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The Necessity of the Past and Modal-Tense Logic Incompleteness

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The idea that there is a variety of necessity, i.e., the necessity of unpreventability, unalterability, or irrevocability, for which it is true to say that the past is necessary is a notion of great antiquity[1] which still possesses considerable intuitive appeal. However, this idea proves difficult to express adequately in a modal propositional logic that possesses both tense and alethic modal operators. The obvious candidate for a thesis expressing the necessity of the past, and the one normally so employed (see [7], p. 117), is

A1 $\quad Pp \supset LPp$.

In this paper I explore several problems connected with the use of Thesis A1 to express the concept of the necessity of the past in a mixed modal-tense logic. Section 1 consists of a brief rehearsal of a "philosophical" difficulty encountered in employing A1 to express the necessity of the past: it proves difficult to isolate this necessity from the remainder of time, i.e., to avoid a form of fatalism. This problem with A1 has been previously recognized. In Sections 2 and 3, I discuss several more strictly logical problems with A1. Section 2 pertains to Arthur Prior's use of A1 in his modal-tense logical reconstruction of the famous "Master" argument of Diodorus Cronus. It is shown that Prior's modal-tense logical version of the conclusion of the Master can be

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semantically derived from premises which do not entail the discreteness of time. However, one of the conditions necessary for this semantical derivation, namely, the irreflexivity of the temporal ordering relation, is not "directly" syntactically expressible in a modal-tense logic. It is this "vagary" of modal-tense logic that has introduced a red herring into the scholarly discussion of the adequacy of Prior's formulation of the Master: the issue of Diodorus' beliefs concerning the discreteness of time.

The argument of Section 2 involves a proof that a modal-tense logic representing Prior's formulation of the Master—minus the assumption of the discreteness of time—is incomplete with respect to the class of (extended) standard frames characterized by an irreflexive temporal ordering. Section 3 presents a "genuine" incompleteness result for an intuitively reasonable modal-tense logic with A1 as the only axiom "mixing" the alethic and temporal operators. This logic, in other words, is complete with respect to *no* class of (extended) standard frames. It is also shown, however, that the logic resulting from the addition of A1 to the combined minimal normal alethic and tense logics (K and K_t, respectively) *is* complete. The results of Section 3 have no direct connection with the Master or Prior's modal-tense logic formulation of it. Rather, these results are offered as initial contributions to the logical study of A1, a modal-tense logic thesis whose philosophical interest warrants its further logical study, in my view.

1 The principal philosophical problem encountered in employing A1 as an axiom in a mixed modal-tense logic is that uniform substitution permits wffs with future temporal significance to be substituted for the propositional variable in A1. The following syntactic derivation illustrates the ease with which the necessity of the past may then be "transmitted" to the future:

I.	$(\sim Pp \wedge \sim p \wedge \sim Fp) \supset P{\sim}Fp$	(Theorem of tense logic for forwards-linear time with backwards-seriality)[2]
II.	$Pp \supset LPp$	(A1)
III.	$P{\sim}Fp \supset LP{\sim}Fp$	(2, sub. Fp/p)
IV.	$LP{\sim}Fp \supset \sim MHFp$	(P-H and L-M duality)
V.	$L(p \supset HFp)$	(Axiom of Lemmon's "minimal" tense logic K_t plus application of "L-necessitation" rule)
VI.	$L(p \supset q) \supset (Lp \supset Lq)$	(Axiom of minimal "normal" modal logic K)[3]
VII.	$L(p \supset q) \supset (Mp \supset Mq)$	(6, PC and L-M duality)
VIII.	$L(p \supset HFp) \supset (Mp \supset MHFp)$	(7, sub. HFp/q)
IX.	$Mp \supset MHFp$	(5, 8 modus ponens)
X.	$\sim MHFp \supset \sim Mp$	(9, contraposition)
XI.	$(\sim Pp \wedge \sim p \wedge \sim Fp) \supset \sim Mp$	(1, 3, 4, 10, transitivity of '$\supset$')
XII.	$Mp \supset (Pp \vee p \vee Fp)$	(11, contra. and De Morgan's).

In effect, A1 plus unrestricted substitution and some fairly intuitive assumptions concerning time yield a version of the principle of plenitude, that is, a no-unactualized-possibilities principle which is often taken to be an expression of fatalism.[4]

Various strategems for blocking fatalist arguments similar to the preceding one—as well as the historical antecedents of these strategems—have been discussed by contemporary philosophers and logicians, perhaps most notably by Prior ([7], pp. 113-136). I shall not here pursue this matter further but, rather, turn to some more strictly logical problems generated by the use of A1 in certain modal-tense logic contexts.

2 The concept of a temporal account of the alethic modalities of necessity, possibility, and their contradictories was not uncommon in antiquity. Various formal systems combining both tense and alethic modal operators have also been investigated by contemporary philosophers and logicians. Perhaps the best-known contemporary studies in this area are those that have centered on the "Diodorean" modalities: present possibility is equated with present or sometime-hereafter truth; present necessity with present and always-hereafter truth.

