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Tense Logic and Time

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Introduction A new wave of research in tense logic is under way, investigating temporal ontologies other than the traditional ones, inspired by logical concerns different from the usual ones. Some recent papers exemplifying this trend are [2], [3], [5], [6], and [8]. The purpose of the present paper is to give a systematic survey of some important issues in the logical study of Time, as well as some new results pointing at, and hopefully strengthening, the new lines of research.

1 Traditional tense logic The pattern set by a typical tense logic in the style of Arthur Prior has become classical in philosophical logic. Inferences are studied using some formal logical language with certain (tense) operators, and that *language* is interpreted in temporal *structures* through a suitable *truth definition*. The motivation for the various choices made with respect to each of these three conceptual "degrees of freedom" came from various sources: philosophical, linguistic, and also purely logical. Accordingly, criticism of Prior's research program has also emanated from all of these sources—for a survey, cf. [7].

If there is a criticism of traditional tense logic behind the present paper, it is not so much that there is anything wrong with this enterprise as that its scope needs to be enlarged to become a truly logical study of time. Tense logicians often introduce their structures $\langle T, < \rangle$ ("points in time", "earlier than") as a matter of course: the semantic action occurs elsewhere. But, in a systematic logical study of temporal ontology, this first act already embodies various choices to be investigated systematically: choices of temporal *individuals*, *relations*, and *postulates*. Shall the individuals be points, periods, or even events? Which are the basic relations: precedence, simultaneity, overlap, inclusion? Can fundamental temporal postulates be logically derived for these, e.g., linear order? We possess a wealth of temporal ideas and intuitions, and logic should help us in organizing and relating these.

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*2 **Revisionist tense logic*** Traditional tense logics have usually taken over the scientific picture of durationless points in time, ordered by precedence ("earlier than", "before"). But, even assuming this basic picture, an enlightened Priorean approach might consider various options here. An outline of the resulting enterprise follows.

*2.1 **Temporal structures*** The traditional models are *point structures* $\langle T, < \rangle$, with $<$ a binary relation on the set T. Prime examples are the number lines $\mathbf{Z}$ (integers), $\mathbf{Q}$ (rationals), and $\mathbf{R}$ (reals). Why does one choose precedence for a basic relation here? A "logical fable" explaining its genesis in terms of context-dependent temporal arrangement (late/not late) inducing a context-free temporal comparative ("later") may be found in [8]. We shall see later how much mileage can be got from this analysis. Why does one choose precedence as the only basic relation here? This is more difficult to motivate, at least as soon as one takes less classical temporal structures into account, such as Minkowski Space with the relation of "possible causal precedence". (This relation holds between a point and all points in the interior of its forward light cone.) For instance, simultaneity ("space-like separation") might then seem equally important. The fact of the matter is, however, that even here precedence is rich enough to allow for *definitions* of all other significant (spatio-) temporal relations; cf. [9].

*2.2 **Temporal postulates*** Not every point structure qualifies as a possible model of time. Obviously, our intuitions, however diverse, impose some basic restrictions. These come in various kinds, some more concrete, others more volatile.

Direct axioms Some straightforward conditions on precedence are axioms for comparatives, derived in [8] from the context construction mentioned above. $\langle T, < \rangle$ is to satisfy

transitivity: $\forall xyz(x < y < z \rightarrow x < z)$
irreflexivity: $\forall x(\neg x < x)$,

as well as

connectedness: $\forall xyz(x < y \rightarrow (x < z \vee z < y))$.

(Another way to view connectedness is as transitivity of nonprecedence.) Structures of this kind can be represented as a linear order of point sets inside which all points are mutually nonpreceding ("simultaneous").

What other conditions on temporal structures could be motivated through logic alone? Notice that all principles given here have a *universal* form. Perhaps logical analysis should, at least, give us all basic temporal postulates of this kind—the remaining, existential principles requiring knowledge about the actual furniture of the world. One obvious strengthening of this kind would be the following

linearity: $\forall xy(x < y \vee y < x \vee x = y)$,

i.e., simultaneity collapses into identity.

We seem to be crossing a border here. Linearity, and indeed already connectedness, exclude relativistic temporal structures as mentioned earlier. Do we want logic to make physically restrictive choices as to the nature of time? It seems that, beyond the above first two requirements of so-called *strict partial order*, all further additions express some "dimensionality restriction" upon time, constraining physical, rather than logical space. We mention two relevant points from [8]:

Proposition *Strict linear order is the universal theory of classical time.*

Conjecture: Strict partial order is the universal theory of all finite-dimensional Minkowski space-times.

(Actually, the universal theory of ordinary four-dimensional relativistic space-time contains certain universal "spatial" principles, in addition to temporal strict partial order; but this matter will not concern us here.)

Global intuitions An analysis of a concept in logical semantics need not always result in explicit direct axioms. Our intuitions may be much more global, and yet exact. One recurrent idea as to the *kind* of structure that qualifies as a temporal model is "homogeneity": the pattern of time is the same everywhere. Formally, this may be captured by requiring that point structures $\langle T, < \rangle$ be *homogeneous*:

> for every two $t, t' \in T$, there exists some $<$-automorphism of $\langle T, < \rangle$ sending t to t'.

Notice that both classical structures, such as $\mathbf{Z}$ or $\mathbf{Q}$, and relativistic ones, such as Minkowski space-time, satisfy this postulate. (In the general theory of relativity it may fail, however.)

