformity with the classical truth-tables. A valuation is called *classical* if it is total, bivalent, and normal. In addition to bivalent languages we shall discuss in detail only those taking three values.

For convenience we refer to the usual values 0, 1, and 2 by $F$, $T$, and $N$, respectively, and we identify a total valuation $V$ over $\{T, F, N\}$ with the partial valuation $V'$ that is just like $V$ except that $V'$ is undefined for $A$ iff $V(A) = N$. Logical entailment for a language is always defined by reference to a set of *designated values*, which must be some subset of all values assigned by some valuation of the language. We let $X$ and $Y$ range over subsets of *Sen*, and $A$, $B$, and $C$ over *Sen* itself. The entailment relation is then defined: $X$ *entails* $A$ in $\langle Sen, Val \rangle$ relative to a set of designated values $D$ iff all valuations in *Val* assign a member of $D$ to every sentence in $X$ only if they assign a member of $D$ to $A$. If a valuation ranges over a number $n$ we say it is *n-valued*, and if some valuation for a language is $n$-valued and all its valuations are $m$-valued for some $m \leqslant n$ then we say the language itself is *n-valued.*

What is interesting for our purposes is that the three values $T$, $F$, and $N$ may be given the same epistemic interpretation in all the projections we shall discuss. We might, for example, interpret the values as representing respectively the state of a sentence's being justified, the state of its being refuted, and the state of its being neither. (This reading would have the advantages of being a fairly obvious abstraction from Kleene's narrower readings in terms of decidability as well as being phrased in terms of common and central concepts from epistemology.) But what is of interest here is not the details of a possible epistemic reading so much as the fact that these details are irrelevant to explaining how the projections differ. A reading appropriate to one is equally appropriate to the others.

The semantic theories we shall study also agree in the general outline of the principles of projection in terms of which the truth-value of a whole is calculated from those of its parts. If the parts are bivalent, then classical semantics is applied. If one or more part is indeterminate then classical semantics is again used but via the intermediate notion of a classical idealization of the indeterminate state. Such an idealization agrees on all bivalent assignments to parts and in addition assigns classical values to those parts that are indeterminate. The value of the whole is then defined as the classical value, if there is one, that is unanimously assigned by the idealizations, and is indeterminate if there is no unanimous assignment. Informally, such an idealization would amount to a limiting epistemic state in which all sentences were either justified or refuted.

Within this general framework it is possible to distinguish the various projections in terms of how much of the structure of the parts of a sentence are relevant to determining the value of the whole. The various theories seal off the structure of parts in a characteristic fashion. In the Łukasiewicz and supervaluational approaches they do so in a way that is sensitive to the values that have already been assigned to shorter sentences, and the Łukasiewicz projection is sensitive in addition to the grammatical structure of the whole sentence under evaluation. Kleene's valuations, on the other hand, are totally insensitive to the values of previously evaluated smaller sentences and are unaffected by the grammar of the whole. They merely treat every part, regardless of the whole, as if it were totally lacking in structure. We represent this lack of structure by atomic sentences. Łukasiewicz's valuations, on the other hand, sometimes do and sometimes do not respect the structure of the parts and their previously assigned values. Whether it is respected depends on the grammar of the whole sentence under evaluation. Wholes other than the conditional are evaluated just as in Kleene's theory, but in the case of the conditional, structure is respected if not doing so could possibly cause some obviously logical truth like $A \rightarrow A$ to be rejected as sometimes nontrue. Supervaluations differ from both matrix accounts in respecting all the logical structure of a part and the values of shorter sentences, regardless of the grammar of the sentence under evaluation. With these remarks as explanation, we can define the formal notions designed to capture them precisely.