In his development of a semantic model for the analysis of the Diodorean modalities, Prior conceived of time as forwards-linear ([7], pp. 22-23). The Diodorean modal system could then be axiomatized by a tense logic with axioms for forwards-linear time specified in terms of the "simple future" operator F (G defined by $G\phi \equiv \sim F \sim \phi$) and the definitions $M\phi \equiv \phi \vee F\phi$ (for possibility) and $L\phi \equiv \phi \wedge G\phi$ (for necessity). However, it is also the case that the Diodorean modalities, so conceived, can be axiomatized by axioms containing M (with $L \equiv \sim M \sim$) as the sole modal operator. If the assumption of discreteness of time is not made, the System $S4.3$ (the Lewis $S4 + L(Lp \supset Lq) \vee L(Lq \supset Lp)$) axiomatizes the Diodorean modalities; if the assumption of temporal discreteness is made—and, in fact, the assumption is implicit in Prior's semantic model for the modalities—the system of $S4.3 + L(L(p \supset Lp) \supset p) \supset (MLp \supset p)$ is required ([7], pp. 23-31; [4], pp. 260-267).

Diodorus seems to have derived his account of the modalities, $M\phi \equiv \phi \vee F\phi$, (at least in part, i.e., from left to right) from the Master argument. Of the various reconstructions of Diodorus' argument, the tense-logical version of Prior is surely the most elegant.[5] The first premise of the Master, expressing the necessity of the past becomes our

A1 $\quad Pp \supset LPp.$

The argument's second premise, expressing the principle of *reductio ad impossibile,* is, in Prior's reconstruction,

2b $\quad L(p \supset q) \supset (\sim Mq \supset \sim Mp).$

This is obviously derivable from a distinctive axiom for normal modal logics:

2 $\quad L(p \supset q) \supset (Lp \supset Lq).$

From these premises Diodorus claimed to derive a principle of "no unactualized possibilities," i.e., in Prior's formulation

3 $\quad (\sim p \wedge \sim Fp) \supset \sim Mp,$

which is PC-equivalent to the left-to-right direction of the Diodorean "definition" of possibility as present-or-future truth:

3b $Mp \supset (p \vee Fp)$.

In order to effect a syntactic derivation of 3 from 1 and 2b, Prior found it necessary to appeal to two other temporal-modal premises:

4 $L(p \supset HFp)$
5 $(\sim p \wedge \sim Fp) \supset PG\sim p$.

Premise 4 is simply the "L-necessitation" of one of the past-future "mixing" postulates of Lemmon's minimal tense-logic K_t. Premise 5, which will be discussed in more detail, proves to be semantically very powerful. A question of some interest is whether 3, the modal-tense logic version of the conclusion of the Master, can be derived in some appropriate modal-tense logic without Premise 5. In order to consider this question, let us consider a normal modal-tense logic, which I call MA, that can be specified as follows. To Lemmon's minimal tense logic K_t, adjoin a rule of "alethic necessitation" ($\vdash p \Rightarrow \vdash Lp$) plus the following axioms:

A1 $Pp \supset LPp$ (Premise 1 of Master)
A2 $L(p \supset q) \supset (Lp \supset Lq)$ (Premise 2 of Master)
A3 $Hp \supset HHp$
A4 $(Fp \wedge Fq) \supset (F(p \wedge q) \vee F(p \wedge Fq) \vee F(Fp \wedge q))$
A5 $Hp \supset Pp$.

Of the added tense axioms, A3 yields the transitivity of the temporal ordering or "accessibility" ("alternative") relation, A4 its forwards-linearity, and A5 its backwards-seriality (i.e., $\forall t \exists t'(t' < t)$). The following is a brief review of the standard (Kripke) semantic theory for a modal-tense logic such as MA.

A standard (Kripke) frame is defined as an ordered pair $\mathscr{F} = \langle W, R \rangle$, where W is some nonempty set and R is some dyadic relation on W (i.e., $R \subseteq W^2$). R thus serves as the "accessibility" or "alternative" relation for a ("dual" pair of) modal operator(s). But in a temporal-modal logic such as MA, there are three dual pairs of modal operators. Although the accessibility relations for all three pairs are to be defined on a single set W, the temporal and alethic modal accessibility relations may be distinct. Hence, an *extended standard frame* is an ordered triple $\mathscr{F} = \langle W, R_T, R_A \rangle$. W is a nonempty set, intuitively, a set of "possible times". R_T and R_A are each dyadic relations on W. They serve as the accessibility relations for the temporal and alethic modal operators, respectively. Henceforth, I abbreviate 'R_T' by '$<$' and 'R_A' by 'R'. An *extended model* is standardly defined as an ordered pair $\mathfrak{M} = \langle \mathscr{F}, V \rangle$, where $\mathscr{F}$ is an extended frame and V is function from the set of propositional variables or sentence letters into the power set of W. V, in other words, assigns a set of "possible times" to each propositional variable. The recursive or inductive definition of truth at a point ("possible time") in a model includes a separate clause for each of the three undefined modal operators:

(A) $\models_t^{\mathfrak{M}} P\phi$ iff $\exists t'(t' < t \wedge \models_{t'}^{\mathfrak{M}} \phi)$
(B) $\models_t^{\mathfrak{M}} F\phi$ iff $\exists t'(t < t' \wedge \models_{t'}^{\mathfrak{M}} \phi)$
(C) $\models_t^{\mathfrak{M}} M\phi$ iff $\exists t'(Rtt' \wedge \models_{t'}^{\mathfrak{M}} \phi)$.