Homogeneity does have direct consequences, in the form of "radical choices". For every two points will show the same precedence behaviour; and hence we have, e.g., isolation-or-succession, discreteness-or-density: either all points are isolated, or there exists a precedence pair (and hence *all* points have temporal predecessors and successors); either all points have an immediate predecessor (because one point has), or the ordering is dense.

Even so, very many point structures are homogeneous strict partial orders. Can the homogeneity intuition be strengthened so as to restrict the field to surveyable proportions? After all, space seems to satisfy a much stronger constraint, viz., that any two *pairs* of distinct points can be connected through some automorphism. And indeed, $\mathbf{Q}$ satisfies a similar principle, provided that both pairs lie in the same order. On the other hand, $\mathbf{Z}$ fails this test since the number of intermediate points may be different. Some reflection shows that the proper generalization, then, is the following principle of *indistinguishability*:

> for every two finite sequences $t_1, \ldots, t_n$; $t'_1, \ldots, t'_n \in T$ with the same type (i.e., verifying the same first-order formulas), there exists some $<$-automorphism of $\langle T, < \rangle$ sending t_1 to $t'_1, \ldots, t_n$ to t'_n.

Thus, sequences of points in time that cannot be distinguished in our language cannot be distinguished at all. This is the familiar model-theoretic notion of 'homogeneity', about which various logical results are known. Thus, nontrivial model theory turns out to be relevant to the logical study of time. For instance, the theorem that every countable model has an elementary extension satisfying indistinguishability (cf. [1]) now tells us that the latter postulate has no "direct" consequences.

For an important special case, we can now classify all possible temporal models.

Theorem *The countable connected strict partial orders satisfying homogeneity and indistinguishability consist of linear orders of simultaneous point sets (all of the same size), whose pattern is one of the following four*: $\mathbf{1}$, $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{Q} \odot \mathbf{Z}$.

Here, $\mathbf{1}$ is one single isolated point, and $\mathbf{Q} \odot \mathbf{Z}$ is the structure consisting of $\mathbf{Q}$ (dense time at its macrolevel) with each point replaced by a copy of $\mathbf{Z}$ (discrete time at its microlevel).

Proof: The linear pattern of the simultaneity sets was observed before. That these all have the same size follows from homogeneity (plus countability). Now, the global order is either $\mathbf{1}$, or all its points have predecessors and successors, as was noted earlier. Moreover, it is either dense or discrete. In the former case, the global order is a countable unbounded dense linear order, which can only be $\mathbf{Q}$, by Cantor's Theorem. In the latter case, the global order consists of a countable number of copies of the integers, by a standard argument. Now, either there is only one such copy $\mathbf{Z}$, which is one of the above four possibilities, or there are more, say forming a linear order L. It remains to be shown that L must be isomorphic to the rationals, which accounts for the fourth case of $\mathbf{Q} \odot \mathbf{Z}$.

Suppose that L has more than one element. By homogeneity, then, it cannot have final points. (One has to argue about moving copies of the integers now.) Moreover, it is dense. For if $1_1 < 1_2$, then consider $1_3 < 1_1$. Notice that, in discrete linear orders, all pairs of points from different copies of the integers verify the same first-order formulas. Therefore, this fact holds for (any choice of points from) the pairs $(1_1, 1_2)$, $(1_3, 1_2)$. (As we are concerned with automorphisms, moving one point means rigidly moving its surrounding copy of $\mathbf{Z}$.) By indistinguishability, then, some automorphism maps 1_3 to 1_1, leaving 1_2 fixed, i.e., 1_1 moves between its original position and 1_2. Thus, there must have been a point between 1_1 and 1_2. So L is again an unbounded dense linear order that is countable, i.e., an isomorph of $\mathbf{Q}$. □

Question: Give a similar characterization of countable temporal structures *without* assuming connectedness.

The connection between global intuitions concerning time and notions of classical model theory deserves further systematic exploration.

3 Revolutionary tense logic Starting from the point structures of the preceding section, the orthodox road in tense logic goes *from points to intervals* of time, and thence *to events* as intervals plus linguistic description of what is going on during these. But, for various philosophical and linguistical reasons, the proper order of analysis might well be the other way around. Events form the stock of our primary experience, periods are already abstract substrata underlying simultaneous events, and points are ideal limiting cases of periods. Thus, the heterodox road *from events to periods to points* deserves closer scrutiny.

In this paper, the richer spatio-temporal, possibly even causal, notion of 'event' will not be studied. But we will consider a temporal ontology based upon extended "periods" as its primitive individuals. Rather than construct an "interval tense logic" upon the pattern of already existing Priorean models, we will consider this new ontology by itself, noticing the new types of question it engenders.

3.1 Temporal structures Having chosen the *individuals*, there arises the choice of appropriate primitive temporal *relations.* As in the earlier case, *precedence* seems a fundamental notion. But, moreover, the extendedness of periods should now show up in the relational pattern. Various preferences have been expressed in the literature on this issue, resulting in two main candidates: *inclusion* and *overlap*. In this paper, we opt for the former. Thus, a *period structure* will be a triple $\langle I, <, \sqsubseteq \rangle$. Prime examples are $INT(\mathbf{Z})$, all closed bounded integer intervals with total precedence and set inclusion, and $INT(\mathbf{Q})$, all open bounded rational intervals (with rational boundaries) ordered by these same relations. For the ontological purists, the underlying point set structure should be forgotten, of course.