Let a *sealing-off function* be any mapping from triples consisting of a sentence, a sentence, and a valuation into a valuation. If $f$ is defined for a triple $\langle A, B, V\rangle$, we understand $A$ to be an immediate part of $B$, $V$ to be a valuation just like the valuation being defined in what it assigns to sentences shorter than $B$, and $f(A, B, V)$ to be a grammatical proxy for $A$ embodying just as much structure of $A$ as is relevant to its evaluation in that context. We say that an $m$-valued valuation $V$ is a completion of an $n$-valued valuation $V'$ relative to a sealing-off function $f$ and a sentence $A$ iff $m \leqslant n$ and for any immediate part $B$ of $A$, if $V'(B) \leqslant m$, then $V'(B) = V(f(B, A, V'))$. Let $A[f, V]$ be the result of substituting $f(B, A, V)$ in $A$ for every occurrence of any immediate part $B$ of $A$. We say that the assignment $V(A)$ is *induced* by a set $S$ of completions of $V$ relative to $A$ and sealing-off function $f$ iff (for any $X$) ($V(A) = X$ iff, for all $V'$ in $S$, $V'(A[f, V]) = X$). An $m$-valued language $\langle Sen, Val\rangle$ is said to be a *completion language* relative to a sealing-off function $f$ and set $Val'$ of $n$-valued valuations iff for any $V$ of $Val$ and any sentence $A$, there is a subset of $S$ of $Val'$ such that, relative to $A$ and $f$, $S$ is the set of all elements of $Val$ that are $n$-valued completions of $V$ and $V$ is induced by $S$. The particular sealing-off functions characteristic of the languages we are discussing are defined as follows. We call them *f1, f2,* and *f3,* and assume a numbering of *Sen.*

(1) $f1(A, B, V)$ is the $i$-th atomic sentence where $A$ is the $i$-th sentence.
(2) $f2(A, B, V)$ is the $i$-th atomic sentence if $A$ is the $i$-th sentence and

either $B$ is a negation, conjunction, or disjunction, or $V$ assigns different values to the immediate parts of $B$; on the other hand, $f2(A, B, V)$ is the first atomic sentence if $B$ is a conditional and $V$ assigns the same values to the immediate parts of $B$.

(3) $f3(A, B, V) = A$.

Let *Cval* be the set of all classical valuations. Then the *Kleene, Łukasiewicz,* and *van Fraassen languages* are the three-valued completion languages defined relative to *Cval* and the functions *f1, f2,* and *f3* respectively. By working through the definitions, it is straightforward to establish that the sets of acceptable valuations of the first two languages are, respectively, the set of three-valued valuations defined over atomic sentences and conforming to the Kleene tables for the strong connectives and the set of such valuations conforming to Łukasiewicz's tables.

To state how the third language is equivalent to a superlanguage we must first define the basic notions of supervaluational theory. By a *base* relative to a set *Val* of valuations is meant any family of subsets of *Val.* Then a *supervaluation established* by some set $B$ in some base $\underline{B}$ is defined as that valuation obtained by taking the intersection of $B$. A *superlanguage* is then defined as any language ⟨*Sen, Sval*⟩ such that for some set *Val* of valuations and some base $\underline{B}$ of *Val, Sval* is the set of all supervaluations established by some element of $\underline{B}$. Let us say that one language is *part* of another if the set of valuations of the first is included in that of the second, and that two languages are *equivalent* if each is a part of the other. It then follows that the van Fraassen language is equivalent to what we may call the *classical superlanguage,* defined as that superlanguage constructed relative to the set *Cval* of classical valuations and the base $\underline{P}$(*Cval*) consisting of all subsets of *Cval.*

*3* The preceding section establishes one way in which classical logic and even classical two-valued semantics may be understood as epistemic. Such theories might be used by philosophers who, for example, reject the correspondence theory of truth or a "realistic" reading of classical metatheory. It shows that one can simultaneously reject such readings in favor of a more epistemic one and still not be forced to reject classical entailment. In this section I shall show that supervaluations, which of the three sorts discussed seem most successfully to translate into formal ideas the guiding motivation, also have a close kinship to intuitionistic semantics. In particular it will be shown that Beth's intuitionistic language defined in terms of possible worlds has a three-valued formulation that makes it equivalent to a general superlanguage, one which moreover contains the classical superlanguage as a proper part.[5] Informally, this result amounts to showing that intuitionistic semantics which is commonly understood to be in some sense epistemic rather than realistic may be understood to be so in a fairly clear sense, in the same sense as the earlier theories derived from Kleene, Łukasiewicz, and van Fraassen.