An (extended) model *validates* a wff $\phi(\models^{\mathcal{M}} \phi)$ iff $\forall t \in W$, $\models_t^{\mathcal{M}} \phi$. An (extended) frame *validates* a wff $\phi(\models^{\mathcal{F}} \phi)$ iff $\forall \mathcal{M}$ such that $\forall \mathcal{M} = \langle \mathcal{F}, V \rangle$, $\forall t \in W$, $\models_t^{\mathcal{M}} \phi$. That is, a frame validates a wff ϕ iff every model "constructible" on it validates ϕ. A class $\mathcal{C}$ of (extended) frames validates a wff $\phi(\models^{\mathcal{C}} \phi)$ iff every $\mathcal{F} \in \mathcal{C}$ validates ϕ. Finally, a class $\mathcal{C}$ of (extended) frames *determines* a modal logic Σ just in case $\forall \phi$, $\Sigma \vdash \phi$ iff $\models^{\mathcal{C}} \phi$. The left-to-right direction of the preceding biconditional is normally referred to as specifying the *soundness* of Σ with respect to $\mathcal{C}$; the right-to-left direction as specifying the *completeness* of Σ with respect to $\mathcal{C}$. A logic that is complete, in this sense, with respect to some class of (extended) standard frames is complete *simpliciter*. It is known that there are modal logics that are not complete *simpliciter* (see [2], [3], [11]-[14]). Such logics, in other words, are not determined by any class of (extended) standard frames. I shall later show that a slight modification of *MA* yields a modal-tense logic that is incomplete in this sense. *MA* itself can be shown to be "incomplete" in a derivative and perhaps, from the logical viewpoint, not very interesting sense: *MA* is not determined by any class of extended frames possessing an irreflexive temporal ordering. This fact is significant, however, for Prior's reconstruction of the Master.

Prior's Premise 5, as was noted, is a semantically "loaded" postulate. It is easily shown to be equivalent to what is sometimes referred to as "Hamblin's Axiom":

Hamb $\quad (p \wedge Gp) \supset PGp.$

Either of these postulates defines the following first-order condition on (extended) standard frames:

5′ $\quad \forall t \exists t'((t' < t) \wedge \forall t''(t' < t'' \supset (t = t'' \vee t < t'')))$.[6]

It will be noted that this condition entails, in addition to temporal backwards-seriality and forwards-linearity, the existence of an immediate temporal predecessor for each point.

Although Prior was troubled by this last consequence of Premise 5, he seems to have been reassured by Becker, who pointed out that we have evidence that Diodorus conceived of time as discrete.[7] Diodorus thus would have had no difficulty in accepting the thesis, which, as Prior puts it, commits one to the existence of a "moment just past" ([7], p. 49). A conception of time as discrete was by no means universal in antiquity, however. Most of the Stoics, for example, held that time is dense.[8] It thus seems that if Diodorus had expressly employed some premise recognized as entailing temporal discreteness, *that* premise would have been the one denied by Stoic critics of the argument. Such critics seem to have accepted the validity of the argument; they reportedly felt compelled to deny either our A1 or Premise 2 in order to escape what they took to be the fatalistic consequences of accepting the conclusion of the argument.[9]

As it turns out, however, the denseness-discreteness issue is a red herring introduced by Prior's strictly syntactic approach to the reconstruction of the argument. It can be proved that, given the assumption of the irreflexivity of the temporal ordering relation, every extended standard frame satisfying this condition *and* validating each theorem of *MA* also validates wff 3, the conclusion of the Master. The salient features of the temporal ordering relation used in this proof, namely, backwards-seriality and forwards-linearity, are entailed by Prior's added Premise 5. However, discreteness of the temporal ordering is *not* dictated by *MA*. Thus, there is a "semantic entailment" of 3 by *MA* "within the field of temporally irreflexive frames" not relying on any temporal discreteness assumption.

Proof: A. Wff XII, of the derivation toward the beginning of this paper, is obviously an *MA* theorem. XII and A1 define, respectively, the following "first-order" conditions on extended standard frames:

XII′ $\forall t \forall t'(Rtt' \supset (t < t' \vee t = t' \vee t' < t))$
A1′ $\forall t \forall t'(Rtt' \supset \forall t''(t'' < t \supset t'' < t'))$.

B. Suppose that Rtt' and $t' < t$. Then, by A1′, it follows that $t' < t'$. But this contradicts the irreflexivity assumption. So, $t' \not< t$. Then, by XII′, it follows that $(t = t') \vee (t < t')$. So, (3′) $\forall t \forall t'(Rtt' \supset (t < t' \vee t = t'))$. However, this wff is just the first-order condition defined by 3, $(\sim p \wedge \sim Fp) \supset \sim Mp$: any extended frame validating it will validate 3.