There exist several definitional connections between these various primitives that one could argue for. For instance, obviously,

$$xOy \text{ ("}x \text{ overlaps } y\text{")} \leftrightarrow \exists z(z \sqsubseteq x \;\&\; z \sqsubseteq y)$$

(we are thinking of nonempty periods), and

$$x \sqsubseteq y \leftrightarrow \forall z(zOx \rightarrow zOy).$$

But, more debatably, inclusion might also be definable in terms of precedence. What else is a subperiod than a devoted follower of its superior's preferences?

$$x \sqsubseteq y \leftrightarrow \forall z((z < y \rightarrow z < x) \;\&\; (y < z \rightarrow x < z)) \qquad (*)$$

We shall see presently which constraints would be imposed on inclusion in this manner.

3.2 Temporal postulates An investigation of conditions on temporal period structures is an exploration of a new world. We are not used to studying this kind of relational pattern. Nevertheless, the general procedure of Section 2.2 still recommends itself.

Direct axioms The minimal constraint of *strict partial order* remains equally plausible for precedence among periods. As for inclusion, the *partial order* axioms seem obvious: *transitivity*, together with

reflexivity: $\forall x x \sqsubseteq x$
antisymmetry: $\forall xy(x \sqsubseteq y \sqsubseteq x \rightarrow x = y)$.

As each partial order may be represented as a set inclusion structure in a standard way, these three conditions comprise the complete first-order theory of set inclusion. (It may be proved, by the way, that the complete theory of set theoretic overlap is given by *symmetry*: $\forall xy(xOy \rightarrow yOx)$ and *quasi-reflexivity*: $\forall xy(xOy \rightarrow xOx)$.)

In addition to these pure principles, several mixed principles ensure a certain integration of precedence and inclusion pattern. First, in line with the mentioned definition of inclusion, there is a principle of *monotonicity*:

$$\forall xyz(x \sqsubseteq y < z \rightarrow x < z)$$
$$\forall xyz(z < y \sqsupseteq x \rightarrow z < x).$$

Also, there is a principle of *convexity*, stating that periods should be uninterrupted stretches:

$$\forall xyzu(u \sqsupseteq x < y < z \sqsubseteq u \rightarrow y \sqsubseteq u).$$

Further additions of an existential nature can be made, but new purely universal constraints do not suggest themselves readily. One logical explanation for this phenomenon (and the virtue of logical philosophy is precisely that it affords such precise answers) lies in the following result:

Theorem *On the strict partial orders $\langle I, < \rangle$, the complete universal theory of the period structures $\langle I, <, \sqsubseteq \rangle$ defined by (*) is axiomatized by transitivity and reflexivity for inclusion, together with monotonicity and convexity.*

Proof: That all four principles do follow requires an easy deduction. (E.g., for convexity, assume that $u \sqsupseteq x < y < z \sqsubseteq u$. Suppose that $v < u$. Then $v < x$ (by (*)), and so $v < y$ (transitivity). The case of $u < v$ goes analogously.)

The other way around involves an elementary representation argument, of which we only mention the main steps. Consider any structure $\langle I, <, \sqsubseteq \rangle$ such that $<$ is a strict partial order, with $\sqsubseteq$ satisfying the four mentioned conditions. First, contract the structure by identifying mutually including periods. The contraction is a strong homomorphism respecting the former $<, \sqsubseteq$-theory; but, in addition, $\sqsubseteq$ is now antisymmetric. Next, represent this structure as a set of nonempty convex sets on some underlying strict partial order, with total precedence and set inclusion. (The method available is that of Section 4.2.) Finally, enlarge this structure to one in which the equivalence (*) is satisfied. (One half is automatic, thanks to monotonicity.) Now, we make noninclusions show up in failure of "obsequiousness" by adding suitable points.

Now suppose that some universal principle about $<, \sqsubseteq$ does not follow from our axioms. Then it fails in some model for these, by the completeness theorem. Since this failure merely involves the existence of some recalcitrant finite diagram, the principle fails in all extensions of the model–in particular, in the one constructed above where (*) holds as well. □

When more than purely universal principles are considered, various candidates arise, of which an exhaustive study is made in [8], leading to characterizations of the first-order theories of $INT(\mathbf{Z})$ and $INT(\mathbf{Q})$. Three examples deserve mention here. The first is

*linearity**: $\forall xy(x < y \vee y < x \vee \exists z(z \sqsubseteq x \;\&\; z \sqsubseteq y))$.

This is the obvious analogue of the earlier linearity postulate for points. Its *general* validity is extremely doubtful (cf. [3]). Notice that in terms of overlap this would be a purely universal principle. Indeed, it occurs in Kamp's primary axiom set for $<, O$.

Question: Is this Kamp axiom set deductively equivalent to the above minimal postulates together with linearity*?

Passing on to higher quantifier complexity, there are two principles concerning noninclusion and nonprecedence which occur in various guises in the literature.

freedom: $\forall xy(x \sqsubseteq y \vee \exists z \sqsubseteq x \neg zOy)$
liberty: $\forall xy(x < y \vee \exists z \sqsubseteq x \exists u \sqsubseteq y \neg zIu)$,
where 'zIu' stands for $\exists z' \sqsubseteq z\ \exists u' \sqsubseteq u\ z' < u'$.