Beth valuations are two-valued, and little attention has been given to three-valued semantics for intuitionistic logic, perhaps because of Gödel's proof that there is no finitely valued matrix characteristic of intuitionistic logic. But intuitionistic entailment can be characterized by a non-truth-functional three-valued semantics in which valuations are rather trivial reformulations in three values of Beth's bivalent assignments. Given a Beth-valuation we define a three-valued correlate by assigning $T$ to the sentences proven (assigned $T$ in the original valuation), assigning $F$ to those refuted (those whose negations are $T$ in the original valuation), and $N$ to the rest.

It is clear I think that recasting a Beth valuation in three values is trivial, both formally and conceptually. Formally each Beth valuation determines a unique three-valued version and vice versa. Conceptually, no formal representative of any informal idea is lost; any formal distinction in the bivalent version can be paired with a formal correlate under the three-valued assignment, and vice versa. The interest of the reformulation lies in the fact that it reveals in a striking way the kinship of Beth semantics with the three-valued epistemic theories we have discussed earlier.

Motivationally the three-valued version has the virtue that it gives a straightforward formal representation to one of the ways intuitionistic semantics is commonly explained. Intuitionism, it is often explained, is based on a special view of mathematical truth in which true means proven, and the law of excluded middle is rejected because it is possible for a sentence to be neither proven nor refuted in the sense that its negation is proven. These facts are, of course, formally represented in the two-valued account by allowing for the case in which neither $A$ nor $\sim A$ is true. The three-valued version is a bit finer-grained in that it distinguishes between the cases in which $A$ is refuted and those in which it is unproven and unrefuted; it reserves a special truth-value for each. The formal idiom captures the informal ideas somewhat more directly. It also has the attractive feature of directly linking the concepts of falsity and negation by marking off a special subspecie of non-truth which holds of a sentence exactly when its negation is true. But the three-valued statement of the theory would scarcely be worth the trouble if it did not also enable us to see Beth semantics as a member of the same general family of epistemic theories outlined earlier.

The whole enterprise of a formal semantics for intuitionistic logic may at first seem bizarre if not incoherent. Intuitionistic truth is supposed to amount to the epistemic state of possessing a proof. But the enterprise of semantics is usually understood as realistic and nonepistemic. Indeed the usual semiotic definition of 'semantics' is the study of the relation of signs to the world. Sometimes even the technical distinctions of a purely formal semantics are given realistic import as in those interpretations of Frege that read him as ascribing a realistic ontological status to truth-values. Certainly the use of possible worlds in intuitionistic semantics invites one to read the theory as if 'possible world' here means the same as it does in more standard modal semantics, and it suggests that concern about the ontological status of these worlds is no less appropriate to intuitionistic semantics than it is to standard realistic theories. But if intuitionistic semantics is to be true to its motivation these realistic interpretations are inappropriate. One way writers like Kripke and Beth try to motivate the possible worlds used in their semantics is by explaining them as representatives of states of information.

We now state the usual nonconstructive semantics for intuitionistic logic developed in the manner of Beth. By a world structure is meant any pair $\langle K, R\rangle$ such that

(1) $R$ is a partial ordering on a set $K$
(2) There is a unique maximal element $e$ in $K$ (i.e., $e$ is in $K$ and all elements of $K$ are $R$ predecessors of $e$)
(3) for each element $k$ of $K$, there is a unique finite chain $k_n, \ldots, k_1$ such that $k = k_n$, $e = k_1$, and each element of the chain is an $R$ immediate successor of the previous element.