Showing that 3 is not an *MA* theorem proves an even simpler matter. Consider the following extended standard frame:

As indicated in the diagram, $< = \{\langle t_0, t_0\rangle, \langle t_0, t_1\rangle\}$. Let $R = \{\langle t_0, t_0\rangle, \langle t_0, t_1\rangle, \langle t_1, t_1\rangle, \langle t_1, t_0\rangle\}$. Now define a model on this frame such that for the sentence letter p, $V(p) = \{t_0\}$. It is easily verified that this frame validates all *MA* axioms. Since *MA*'s rules preserve "standard frame validity", it follows that all *MA* theorems are validated on the frame. However, 3, $(\sim p \wedge \sim Fp) \supset \sim Mp$, is falsified at t_1 in the model constructed on the frame. (It can be seen that this model also falsifies an instance of Prior's added Postulate 5, $(\sim p \wedge \sim Fp) \supset PG\sim p$, at t_1.)

The preceding results show that, given the irreflexivity of the temporal accessibility relation, the conclusion of the Master semantically follows from forwards-linearity and backwards-seriality without the additional assumption of discreteness of the temporal ordering. Yet the conclusion of the Master is not a theorem of *MA*. So *MA* is not complete with respect to any class of extended frames with an irreflexive temporal ordering. From a logical perspective, it is not surprising that there should be various tense logics that are incomplete in this sense. For example, adding the "reflexivity" axiom $Gp \supset p$ to Lemmon's K_t yields a logic which is complete with respect to no class of irreflexive frames: every irreflexive standard frame that validates $Gp \supset p$ also validates $p \wedge \sim p$ (trivially, because there is no standard frame satisfying the antecedent). But $p \wedge \sim p$ is not a theorem of the resulting logic.[10]

However, the irreflexivity-incompleteness of *MA* is significant for Prior's modal-tense logical reconstruction of the Master. The salient facts are the following. Irreflexivity is a fundamental component of our conception of temporal order (except in some very special cases, such as circular time).

Irreflexivity is a property of the temporal ordering or accessibility relation that is not "directly expressible" ("by itself") by means of any standard tense logic postulate ([5], p. 78; [8], pp. 124-127). Nonetheless, it is known that we can impose an irreflexivity requirement on the temporal accessibility relation of a standard frame if we are willing to accept some additional semantic concomitants along with irreflexivity. As we have shown, had Prior presented his argument semantically, he could have obtained (the first-order condition defined by) the Master's conclusion, wff 3, from the following: (a) the irreflexivity of the temporal ordering relation; (b) the Master's premises; (c) the minimal tense logic K_t plus postulates defining the forwards-linearity and backwards-seriality of the temporal ordering. His syntactic approach, however, makes it impossible for him to capture the temporal irreflexivity condition "by itself", with a tense logic postulate. His Premise 5 is the additional premise which "does the semantic job" of irreflexivity;[11] but the "extra semantic baggage" that it brings along is the red herring of backwards temporal discreteness. It thus seems clear that the whole issue of the connection between temporal discreteness and the Master arises from a vagary of tense logic, i.e., the fact that no tense-logic postulate defines temporal irreflexivity *simpliciter*. Consequently, it seems exceedingly improbable that temporal discreteness should have figured in Diodorus' formulation of the Master, since he certainly did not have the benefit of tense logic, in the syntactic sense of that term.

The result of this section yields, I believe, a cautionary moral concerning the use of formal logical techniques in studying arguments in the history of philosophy. The rigor supplied by formal logical techniques can be quite helpful in extracting the "conceptual meat" from such arguments, but we can occasionally be misled by the techniques. This is philosophical folk wisdom. As folk wisdom, its truth may be regarded as dubious. Yet, in the case of the Prior/Becker modal-tense logic analysis of the Master, I believe we have clear evidence of its truth.

In the next and final section of this paper, I present some logical results pertaining to the necessity-of-the-past modal-tense logic thesis A1. In view of the philosophical interest of A1 and the problems that can be generated by its philosophical employment (illustrated in the present section by the Prior reconstruction of the Master), such investigations seem worthwhile.

3 Let us create a new modal-tense logic *MAIrr* (for *MA-Irreflexive*) by omitting the transitivity and backwards-seriality Axioms A3 and A5 and substituting for them a new tense axiom:

AIrr $\quad Pp \supset P(p \wedge {\sim}Pp)$.

AIrr can be shown to be equivalent to a temporal form of "Löb's axiom", namely

LöbT $\quad H(Hp \supset p) \supset Hp$.

As Boolos notes, Löb's axiom defines the transitivity of the accessibility relation of the operator in terms of which it is stated together with the "well-foundedness" of that relation (or its inverse) ([1], pp. 5, 81-82). For LöbT, these conditions amount to the transitivity of $<$ (the wff $Hp \supset HHp$ is, in fact, syntactically derivable from LöbT in K_{ti} cf. [1], p. 30) and the following condition: for every nonempty subset S of the set W of a standard frame $\mathcal{F} = \langle W, <\rangle$ there is an element t of S such that there is no element t' of S for which $t' < t$. This condition is captured by a set theoretic condition Thomason has associated with AIrr (see [11], p. 154).