Freedom says that if $x \not\sqsubseteq y$, then x can still be refined to some z disjoint from y. In terms of information periods giving ranges of eventual points in time, this is a reasonable postulate. (It also occurs, e.g., in the area of forcing conditions in set theory.) Liberty expresses something similar for nonprecedence. It may also be read more positively, however, as

$$\forall xy(\forall z \sqsubseteq x \forall u \sqsubseteq y \exists z' \sqsubseteq z \exists u' \sqsubseteq u\ z' < u' \rightarrow x < y),$$

i.e., "cofinal precedence implies total precedence" (cf. [5]).

In the general case, we will stick with the earlier minimal axioms on period structures. The possible uses of freedom and liberty will become apparent later on.

Global intuitions Again, period pictures of time may carry their own more global connotations. Whether the earlier postulate of *homogeneity* still holds is debatable: it is no longer valid in the atomic period structure $INT(\mathbf{Z})$, with its heterogeneous intervals. On the other hand, the multiparameter principle of indistinguishability formulated in Section 2.2 seems to apply to intervals as well.

The period ontology also inspires new global intuitions. For instance, on this more "continuous" view of time, there is a metaphysical tendency to postulate homogeneity in a more vertical direction: every period mirrors the universe! Formally, this becomes a postulate of *reflection*:

every period $i \in I$, when considered as the period structure $\langle\{j \mid j \sqsubseteq i \ \&\ j \neq i\}, <, \sqsubseteq\rangle$, is isomorphic to the whole structure $\langle I, <, \sqsubseteq\rangle$.

Thus, the famous sequence of mirrors arises, ranging from the infinitely great to the infinitely small.

"Direct" effects of reflection are harder to measure than those of homogeneity. Nevertheless, this principle would evidently exclude atomic, indivisible periods.

In some sense, "vertical" reflection and "horizontal" homogeneity seem related. That the two intuitions are still independent is shown by the following period structures.

Example: Reflection does not imply homogeneity. Consider all open subintervals of the rational interval $(-1, +1)$ with the obvious precedence and inclusion. Reflection holds; but, e.g., $(-1, 0)$ cannot be mapped automorphically onto $(0, +1)$.

Example: Homogeneity does not imply reflection. Consider all open subintervals with rational boundaries of the irrational interval $(-\sqrt{2}, +\sqrt{2})$. (Notice that none "reach" the boundaries.) Homogeneity holds; but, e.g., $(-1, +1)$ is not isomorphic to the whole structure, since it can be split up completely into two wings $(-1, 0)$, $(0, +1)$, which the whole interval cannot.

Many matters remain to be investigated here.

Question: Classify the countable linear* period structures satisfying indistinguishability and reflection.

4 Peaceful coexistence When a paradigm has been successful for as long as the point approach, or a competitor as resilient as the period picture, it would be a philosophical perversion to open hostilities. The task of logical analysis is to show how the two ontologies are related, rather than to produce ammunition for controversy. Indeed, there are two broad reasons for being interested precisely in the interplay between points and periods. We want to understand the connection between the "discrete" and the "continuous" point of view in science; but also, we want to see how the former more "scientific" notion relates to the latter more "common sense" view.

4.1 From points to periods Given a strict partial order $\langle T, < \rangle$, the most obvious induced periods are the nonempty *convex* subsets X of T, satisfying the following condition of uninterruptedness:

$$\text{for all } t_1, t_2 \in X, t \in T, t_1 < t < t_2 \text{ only if } t \in X.$$

Singleton sets are convex, while the latter class is closed under the formation of intersections. The finite unions of convex sets (themselves not necessarily convex) form a Boolean Algebra (cf. [8]). Thus, one may either opt for periods as substrata of single events, or, allowing "repetitive events", increase this class in a simple, elegant way.

In general, it need not be assumed that *all* convex sets are substrata of events—whence we will consider *convex interval structures* $\mathcal{T} = \langle \mathbf{T}, <, \mathcal{I} \rangle$ consisting of a point structure $\langle T, < \rangle$ with just some set of convex sets $\mathcal{I}$. The *induced period structure* $\mathsf{P}(\mathcal{T})$ then is the triple $\langle \mathcal{I}, <, \subseteq \rangle$, where $X < Y$ if $\forall t \in X, t' \in Y, t < t'$ (total precedence).

It may be checked that all earlier axioms for period structures are satisfied here. Moreover, the methods of proof in Section 4.2 below will yield as a corollary that these are in fact all valid principles.

Theorem *The minimal period principles axiomatize the complete first-order theory of the period structures induced by convex interval structures.*

Thus again, the above minimal choice turns out to have a stable motivation.

When "full" convex interval structures are considered, containing all possible convex sets, additional validities arise. For instance, freedom and liberty become valid (thanks to the existence of suitable singletons), and so does *atomicity*.

Question: Do these principles combined axiomatize the complete first-order theory of full induced convex period structures?

4.2 ***From periods to points*** When periods form the primary stock of temporal individuals, durationless points may arise as limits to which chains of ever smaller periods converge. Upon some reflection, the appropriate concept is not so much that of a "chain", but of a "funnel". Technically, a *filter* is a set F of periods satisfying the following two conditions:

(i) if $x, y \in F$, then there exists some $z \in F$ with $z \sqsubseteq x$, $z \sqsubseteq y$ (i.e., all pairs in F are compatible) and
(ii) if $y \sqsupseteq x \in F$, then $y \in F$.