The chain meeting condition (3) relative to $k$ is said to be the *branch ending with k*. A branch $k_n, \ldots, k_m, \ldots, k_1$ is said to *contain* a branch $k_m, \ldots, k_1$. We let $\langle K, R\rangle$ range over world structures, and $k$, $k'$, $k''$ over elements of $K$. A subset $K'$ of $K$ is said to *bar* $k$ relative to $\langle K, R\rangle$ iff there is some branch $b$ of $\langle K, R\rangle$ ending with $k$ such that for any branch $b'$ of $\langle K, R\rangle$ containing $b$ there is an element $k'$ of $K'$ that is a strict $R$ predecessor of $k$. A *valuational assignment relative to* $\langle K, R\rangle$ is any function $V$ from $K \times Sen$ into $\{0, 1\}$ such that for any sentence $A$ and world $K$:

(1) if $A$ is atomic, then

   (a) $V(k, A) = 1$ only if $V(k', A) = 1$ of all $R$ predecessors $k'$ of $k$
   (b) if some subset $K'$ of $K$ bars $k$ and all $k'$ of $K'$ are such that $V(k', A) = 1$, then $V(k, A) = 1$

(2) if $A$ is molecular,

   (a) if $A$ is some $\sim B$, $V(k, A) = 1$ iff, for all $R$ predecessors $k'$ of $k$, $V(k', A) \neq 1)$
   (b) if $A$ is some $B \& C$, $V(k, A) = 1$ iff $V(k, B) = V(k, C) = 1$
   (c) if $A$ is some $B \vee C$, $V(k, A) = 1$ iff there is some subset $K'$ of $K$ that bars $k$ such that for any $k'$ of $K'$, either $V(k', B) = 1$ or $V(k', C) = 1$
   (d) if $A$ is some $B \rightarrow C$, $V(A) = 1$ iff, for all $R$ predecessors $k'$ of $k$, $V(k', B) = 1$ only if $V(k', C) = 1$.

We let $V$ range over the valuational assignments of $\langle K, R\rangle$, and relativize this notion to a particular world by speaking of the *world valuation* $V_k$ relative to $\langle K, R\rangle$, meaning by this the function from sentences to $\{0, 1\}$ such that for any sentence $A$, $V_k(A) = V(k, A)$. We complete this semantics by defining the *Beth intuitionistic language,* $\underline{LI}$, as that language in which the set of acceptable valuations is the set of all world valuations relative to some world structure and some valuational assignment over that world structure. The *intuitionistic entailment* relation between sets of sentences and sentences is that which preserves the value 1, i.e., $X \models_I A$ iff, whenever any acceptable valuation of $\underline{LI}$ assigns 1 to all elements of $X$ it also assigns 1 to $A$. Two well-known properties of this language are that all its acceptable valuations are normal, and that if $k$ is the least element of $\langle K. R\rangle$ then $V_k$ is classical. To state another well-known property, we first define a sentence $A$ as being *finitely inevitable* relative to $\langle K, R\rangle$, $V$, and $k$ iff some subset $K'$ of $K$ bars $k$ and for all $k'$ of $K'$, $V_{k'}(A) = 1$. By induction it is then easy to establish that for any sentence $A$, any $\langle K, R\rangle$, any $V$ and any $k$, $V_k(A) = 1$ iff $A$ is finitely inevitable relative to $\langle K, R\rangle$, $V$, and $k$.

I would like now to reformulate this customary semantics in a three-valued fashion. It will be no more truth-functional than its progenitor but it yields the same entailment relation and does so by means of formal distinctions open in a clearer way to epistemic interpretation. Let the three-valued valuation generated by a world valuation $V_k$ relative to $\langle K, R\rangle$ be that function indicated by $V_{k*}$ that maps sentences into $\{T, F, N\}$ as follows: for any sentence $A$,

$V_{k*}(A) = T$ iff $V_k(A) = 1$
$V_{k*}(A) = F$ iff $V_k(\sim A) = 1$
$V_{k*}(A) = N$ otherwise.