AIrr′ $\quad (\forall S \subset W)(\forall t \in S)(\exists t' \in S)((t' < t \vee t' = t) \wedge \forall t''(t'' < t' \supset t'' \notin S))$.

It is easily shown that either of these conditions entails the irreflexivity of the temporal ordering relation on a standard frame. Simply instantiate S by the singleton formed from each element of W. Consequently, for each t, $t \not< t$.

Let us now consider the following formula, closely related to wff 3, the conclusion of the Master:

3⁻ $\quad P(q \vee \sim q) \supset ((\sim p \wedge \sim Fp) \supset \sim Mp)$.

This wff corresponds to a first-order condition on the temporal and alethic accessibility relations of extended standard frames slightly weaker than that defined by 3, namely,

3⁻′ $\quad \forall t(\exists t'(t' < t) \supset \forall t''(Rtt'' \supset (t < t'' \vee t = t'')))$.

Claim 1 *Every extended frame that validates each theorem of MAIrr also validates wff* 3⁻.

Proof: Axioms A1 and A4 of *MAIrr* define, respectively, the following first-order conditions on any extended frame validating all *MAIrr* theorems.

A1′ $\quad \forall t \forall t'(Rtt' \supset \forall t''(t'' < t \supset t'' < t'))$

A4′ $\quad \forall t \forall t' \forall t''((t < t' \wedge t < t'') \supset (t' < t'' \vee t' = t'' \vee t'' < t'))$.

And, as we have seen, Axiom AIrr entails a temporal irreflexivity condition, $\forall t(t \not< t)$, for any standard frame on which it is valid. It is easily shown that the first-order condition 3⁻′ is *LPC* derivable from A1′, A4′, and the irreflexivity condition. Thus, any extended standard frame validating each theorem of *MAIrr* will also validate wff 3⁻.

Claim 2 *Wff* 3⁻ *is not a theorem of MAIrr.*

In order to establish Claim 2, I employ the concept of a general frame. An extended general frame (secondary frame, "first-order structure")[12] $\langle \mathcal{F}, \mathcal{W}\rangle$ is "based" on an extended standard frame $\mathcal{F}$ where $\mathcal{W} \subseteq \mathbf{P}W$ and $\mathcal{W}$ satisfies the following conditions:

(i) $\mathcal{W} \neq \phi$

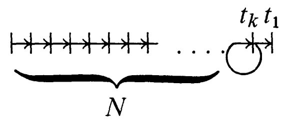
(ii) $\mathcal{W}$ is closed under complement- and intersection-taking (with respect to W of the standard frame $\mathcal{F}$)

(iii) $\mathcal{W}$ is closed under each of the following operations:
$\underline{r}_a(X) = \{t \in W: (\exists t' \in X)Rtt'\}$
$\underline{r}_f(X) = \{t \in W: (\exists t' \in X)\, t < t'\}$
$\underline{r}_p(X) = \{t \in W: (\exists t' \in X)\, t' < t\}$.

$\mathcal{W}$ constitutes the range of the valuation function V of any model constructed from the general frame $\langle \mathcal{F}, \mathcal{W}\rangle$.

Since the rules of *MAIrr* preserve validity on a general frame, it follows that a general frame validating each *MAIrr* axiom validates each *MAIrr* theorem. I propose to specify a structure $\langle \mathcal{F}, \mathcal{W} \rangle$ that is a general frame which validates each *MAIrr* axiom but does not validate wff 3^-.

The structure consists of a set W of points temporally ordered in a manner indicated by the following diagram:



N is a subset of W having the temporal order of the natural numbers. In general, the temporal accessibility relation on W is to be the normal transitive, irreflexive $<$, augmented by the pair $\langle t_k, t_k \rangle$. For the alethic modal accessibility relation R, let $R = \leqslant \cup \{\langle t_1, t_k \rangle\}$, i.e., the normal temporal partial ordering augmented by the pair $\langle t_1, t_k \rangle$. Let S be the set of all subsets of W which contain t_k and which are cofinite in W (i.e., $S = \{X \subseteq W\colon t_k \in X$ and $W - X$ is finite$\}$). Then, let $\mathcal{W}$ be the closure of S under complement-taking (with respect to W).

In order to establish Claim 2, it suffices to establish (Subclaim A) that $\langle \mathcal{F}, \mathcal{W} \rangle$ is a general frame, (Subclaim B) that $\langle \mathcal{F}, \mathcal{W} \rangle$ validates all *MAIrr* axioms, and (Subclaim C) that there is a model constructible on $\langle \mathcal{F}, \mathcal{W} \rangle$ which falsifies wff 3^- at point t_1.

Subclaim A *Since $\mathcal{W}$ is obviously nonnull and closed under complementation, it suffices, in order to establish that $\langle \mathcal{F}, \mathcal{W} \rangle$ is a general frame, to show that $\mathcal{W}$ is closed under (finite) intersection and the operations $\underline{r}_f$, $\underline{r}_p$, and $\underline{r}_a$.*

At the urging of a reader of this paper, I suppress these proofs, which are straightforward but moderately tedious.