One may think of a filter as a "partial point", in some stage of approximation. Now, given any period structure $\mathscr{I} = \langle I, <, \sqsubseteq \rangle$, one defines its *filter representation* $\mathbf{T}(\mathscr{I})$ to be the convex interval structure $\langle T, <, \mathscr{J} \rangle$, with T the set of filters on $\mathscr{I}$, $F_1 < F_2$ iff $\exists x \in F_1\ \exists y \in F_2\ x < y$, and $\mathscr{J}$ equal to $\{\{F \in T \mid x \in F\} \mid x \in I\}$.

If $\mathbf{T}(\mathscr{I})$ really is to be a convex interval structure, two things are to be checked:

1. $\langle T, < \rangle$ is a strict partial order. Transitivity: if $F_1 < F_2 < F_3$, then, say, $x < y_1$, $y_2 < z$ with $x \in F_1, y_1, y_2 \in F_2, z \in F_3$. By the definition of 'filter', some $y \in F_2$ is included in both y_1, y_2. By monotonicity then, $x < y < z$, whence $x < z$, by transitivity. Thus, $F_1 < F_3$. Irreflexivity is proved analogously.

2. $\mathscr{J}$ consists of convex subsets of $\langle T, < \rangle$. If $F_1 < F_2 < F_3$, then as above, $x < y < z$ for some $x \in F_1$, $y \in F_2$, $z \in F_3$. Now, if $u \in F_1$, $u \in F_3$, then we may choose $x \sqsubseteq u$, $z \sqsubseteq u$, again thanks to compatibility for filters. Hence by convexity, $y \sqsubseteq u$; whence $u \in F_2$.

In fact, this procedure gives us a *representation* of period structures as convex interval structures.

Theorem *The function sending $x \in I$ to $\{F \mid x \in F\}$ is an isomorphism between $\mathscr{I}$ and* $\mathbf{PT}(\mathscr{I})$.

A proof of this simple result is found in [8].

One objection to the filter representation is that filters need not be finest, or maximal, approximations to points. (In a thoroughly partial perspective, where new periods may still be discovered extending the filter in unexpected directions, this prudence is rather a bonus, of course.) Thus, we may also consider the *maximal filter representation* $\mathbf{T}^{+}(\mathscr{I})$, in which only maximal filters are employed, which cannot be properly extended to nontrivial filters any longer.

Theorem *The function sending $x \in I$ to $\{F \mid x \in F\}$ is now a homomorphism from $\mathscr{I}$ onto* $\mathbf{PT}^{+}(\mathscr{I})$.

For a proof, compare the above reference.

It is only when freedom and liberty hold in $\mathscr{I}$ that this homomorphism is guaranteed to become an isomorphism. But then again, not all period structures of the form $\mathbf{PT}^+(\mathscr{I})$ satisfy these two constraints.

Question: Find necessary and sufficient axioms for the maximal filter representation to succeed.

Even in this state, both representations are useful and instructive. For instance, the filter theorem has the completeness theorem of Section 4.1 as an immediate corollary.

Two examples of the maximal filter representation may be helpful, involving the two prime examples of Section 3.1.

Example: $\mathbf{T}^+(INT(\mathbf{Z}))$ is isomorphic to $\mathbf{Z}$ with its bounded convex subsets.

Thus, the construction has a fixed point in this case.

Example: $\mathbf{T}^+(INT(\mathbf{Q}))$ is isomorphic to $\mathbf{R}$ with all its rational points replaced by discrete jumps (corresponding to "instantaneous changes").

The relevant arguments are again in [8]. Finally, one ontological worry could be that the present representations produce ever larger structures when iterated. But actually, the above reference contains an assurance that stability is attained quite soon:

$\mathbf{PT}^+(\mathscr{I})$ is isomorphic to $\mathscr{I}$,
$\mathbf{T}^+\mathbf{P}(\mathscr{T})$ need not be isomorphic to $\mathscr{T}$; but
$\mathbf{T}^+\mathbf{PT}^+\mathbf{P}(\mathscr{T})$ is always isomorphic to $\mathbf{T}^+\mathbf{P}(\mathscr{T})$.

4.3 Transfer of information One obvious question now is if certain properties of, or relations between, temporal structures in one ontological realm are preserved among their counterparts in the other ontology. Here are some relevant observations.

Going from points to periods, *first-order equivalence* need not be respected. For instance, $\mathbf{Z}$ and $\mathbf{Z} \oplus \mathbf{Z}$ (i.e., two consecutive copies of $\mathbf{Z}$) have the same first-order theory of precedence, but their full convex interval structures are not elementarily equivalent. That of $\mathbf{Z}$ gives each period with an upper bound a final atom, while that of $\mathbf{Z} \oplus \mathbf{Z}$ does not. Another example of non-transfer concerns a global property, viz., *homogeneity*. As was remarked earlier on, $\mathbf{Z}$ is homogeneous, while $INT(\mathbf{Z})$, or its full convex interval structure is not. But, as we shall see, transfer need not result in *identical* properties, of course.

Going from periods to points, transfer problems have been implicit in the literature. Which conditions on period structures guarantee that its point representation will have certain desired properties? Here is an example of a target correspondence.