By *the three-valued intuitionistic language based on Beth*, briefly $\underline{L}I3$, let us mean the language in which the set of acceptable valuations is the set of all $V_{k*}$ such that $V_k$ is an acceptable valuation of $\underline{L}I$. Let entailment for $\underline{L}I3$ be defined as that relation that holds between a set of sentences and a sentence exactly if all acceptable valuations assign $T$ to the set only if they assign $T$ to the sentence. Clearly this entailment is exactly coextensive with that of $I$. Moreover by working through the definitions of $V$, $V_k$, and $V_{k*}$ it is straightforward to confirm that any such $V_{k*}$ projects values according to the following non-truth-functional tables:

| | $\sim$ | & | $T$ | $F$ | $N$ | $\vee$ | $T$ | $F$ | $N$ | $\rightarrow$ | $T$ | $F$ | $N$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | $F$ | | $T$ | $F$ | $N$ | | $T$ | $T$ | $T$ | | $T$ | $F$ | $N$ |
| $F$ | $T$ | | $F$ | $F$ | $F$ | | $T$ | $F$ | $N, T$ | | $T$ | $T$ | $T$ |
| $N$ | $N, F$ | | $N$ | $F$ | $N, F$ | | $T$ | $N, T$ | $N, T$ | | $T$ | $N, F$ | $N, T$ |

The Three-Valued Intuitionistic Projection Derived from Beth's Semantics

**Theorem** *$\underline{L}I3$ is equivalent in the sense of Herzberger to a superlanguage.*

*Proof:* For an acceptable valuation of $\underline{L}I3$ we define the element of a base that establishes it. Consider $V_{k*}$ for a fixed world structure $\langle K, R\rangle$, valuational assignment $V$ on $\langle K, R\rangle$, and $k$ in $K$. Define:

$C = \cup\{K'\colon K' \subseteq K,\ K'$ bars $k$ and for some $A$ and all $k''$ in $K'$, $V_{k''}(A) = 1\}$, and
$C' = \{k'\colon k'$ is in $C$ and there is no $k''$ in $C$ such that $k'Rk''$ and $k' \neq k''\}$.

Clearly both $C$ and $C'$ bar $k$, and if for all $k'$ in $C$, $V_{k'}(A) = 1$, then for all $k''$ in $C'$, $V_{k''}(A) = 1$. Define:

$V(C') = \{V_{k'}\colon k'$ is in $C'$ and $V_{k'}$ is a world valuation$\}$, and
$E(V_{k*}) = \{V_{k'*}\colon k'$ is in $C'$ and $V_{k'}$ is a world valuation$\}$.

(These last two notions are defined relative to the given $\langle K, R\rangle$ and $V$.) To show $E(V_{k*})$ establishes $V_{k*}$ it suffices to show:

(1) $V_{k*}(A) = T$ iff for any $V_{k'*}$ in $E(V_{k*})$, $V_{k'*}(A) = T$; and
(2) $V_{k*}(A) = F$ iff for any $V_{k'*}$ in $E(V_{k*})$, $V_{k'*}(A) = F$.

Recall that $V_{k*}(A) = T$ iff $V_k(A) = 1$ iff $A$ is finitely inevitable for $k$. Since proof of (2) involves all the moves of the proof of (1) we present just it. *If*

*Direction*: Let $V_{k*}(A) = F$. Then $V_k(\sim A) = 1$ and $\sim A$ is finitely inevitable for $k$. Hence for some subset of $K'$ of $K$, $K'$ bars $k$ and for any $k'$ in $K'$, $V_{k'}(\sim A) = 1$. Since $K' \subseteq C$, we know that for all $k''$ in $C'$, $V_{k''}(\sim A) = 1$. Hence for all $V_{k''*}$ in $E(V_{k*})$, $V_{k''}(A) = F$. *Only-if Direction*: Let for all $V_{k'*}$ in $E(V_{k*})$, $V_{k'*}(A) = F$. Then for all $V_{k'}$ in $V(C')$, $V_{k'}(\sim A) = 1$. Since $C'$ bars $k$, $V_k(\sim A) = 1$ and $V_{k*}(A) = F$. Thus (2) is established, and likewise (1). We now define a language $\langle Sen, Sval\rangle$ where *Sval* is the set of supervaluations established by some members of the base $\underline{B}$ constructed as follows:

$\underline{B} = \{B$: for some world structure $\langle K, R\rangle$, some valuational assignment $V$ on $\langle K, R\rangle$, and some $k$ in $K$, $B = E(V_{k*})$ relative to $\langle K, R\rangle$, $V$, and $k\}$.