Subclaim B *The general frame $\langle \mathcal{F}, \mathcal{W} \rangle$ validates all axioms of MAIrr.*

Proof: The fact that the temporal ordering of $\langle \mathcal{F}, \mathcal{W} \rangle$ is forwards-linear, together with the fact that $\langle \mathcal{F}, \mathcal{W} \rangle$ is a general frame, guarantee that all axioms of *MAIrr*, except A1 and AIrr, are validated by the general frame.

For A1: For any point $t \neq t_1$, the truth value of A1 in any model constructible on $\langle \mathcal{F}, \mathcal{W} \rangle$ will be identical to that of the wff $Pp \supset (Pp \wedge GPp)$ at that point in the model. But the latter wff is validated by any frame with a transitive temporal ordering (which our general frame possesses); so A1 will be true at any point $t \neq t_1$ in any model constructible on $\langle \mathcal{F}, \mathcal{W} \rangle$. If $t = t_1$ and we consider a model on $\langle \mathcal{F}, \mathcal{W} \rangle$ making Pp true at t_1, then the model must make p true either at t_k or at some point t' in N temporally preceding t_k. In the latter case, Pp will then be true both at t_k and t_1 and, consequently, Pp will be true at all t such that Rt_1t; LPp true at t_1; and A1 true at t_1. In the former case (in which p is true at t_k) Pp will be true at t_1 *and* at t_k (since $t_k < t_k$); and the same consequences follow.

For AIrr: By substituting $(\forall S \in \mathcal{W})$ for $(\forall S \subseteq W)$ in Thomason's condition AIrr$'$, we can, in effect, obtain a "second-order" general-frame condition for

AIrr. The modified AIrr′ then says that any nonempty subset of W that can serve as a "truth set" for a wff (that is, any element of $\mathcal{W}$) must have a temporally initial member. $\mathcal{W}$, as we have defined it for $\langle \mathcal{F}, \mathcal{W} \rangle$, satisfies this condition.

Subclaim C *Wff* 3^- *can be falsified at point* t_1 *on* $\langle \mathcal{F}, \mathcal{W} \rangle$.

Proof: Define a model $\mathcal{M}$ on $\langle \mathcal{F}, \mathcal{W} \rangle$ such that $V(p) = \{t_k\} \cup N$. $P(q \vee \sim q)$ is true at t_1 in this model. So is $\sim p \wedge \sim Fp$. But, since p is true at t_k, Mp is true at t_1 in the model. Hence, 3^- is false at t_1 in this model.

We have now established Claim 2, that 3^- is not an *MAIrr* theorem. This result, together with Claim 1, demonstrates the incompleteness of *MAIrr*.

I conclude this paper with some reflections on this result plus the outline of a completeness result for our "troublesome" necessity-of-the-past postulate A1.

The following general definition of modal completeness, relative to extended standard frames, can be formulated from a definition employed by van Benthem in a recent paper on varieties of modal completeness ([14], p. 127):

Definition A set Σ of modal formulas is complete if, for all modal formulas ϕ

$$\Sigma \vdash_{K_t^+} \phi \text{ iff } \Sigma \vDash_{\mathcal{F}} \phi.$$

A modal formula ϕ is complete if $\{\phi\}$ is.

Here K_t^+ is understood to be that "minimal" combined alethic modal/tense logic of which the alethic modal component is equivalent to the minimal normal modal logic K and the tense component is equivalent to Lemmon's K_t. There are no axioms "mixing" alethic modal and tense operators in K_t^+. And $\mathcal{F}$ is understood to be an extended standard frame. The preceding result shows that the set Σ of wffs, where $\Sigma = \{\text{A1, A4, AIrr}\}$, is incomplete. Specifically, where ϕ is $P(q \vee \sim q) \supset ((\sim p \wedge \sim Fp) \supset \sim Mp)$, $\Sigma \vDash_{\mathcal{F}} \phi$ but not $\Sigma \vdash_{K_t^+} \phi$.

It is clear, in this case, what has happened. The wff AIrr entails the irreflexivity of the temporal ordering *only if* we can assume that each singleton formed from an element of W is a "truth set" to which a propositional variable may be assigned. But in the case of a general frame, we cannot make this assumption. We have an illustration of this phenomenon in our general frame $\langle \mathcal{F}, \mathcal{W} \rangle$. The singleton $\{t_k\}$ is not a member of $\mathcal{W}$; thus, even though AIrr is valid on this general frame, t_k is not a temporally irreflexive point. As it happens, the irreflexivity condition figures essentially in the derivation of a first-order condition ($3^{-\prime}$) defined by a modal formula 3^-. Since the irreflexivity condition does not "carry over" to general frames, there is not a similar derivation of a "second-order" general-frame condition corresponding to $3^{-\prime}$. Thus, in view of the general completeness and soundness results of K_t^+ with respect to *general frames*, 3^- is not K_t^+ derivable.

There is, as it happens, another "second-order" general-frame condition defined by our wff A1, namely,

A1Sec $(\forall S \in \mathcal{W}) \forall t \forall t' (Rtt' \supset \forall t''((t'' < t \wedge t'' \in S) \supset \exists t'''(t''' < t' \wedge t''' \in S)))$.