$\mathbf{T}^+(\mathscr{I})$ is linear if and only if $\mathscr{I}$ itself is linear*. As for a proof, the direction from right to left follows at once by the previous type of argument. The converse requires validity of the $\mathbf{T}^+$-representation however, say, with the help of freedom and liberty. (The argument is still routine.) In the absence of the latter, the above equivalence may not even be valid, for all we know.

An example for which one may even refute any first-order correspondence is that of discreteness for $\mathbf{T}^+(\mathscr{I})$. For, e.g., $\mathbf{T}^+(INT(\mathbf{Z}))$ was discrete, witness an earlier observation. But now, consider the elementarily equivalent companion of $INT(\mathbf{Z})$ consisting of the convex periods with end points of $\mathbf{Z} \oplus \mathbf{Z}$. The filter generated by the sequence (left 0, right 0), (left +1, right −1), (left +2, right −2), etc., is maximal. But although it has successors and predecessors in the $\mathbf{T}^+$-representation (corresponding to atomic filters in the right and left-hand copies of $\mathbf{Z}$), it has no immediate successor or predecessor, i.e., discreteness fails. Thus, generally, $\mathbf{T}^+$ fails to preserve elementary equivalence of period structures.

Correspondences between first-order direct axioms at the two ontological levels turn out to be scarce—and this is understandable, for, after all, the maximal filter representation transforms (simple) first-order statements about (complex) points (viz., subsets of I) into (complex) higher-order statements about (simple) periods.

What about transfer of global, rather than direct properties? Notably, the maximal filter construction would seem to "average out" individual differences between periods, and hence *homogeneity* of the resulting "anonymous" points seems plausible. But the above already provided a counterexample: $INT(\mathbf{Q})$ is homogeneous, while its maximal filter representation obviously is not. Looking in the other direction, we note that the maximal filter representation of $INT(\mathbf{Z})$ was indeed homogeneous, whereas $INT(\mathbf{Z})$ itself is not. In fact, thinking of *necessary* conditions for homogeneity of $\mathbf{T}^+(\mathcal{I})$, one arrives at weaker requirements such as the following:

> for each $x, y \in I$ there exists some $<, \sqsubseteq$-automorphism of $\langle I, <, \sqsubseteq \rangle$ sending x to some period overlapping y.

Question: Find necessary and sufficient conditions on period structures in order that their maximal filter representation be homogeneous.

In this way, global intuitions on point structures may, indirectly, exert their influence on period structures after all.

4.4 Ontological duality Temporal structures may be related in various ways, and these relations are interesting just because our minimal postulates leave room for quite a variety of point or period structures. In the realm of the latter, one obvious relation is the following: As we learn about more events, our *stock* of underlying periods increases. (Again, this is a logical fable, of course.) Say, we pass from I_1 to I_2. Moreover, our *knowledge* about the original stock may increase; say, because new precedence or inclusion judgments are made. Thus, $\mathcal{I}_2$ is a *positive extension* of $\mathcal{I}_1$ if

$$I_1 \subseteq I_2, <_1 \subseteq <_2 \upharpoonright I_1 \text{ and } \sqsubseteq_1 \subseteq \sqsubseteq_2 \upharpoonright I_1.$$

The earlier representations yield counterparts for such relations in the opposite ontology. For instance, if $\mathcal{I}_2$ is a positive extension of $\mathcal{I}_1$, then the relation E of extension between maximal filters in $\mathbf{T}^+(\mathcal{I}_1)$ and $\mathbf{T}^+(\mathcal{I}_2)$ has the following properties. E has domain I_1, since every maximal filter in $\mathcal{I}_1$ still remains a filter in $\mathcal{I}_2$ ($\sqsubseteq$-relationships remaining valid), and hence can be extended to a maximal filter in $\mathcal{I}_2$. Moreover, E is *homomorphic* in the sense of preserving precedence, the reason now being the continuing validity of $<$-relationships, in combination with the definition of precedence among filters.

Another way to view this situation is to consider the restriction map from I_2 to I_1 as a function from the range of E to its domain. Thus, this function is a *partial* map from $\mathbf{T}^+(\mathscr{I}_2)$ *onto* $\mathbf{T}^+(\mathscr{I}_1)$ which is an *anti-homomorphism* in the sense that precedence of images implies precedence of originals. (Notice that the restrictions need not be an ordinary homomorphism.)

Example: The obvious contraction map from $\mathbf{Q} \odot \mathbf{Z}$ (cf. Section 2.2) to its "macro-structure" $\mathbf{Q}$ is an anti-homomorphism that is not a homomorphism. Indeed, $\mathbf{Q}$ cannot be a homomorphic image of the discrete structure $\mathbf{Q} \odot \mathbf{Z}$ at all, as it fails to verify the positive sentence

$$\exists x \exists y (x < y \wedge \forall z (z < x \vee z = x \vee z = y \vee y < z)).$$

Restriction to I_1 need by no means be a total function. For one thing, the I_1-part of a maximal filter in $\mathscr{I}_2$ need not be maximal in $\mathscr{I}_1$. For another, it need not even be a filter: previously disjoint periods may have received a common subperiod in the positive extension.

Finally, the partial restriction map has the following *continuity* property. Inverse images of distinguished convex sets in $\mathbf{T}^+(\mathscr{I}_1)$ (induced by some period in I_1) are the intersections of distinguished convex sets in $\mathbf{T}^+(\mathscr{I}_2)$ with the domain of the mapping.