Then for any acceptable valuation of $\underline{L}I3$, there is an element of $\underline{B}$ that establishes it, and every element of $\underline{B}$ establishes an acceptable valuation of $\underline{L}I3$.

We now present a result that shows the relation of $\underline{L}I3$ to the classical superlanguage defined as that superlanguage relative to the base consisting of all subsets of the set *Cval* of classical valuations. Let us call a world structure $\langle K, R\rangle$ *finitary* if there is a nonempty subset $K'$ of $K$ that consists of $R$ least elements and that bars the maximal element of the structure. A standard result of Beth's bivalent semantics that follows directly from the definitions is that if a world structure is finitary the valuations associated with its leasts elements are classical. Further if some $V_k$ is classical for a world $k$ in a structure, $V_{k*} = V_k$. It follows that in a finitary tree, $V_{k*}$ is established by a set of classical valuations. Thus at least some valuations of $\underline{L}I3$ are classical supervaluations. More precisely, if a world structure $\langle K, R\rangle$ is finitary, then for any valuational assignment $V$ on $\langle K, R\rangle$, and any $k$ in $K$, and world valuation $V_k$, $V_{k*}$ is a classical supervaluation. Moreover it is true also that every classical supervaluation is an acceptable valuation of $\underline{L}I3$.

**Theorem** *The classical superlanguage $\langle Sen, Sval\rangle$ is a part of $\underline{L}I3$.*

*Proof:* We show that for an arbitrary supervaluation established by a set $B$ of classical valuations there exists a world structure $\langle K, R\rangle$, a valuational assignment $\underline{V}$ on $\langle K, R\rangle$, and a $k$ in $K$ such that $S = \underline{V}_{k*}$. We construct the $\langle K, R\rangle$ in question as follows. Let us map $B \cup \{S\}$ one-to-one and onto some set $K$ by a function $P$. We define $R$ as follows: $xRy$ iff $x = y$ or both $x$ is in $B$ and $y = S$. Clearly $\langle K, R\rangle$ is a (finitary) world structure. We define $\underline{V}$ such that if $k = P(S)$ then $\underline{V}(k, A) = S(A)$, and if $k = P(V)$ for some $V$ in $B$, then $\underline{V}(k, A) = V(A)$. Clearly $\underline{V}$ is a valuational assignment on $\langle K, R\rangle$ and each $\underline{V}_k$ for $k$ in $K$ is a world valuation for $k$. Since $S = \underline{V}_k$ where $P(S) = k$, we see that $S$ is an acceptable valuation of $\underline{L}I3$.

It is possible to characterize exactly which valuations of $\underline{L}I3$ are classical supervaluations, and it should not be surprising that they are those representing epistemic situations whose completions conform to the ideal that every sentence is either justified or refuted. Let us define a sentence $A$ to be *finitely decidable* relative to a world structure $\langle W, R\rangle$, valuational assignment $V$, and world $k$ in $K$ iff there is some subset $K'$ of $K$ that bars $k$ and is such that for all elements $k'$ of $K$, either $V_{k'}(A) = 1$ or $V_{k'}(\sim A) = 1$.

**Lemma** *For every classical supervaluation $S$, there is a world structure $\langle K, R\rangle$, valuational assignment $\underline{V}$ on $\langle K, R\rangle$, and element $k$ of $K$ such that $S = \underline{V}_{k^*}$ and all sentences of Sen are finitely decidable for $k$ in $\langle K, R\rangle$ and $\underline{V}$.*

*Proof:* Let $\langle K, R\rangle$ and $\underline{V}$ be defined relative to $S$ as in the last proof, and let $k$ be the maximal element of the structure. Then $S = \underline{V}_{k^*}$, and since $\langle K, R\rangle$ is finitary, the set $K'$ of least elements of $\langle K, R\rangle$ bars $k$ and is such that for any $k'$ in $K'$, $\underline{V}_{k'}$ is classical and hence bivalent.