The first-order condition A1′ defined by A1 entails A1Sec, but the converse is true only if we can assume that each singleton formed from a member of W is an element of $\mathscr{W}$. We cannot make this assumption for any arbitrary general frame. And indeed, another general frame can be constructed validating each axiom of *MAIrr* and preserving the irreflexivity of the temporal ordering which satisfies the second-order condition A1Sec but does not satisfy the first-order condition A1′.[13] It seems that for a general frame to validate all *MAIrr* theorems but *not* to validate wff 3^-, that general frame must violate either the temporal irreflexivity condition or the first-order condition A1′.

The foregoing observation raises the question of whether A1 is, by itself, complete according to the mixed modal-tense logic version of van Benthem's definition of completeness previously stated. The answer is affirmative. A sketch of the proof of this claim follows. Any extension of K_t^+ is complete with respect to what may be called its "Henkin general frame". For the standard frame $\mathscr{F} = \langle W, <, R \rangle$, W is the set of maximally (proof-theoretically) consistent sets of wffs of the logic in question, and $<$ and R are defined on W as follows: $t < t'$ iff, for all modal formulas ϕ, $H\phi \in t' \Rightarrow \phi \in t$; Rtt' iff, for all modal formulas ϕ, $L\phi \in t \Rightarrow \phi \in t'$. The set $\mathscr{W}$ of the Henkin *general* frame is the set of all sets of the form $\{t\colon t \in W \text{ and } \phi \in t\}$, where ϕ is any modal formula.[14]

The heart of the proof is to show that the Henkin general frame satisfies the first-order condition A1′ defined by A1. If it does, it follows that A1 will be validated by the Henkin standard (or "underlying") frame. The logic obtained by adding A1 to K_t^+ is then said to be "canonical". Well-known results entail the completeness (with respect to extended standard frames) of such a canonical logic (see [14], pp. 129-131).

The proof goes as follows (employing only *LPC* and the preceding definitions of the accessibility relations for Henkin frames):

(i) $\forall t \forall \phi (P\phi \supset LP\phi \in t)$
(ii) $\forall t \forall \phi (P\phi \in t \Rightarrow LP\phi \in t)$
(iii) $\forall t \forall \phi (P\phi \in t \Rightarrow \forall t'(Rtt' \Rightarrow P\phi \in t'))$
(iv) $\forall t \forall t'(Rtt' \Rightarrow \forall \phi (P\phi \in t \Rightarrow P\phi \in t'))$
(v) $\forall t \forall t'(Rtt' \Rightarrow \forall \phi (H\phi \in t' \Rightarrow H\phi \in t))$
(vi) $\forall t \forall t'(Rtt' \Rightarrow \forall \phi (H\phi \in t' \Rightarrow \forall t''(t'' < t \Rightarrow \phi \in t'')))$
(vii) $\forall t \forall t'(Rtt' \Rightarrow \forall t''(t'' < t \Rightarrow \forall \phi (H\phi \in t' \Rightarrow \phi \in t'')))$
(viii) $\forall t \forall t'(Rtt' \Rightarrow \forall t''(t'' < t \Rightarrow t'' < t'))$.

We thus know that although {A1, A4, AIrr} is not complete, {A1} is. I conclude the present logical investigation of the necessity-of-the-past modal-tense postulate A1 with a conjecture: {A1, AIrr} is not complete.

NOTES

1. Explicit statement of the position dates at least from Aristotle; for references to its occurrence in the Aristotelian *corpus* see J. Hintikka, *Time and Necessity* (Oxford, 1973), p. 183.

2. Forwards-linearity yields $PFp \supset (Pp \vee p \vee Fp)$. (See Rescher and Urquhart [8], pp. 88-89.) The backwards-seriality axiom and the preceding wff then yield, by contraposition, De Morgan's and the *P-H* duality relation $(\sim Pp \wedge \sim p \wedge \sim Fp) \supset P{\sim}Fp$.

3. A normal modal logic (with respect to the alethic modalities) may be defined as any propositional modal logic containing all *PC* tautologies and wff VI, and which is closed under the operations of uniform substitution, modus ponens, and "necessitation" ($\vdash p \Rightarrow \vdash Lp$). The "minimal" normal modal logic is usually denominated K. For tense logic, Lemmon's K_t serves as a "minimal" normal logic: it is the smallest tense logic containing all *PC* tautologies, "distribution" axioms corresponding to VI (one each for G and H), the "mixing" axioms $p \supset HFp$ and $p \supset GPp$, and which is closed under substitution, modus ponens, and H- and G- "necessitation".