Thus, from periods to points, an analogy exists between positive extensions and partial continuous anti-homomorphisms. A check upon its accuracy is provided by changing perspective once again.

Assume that some function of the above kind runs from the point structure $\mathscr{T}_2 = \langle T_2, <_2, \mathscr{A}_2 \rangle$ to $\mathscr{T}_1 = \langle T_1, <_1, \mathscr{A}_1 \rangle$. Then there is a canonical map from the period structure $\mathbf{P}(\mathscr{T}_1)$ to $\mathbf{P}(\mathscr{T}_2)$, sending each $X \in \mathscr{A}_2$ to its inverse image in $\mathscr{A}_2$. (For convenience, T_2 is supposed to be the whole domain of the function here.) It is easily checked that this canonical map preserves inclusion (by the nature of inverse images), as well as precedence (by the anti-homomorphism clause and the definition of total precedence), while it is also one-to-one. Thus, we have retrieved (an embedding into) a positive extension.

Proposition *There exists an exact duality between positive extension among period structures and anti-homomorphic surjective continuous maps among point structures.*

Similar correspondences may be developed for other important relations in the two ontologies, as the need for them arises.

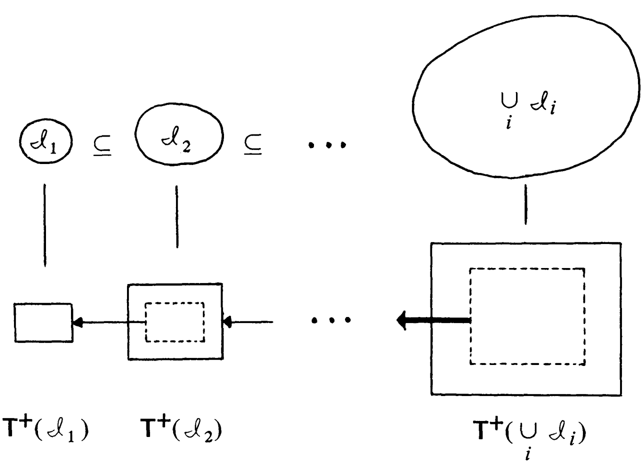
5 Illustrations It is precisely the interplay between the two temporal ontologies which proves useful in applications of the above logical considerations. In line with the dual motivation of the present enterprise, we will consider a more philosophical as well as a more linguistic example, the first derived from [6], the second from [3]. Afterwards, one rather programmatic case will be presented of a more physical nature, inspired by [9].

5.1 From private to public time Bertrand Russell's interest in the connections between common sense notions and those of science led him to consider event or period structures as more typical of the former, reserving point structures for the latter view of time. His reconstruction of the genesis of (scientific) public time was this. There is a growing sequence of private experiences $\mathscr{I}_1$, $\mathscr{I}_2, \ldots$, which are all joined together into some common fund of experience, and the latter is then represented as a point structure, much in the way of our maximal filter representation. (For precise references, and the full story, cf. Thomason's paper [6], which, however, uses a representation by Dedekind Cuts.)

One way to make this more vivid is as follows. $\mathscr{I}_1$ is the period structure of one person's experience. Next comes a second person with her private experience, and the two (if compatible) are joined into the extension $\mathscr{I}_2$. All of the first person's precedence and inclusion judgments are preserved in this way, but the second may have added some of her own to the common part. Thus, $\mathscr{I}_2$ becomes a positive extension of $\mathscr{I}_1$, in the sense of Section 4.4. The process repeats itself, to form a positive extension chain $\mathscr{I}_1 \subseteq \mathscr{I}_2 \subseteq \ldots$. The creation of public time then amounts to the formation of the point representation of the union of this chain: $\mathbf{T}^+(\bigcup \mathscr{I}_i)$.

But there are also private times, obtained by representing finite stages $\mathscr{I}_i$ at once. What is the relation between these private times $\mathbf{T}^+(\mathscr{I}_i)$ and the public time created above? Although Russell himself did not consider this question, it seems a fairly natural one.

The ontological duality of the preceding sections tells us how to formulate an answer. As it turns out, the two roads "private experience, public experience, public time" and "private experiences, private times, public time" are related, but not equivalent. Consider the earlier ascending chain. As it grows, anti-homomorphic connections are established between the corresponding private times, as in Section 4.4. The picture is as follows.



In the time chain, partial anti-homomorphisms run from above to below. Thus, the obvious limit construction upon the time chain consists in considering all "histories". That is, starting with a point at some level, one may trace its various subsequent developments along the anti-homomorphism into the next larger time structure, etc. Thus, points in the limit will be functions t from time structures to points in them, starting from some finite stage, such that every two subsequent stages $t(i + 1)$, $t(i)$ are connected by the relevant morphism. (Such a functional structure is usually known as an *inverse limit.*) Precedence will come out naturally by setting $t < t'$ if, at some stage i (and hence always higher up), $t(i) <_i t'(i)$.

Evidently, each such function t creates a maximal filter in public time. But this construction need not exhaust the latter domain. For there may be maximal filters on $\cup \mathcal{I}_i$ whose restrictions to levels i need not be maximal, or even filters. (The reason is as in an earlier similar observation.) Thus, public time in Russell's sense consists of a core, constituted by successive private times, surrounded by a more global hull of points in time arising from the structure of public experience in its entirety.