**Lemma** *If all sentences of Sen are finitely decidable relative to a world structure $\langle K, R\rangle$, valuational assignment $V$, and element $k$ of $K$, then $V_{k^*}$ is a classical supervaluation.*

*Proof:* Let every sentence be finitely decidable relative to $\langle K, R\rangle$, $V$, and $k$ in $K$ as stipulated in the antecedent of the lemma. Relative to these we define the following:

$$\begin{aligned}
D \quad &= \cup\{K' : K' \subseteq K,\ K' \text{ bars } k, \text{ and for some } A \text{ in } Sen \text{ and any } k' \text{ in } K',\\
&\qquad V_{k'}(A) = 1 \text{ or } V_{k'}(\sim A) = 1\}:\\
D' \quad &= \{k' : k' \text{ is in } D \text{ and for no } k'' \text{ of } K,\ k''Rk' \text{ and } k'' \neq k'\}\\
V(D') &= \{V_{k'} : k' \text{ is in } D'\}\\
E(V_{k'}) &= \{V_{k'^*} : V_{k'} \text{ is in } V(D')\}.
\end{aligned}$$

Clearly $D$ and $D''$ bar $k$. We show first that $V_{k^*}$ is established by $E(V_{k^*})$ and is therefore a supervaluation. We must show: for any $A$,

(1) $V_{k^*}(A) = T$ iff for all $V_{k'^*}$ in $E(V_{k^*})$, $V_{k'^*}(A) = T$, and
(2) $V_{k^*}(A) = F$ iff for all $V_{k'^*}$ in $E(V_{k^*})$, $V_{k'^*}(A) = F$.

We present the proof of just (2). *If Direction:* Suppose $V_{k^*}(A) = F$. Hence $V_k(\sim A) = 1$ and $\sim A$ is finitely inevitable for $k$ relative to some $K'$ that bars $k$, viz., for any $k'$ in $K'$, $V_{k'}(\sim A) = 1$. Since $K' \subseteq D$, we know that for any $k''$ in $D'$, $V_{k''}(\sim A) = 1$. Thus for any $V_{k''}$ in $V(D')$, $V_{k''}(\sim A) = 1$, and accordingly for any $V_{k''^*}$ in $E(V_{k^*})$, $V_{k''^*}(A) = F$. *Only-if Direction:* Let every $V_{k'^*}$ in $E(V_{k^*})$ be such that $V_{k'^*}(A) = F$. Then for all $V_{k'}$ in $V(D')$, $V_{k'}(\sim A) = 1$. Since $D'$ bars $k$, $V_k(\sim A) = 1$ and accordingly $V_{k^*}(A) = F$. Proof of (1) is similar. We show next that $V_{k^*}$ is a classical supervaluation by showing for an arbitrary $V_{k''^*}$ in $E(V_k)$ that it is classical. Since $V_{k''^*}$ is normal we need only show that it is bivalent. Suppose for reductio that it is not, that for some $A$, $V_{k''^*}(A)$ is neither $T$ nor $F$. Thus, $V_{k''}(A) \neq 1$ and $V_{k''}(\sim A) \neq 1$. But $A$ is finitely decidable relative to $k$. Hence for some subset $K'$ of $K$ and any $k'$ in $K'$, $V_{k'}(A) = 1$ or $V_{k'}(A) = 0$. Since $K'$ bars $k$ there is some $k'$ of $K'$ such that $k''Rk'$. Consider one such $k'$. *Case I.* $V_{k'}(A) = 1$. But then since $k''Rk'$, $V_{k''}(A) = 1$. *Case II.* $V_{k'}(\sim A) = 1$. But each case contradicts what we have assumed about $k''$. Hence by reductio, every element of $E(V_{k^*})$ is bivalent.

The two lemmas directly imply a characterization of the subset of $\underline{LI}3$ valuations that coincides with the classical supervaluations.