4. Diodorus' "no-unactualized-possibilities" principle is so interpreted by Cicero in his *De Fato*, for example.

5. Summarized in Prior [7], pp. 32-34. A considerable contemporary literature concerning the Master, unfortunately too lengthy to list in full, has developed. In addition to Prior's work, summarized in [7], the following discussions have received considerable attention: Benson Mates, *Stoic Logic* (Berkeley and Los Angeles, 1953); Oskar Becker, "Zur Rekonstruktion des *Kyrieuon Logos* des Diodoros Kronos," in *Erkenntnis und Verwortung: Festschrift für Theodore Litt,* eds., J. Derbolav, F. Nicholin (Düsseldorf, 1960); W. Kneale and M. Kneale, *The Development of Logic* (Oxford, 1962); P. M. Schuhl, *Le Dominateur et les possibles* (Paris, 1960); J. Hintikka, "Aristotle and the 'Master Argument' of Diodorus," *American Philosophical Quarterly,* vol. 1 (1964), pp. 101-114 reprinted as ch. IX, *Time and Necessity* (Oxford, 1973); R. Blanché, "Sur l'interprétatione du κυριεύων λόγος," *Revue Philosophique de la France et de l'Etranger,* vol. 155 (1965), pp. 133-147; N. Rescher, "A version of the 'Master Argument' of Diodorus," *Journal of Philosophy,* vol. 63 (1966), pp. 438-445.

6. This first-order condition is due to an anonymous reader of this paper.

7. See O. Becker, Note 5 above, pp. 250-263. For Diodorus and the discreteness of time, see Sextus Empiricus, *Adversus Mathematicos,* 10.85ff.

8. Sextus, *Adversus Mathematicos,* 10. See also Plutarch *De Communibus Notitiis Adversus Stoicos* and the discussion in S. Sambursky, *Physics of the Stoics.*

9. As reported in Epictetus, *Dissertationes*, 2.19.

10. The example is that of van Benthem, who also suggested several simplifications and improvements of the preceding proof.

11. Actually, Premise 5 yields a semantical condition which is slightly weaker than irreflexivity but sufficient for the preceding semantic proof. Premise 5 does entail that if a point t has an "immediate" temporal successor t' such that $t' \neq t$ and t' is irreflexive, then t must also be irreflexive.

12. See, for example, van Benthem [13] and [14], Goldblatt [3], or Thomason [12]. In what is perhaps a somewhat unfortunate "inversion" of terminological intuitions in [11] (explained on p. 152), Thomason refers to general frames as "first-order structures".

13. For the point t_k substitute an infinite set Z of points temporally ordered by the normal transitive, irreflexive $<$ and having the order type, relative to this relation, of the signed integers. The temporal ordering of the frame, then, becomes the normal strict linear ordering $<$. For the alethic modal accessibility relation R, let $R = \leqslant \cup \{\langle t_1, t_0\rangle\}$, where t_0 is some element of Z. Let the set S be the set of all subsets of W including Z which are cofinite in W, and let $\mathcal{W}$ be the closure of S under complement-taking. The resulting general frame: (a) validates all *MAIrr* theorems; (b) falsifies 3^- at point t_1 for the model $\mathcal{M}$ such that $V(p) = N \cup Z$; (c) preserves the irreflexivity of the temporal accessibility relation; (d) satisfies second-order condition A1Sec; but (e) does *not* satisfy first-order

condition A1′. This general frame was the one originally used in the incompleteness result of this paper. The general frame of the present text, which satisfies condition A1′ but does not preserve temporal irreflexivity, was suggested by an anonymous reader of the paper.

14. This is a standard account: see, for example, [14] or [10].

REFERENCES

[1] Boolos, G., *The Unprovability of Consistency: An Essay in Modal Logic,* Cambridge University Press, Cambridge, 1979.

[2] Fine, K., "An incomplete logic containing S4," *Theoria*, vol. 40 (1974), pp. 23-29.

[3] Goldblatt, R. I., "First-order definability in modal logic," *The Journal of Symbolic Logic,* vol. 40, no. 1 (1975), pp. 35-40.

[4] Hughes, G. E. and M. J. Cresswell, *An Introduction to Modal Logic,* Methuen and Co., Ltd., London, 1968.

[5] Lemmon, E. J. and D. Scott, *An Introduction to Modal Logic,* ed., K. Segerberg, Blackwell, Oxford, 1977.

[6] McArthur, R. P., *Tense Logic,* D. Reidel, Dordrecht and Boston, 1976.

[7] Prior, A., *Past, Present and Future,* Oxford University Press, Oxford, 1967.

[8] Rescher, N. and A. Urquhart, *Temporal Logic,* Springer, Vienna and New York, 1971.

[9] Sahlqvist, H., "Completeness and correspondence in the first and second order semantics for modal logic," pp. 110-143 in *Proceedings of the Third Scandinavian Logic Symposium,* ed., S. Kanger, North-Holland Publishing Co., Inc., Amsterdam, 1976.

[10] Segerberg, K., *An Essay in Classical Modal Logic,* University of Uppsala, Sweden, 1971.

[11] Thomason, S. K., "Semantic analysis of tense logics," *The Journal of Symbolic Logic,* vol. 37, no. 1 (1972), pp. 150-158.

[12] Thomason, S. K., "An incompleteness theorem in modal logic," *Theoria,* vol. 40 (1974), pp. 30-34.

[13] van Benthem, J. F. A. K., "Two simple incomplete modal logics," *Theoria,* vol. 44 (1978), pp. 25-37.

[14] van Benthem, J. F. A. K., "Some kinds of modal incompleteness," *Studia Logica,* vol. 39, no. 2/3 (1980), pp. 125-141.

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