5.2 ***Representation of temporal discourse*** Another meeting place of common sense pictures and more scientific world views is contemporary formal semantics. It was especially Hans Kamp who has developed an approach in which the tenses of natural language provide systematic clues for an event or period representation of discourse, only to be connected later on with the usual point structures of standard tense logic. The guiding idea then is that, modulo some technicalities, a simple piece of tensed discourse is actually true if its event representation can be *embedded* into the convex bounded open real intervals.

Just when is a finite period structure thus embeddable? (Discourse representations are always *finite* objects.) This question fits in well with the previous concerns of this paper. (Evidently, it is just one of a whole family of logical questions arising in the discourse representation perspective.)

For a start, if a period structure is embeddable in the prescribed sense, then all minimal period postulates of Section 3 will be valid, universal sentences being preserved under transition to submodels. There are additional validities, however. Already for pure inclusion, there is now "one-dimensionality" (or "planarity"):

$$\forall xyz(B_x yz \vee B_y xz \vee B_z xy) \tag{1}$$

where '$B_x yz$' ("betweenness") stands for '$\forall u((y \sqsubseteq u \;\&\; z \sqsubseteq u) \to x \sqsubseteq u)$'. A pure precedence addition is the related "comparability":

$$\begin{gathered}\forall xy(\forall z(y < z \to x < z) \vee \forall z(x < z \to y < z))\\ \forall xy(\forall z(z < y \to z < x) \vee \forall z(z < x \to z < y)).\end{gathered} \tag{2}$$

Finally, a new mixed universal postulate occurs, akin to the connectedness of Section 2:

$$\forall xyz(x < y < z \to \forall u(x < u \vee u < z \vee y \sqsubseteq u)). \tag{3}$$

Notice that convexity already follows from this principle.

Conjecture: Validity of the minimal period postulates in conjunction with (1), (2), (3), is a necessary and sufficient condition for a temporal period structure to be embeddable into the Kamp real interval structure.

Actually, a proof of the pure precedence part of this result can easily be given, but it has been omitted here because of its combinatorial complexity.

All axioms occurring in the above were universal ones, and this is no accident.

Observation: A finite period structure can be embedded into the convex open bounded interval structure of the reals iff it verifies the latter's *universal* first-order theory.

Thus, we are asking in effect for a completeness theorem; and we realize that old themes from philosophical logic may arise in perfectly natural ways in new semantic settings.

5.3 ***Space-time from events*** After these more philosophical and linguistic examples, one would also expect some illumination as to the foundations of physics. Actually, this direction of research has not been developed yet within the present perspective. Hence we can only offer a suggestion illustrating which kind of problem could possibly benefit from the theory advanced in this paper.

In [9], there is a description of Leibniz' project for constructing both time and space out of a primary spatio-temporal event structure, provided with a primitive relation of possible causal precedence. Briefly, the idea is this: Precedence is a connected strict partial order (a form of connectedness is referred to as "Leibniz' Postulate"), representable as a linear sequence ("time") of simultaneity classes ("spaces"), just as in Section 2 above. Winnie points out that inside these spaces all point sequences satisfy the same precedence statements, and hence no structure is left to define a nontrivial geometry over them. Thus, Leibniz' project fails for classical Newtonian time. Only when different principles are adopted for the primary causal structure, so the story continues, say along the lines of Robb's "Causal Theory" of space-time, a construction can indeed be found producing the familiar Minkowski space-time of special relativity.

Now, the period perspective might provide a way out for Leibniz' project after all, for the earlier period structures can be reinterpreted as event structures, with *spatio-temporal* inclusion, without any difficulty. Even the postulates considered above would remain equally plausible. Thus, Leibniz' project could start from a period, rather than a point structure, utilizing its double precedence/inclusion structure to escape from Winnie's refutation.

Question: Extract a viable chronometry and geometry from period structures, by producing suitable definitions for *simultaneity, betweenness*, and *equidistance*.

This task will not exactly be easy. For instance, the notion of betweenness from Section 5.2 (provably) produces only the following geometrical principles:

$$\forall xyz(B_xyz \rightarrow B_xzy)$$
$$\forall xy\ B_xxy$$
$$\forall xyzuv((B_xyz \ \&\ B_yuv \ \&\ B_zuv) \rightarrow B_xuv),$$

which is nowhere near even the minimal logic of this notion.

But then, perhaps the above approach is not quite the correct point of view. Classical (or Minkowski) space-time is a mathematical construct at the point, rather than the period level. Thus, it would be sufficient to have only punctual representations $\mathsf{T}^+(\mathscr{I})$ come out similar to these space-times. Whether this leaves any advantages over Winnie's direct approach in terms of points and precedence only, remains to be seen.

Conclusion This paper has aimed to show how logic can contribute in exploring the variety of temporal ontology. In the process, many types of logical questions emerged, some advanced, some elementary, but most of them outside the scope of traditional tense logic. The guiding philosophical interest has been, not in a clash of temporal paradigms, but in a study of their interrelations. But there is also a more ambitious philosophical goal in the background. Contemporary philosophical logic has two main faces: one directed toward the philosophy of language, the other toward the philosophy of science. The subject of time occurs in both these traditions, once as tense logic (cf. [4]), then again as the philosophy of time (cf. [9]). This unfortunate separation has not been taken for granted here; both interests appeared in this paper. In the end, one would hope, the two traditions will be joined in one integrated logic of time.

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