**Theorem** *$S$ is a classical supervaluation iff for some world structure $\langle K, R\rangle$, valuational assignment $\underline{V}$, and $k$ in $K$, $S = V_{k^*}$ and all sentences of Sen are finitely decidable relative to $\langle K, R\rangle$, $\underline{V}$, and $k$.*

We have investigated four semantic theories that can be interpreted as epistemic. Those characterized by the strong connectives and Łukasiewicz's matrix have the defect that they only imperfectly render into formal terms the idea that epistemic classification may be determined by reference to idealized classical completions of current knowledge. They also reject some classically valid arguments in rather unmotivated ways. Both faults can, I think, be laid to their matrix format. By sacrificing truth-functionality superlanguages both give a forthright representation of evaluation by reference to classical completions and capture a purely classical entailment. But they have the curious feature from the perspective of an epistemic semantics of assuming that in the idealized future every sentence will be either justified or refuted. The intuitionistic version of the epistemic semantics, on the other hand, rejects just this assumption. With it goes classical logic and standard mathematics. It is inappropriate here to attempt to carry evaluation of supervaluations over intuitionism any further. It will suffice to conclude that classical and intuitionistic logics can be seen as rival theories about essentially the same subject matter, and that the rejection of classical logic in favor of intuitionism is not the inevitable consequence of a nonrealistic world view or an epistemic semantics.

## NOTES

1. Hartry Field has explored an epistemic semantics that yields a classical entailment relation in a rather different way from that investigated here. The semantic theories discussed here all consist of recursive definitions of assignments of truth-values and subsequent definition of entailment as a relation preserving designated values. Moreover, they share a common informal interpretation consisting of straightforward translations of truth-values into varieties of justificational success. Field, at least for the epistemic part of his semantics, rejects recursive definitions of truth-value assignments in favor of axiomatizing a set of functions giving to each pair $\langle \Gamma, A \rangle$ of a set $\Gamma$ of sentences and a sentence $A$, a probability. In terms of these probability assignments the set of classically valid arguments is definable. It will suffice for the purposes of the present paper to merely describe the possibility of its rather different sort of epistemic semantics without actually trying to argue in any detail for its superiority. Its main virtue is that it draws into a natural kind various standard and nonstandard semantics by showing their formal similarities and by sketching a common epistemic interpretation. See [4].

2. The original presentation of the strong connectives is found in [7], pp. 332-340, and in an early statement of Łukasiewicz's three-valued logic in [8].

3. That the strong connectives and Łukasiewicz's matrix do not yield a classical entailment relation is well-known. Regardless of whether just $T$ or both $T$ and $N$ are designated the entailment relations of the two matrices are proper parts of classical entailment. If just $T$ is designated, observe that the classically valid argument from to $A$ to $(A \mathbin{\&} B) \vee (A \mathbin{\&} {\sim}B)$ fails; if both $T$ and $N$ are designated, observe that the classically valid argument from $A \mathbin{\&} {\sim}A$ to $B$ fails. Students of these matrices have observed however that if both $T$ and $N$ are designated, the logical truths (sentences which are always designated) coincide exactly with the truths of classical logic, and moreover some of the classically valid arguments that fail are peculiar anyway, involving for example irrelevant sentences as in the paradoxes of material implication. Characterizing just which inferences fail is an interesting problem. I don't here want to suggest that the entailment relations of the

matrices are obviously unintuitive because they fail straightforwardly to coincide with the classical relation, but merely that in a very literal sense they do not preserve all the classical entailments. Whether the entailments they do capture are plausible, even on classical grounds, is a long story. See, for example, [1] and [10].

4. A statement of the theory of supervaluations can be found in [12]. The particular metatheory employed here is based on the generalizations contained in [5]; on the links to Kleene and Łukasiewicz see [11]. An earlier use of a concept of sealing-off to explain Bochvar's internal and external connectives is explored in [6].

5. For statements of the main versions of possible world semantics for intuitionistic propositional logic see [2] and [9]. For a discussion of the motivation of intuitionistic semantics and for a statement of Beth semantics as they are presented here see [3].

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