Notre Dame Journal of Formal Logic
Volume 24, Number 4, October 1983

The Fundamental S-Theorem—A Corollary

ROBERT K. MEYER, ERROL P. MARTIN,
and ROBERT DWYER*

For reasons set out in [4] and [5], it is of immense significance that the system S (for *syllogism*) satisfies the following condition:[1]

Powers property *For every formula A of S, $A \rightarrow A$ is unprovable in S.*

It is of equal and indeed equivalent significance that the system $P\text{-}W$ (for *pure* ticket implication *minus waffly* axioms) satisfies the following condition:

Belnap property *For every pair A,B of distinct formulas of $P\text{-}W$, at least one of $A \rightarrow B$, $B \rightarrow A$ is a nontheorem of $P\text{-}W$.*

Put contrapositively, what the Belnap property says is that if A and B are *provably equivalent* in $P\text{-}W$, then A is the very same well-formed formula as B.

That S and $P\text{-}W$ had their corresponding properties was a long-standing—and recalcitrant—conjecture in the area of relevant logic. Building on work of Belnap, Powers, Dwyer, and Meyer, Martin eventually found a (surprisingly difficult) proof, which Martin and Meyer recount in [4]. As for the significance

*This paper grew out of joint work by Martin and Meyer. Dwyer, with whom we have lost touch, may be surprised to find himself an author. Accordingly, he is not to be held responsible for any of our editorial comments—or, for that matter, for the details of the arguments set out. But key steps in those arguments do rest upon his work. In addition to those already mentioned, we are also indebted, in particular, to P. Thistlewaite, in the course of conversation with whom the truth of the Main Lemma became apparent, and to S. Giambrone, for help in preparing the manuscript.

Received August 13, 1982; revised December 3, 1982

of all this, there is no need to repeat here what we have already said in [4] and [5]. In a nutshell, S supplies sound implicational principles of inference, without *ever* lapsing into the illogical practice of assuming what is to be proved and then deriving it by $A \rightarrow A$. So you may take it for granted that S will prove to be your ticket.

What we shall do here is derive a corollary to the fundamental theorems for *P-W* and S. In fact, the corollary first occurred to us as a *way to prove* these theorems, since if established independently it would suffice to show that S has the Powers property, and *P-W* the Belnap one. (Accordingly, if the reader can find an independent proof, he may find a quick shortcut to the result of [4].)

Now, the shape of the Powers property together with our previous remark on begging the question suggest that formulas of the form $A \rightarrow A$ will be of particular interest for S and *P-W*. Let us permit ourselves some abbreviatory notation, letting the *diagonalization* operator Δ be introduced by contextual definition thus:

DΔ $\Delta A =_{df} A \rightarrow A$.

The following property will now be of interest:

Subformula property (*For P-W*) $\Delta A \rightarrow \Delta B$ *is a theorem of P-W iff A is a subformula of B.* (*For S*) $\Delta A \rightarrow \Delta B$ *is a theorem of S iff A is a proper subformula of B.*

At first glance, the subformula property seems somewhat strange. Why should a *syntactical* property—the subformula *relation*—determine when one identity entails another? When looked at again, however, it seems only natural, for the essence of a vertebrate theory of deduction is that one must do some work to actually derive one formula from another. And if one looks at the sort of work that is involved if one sticks to valid syllogistic modes of reasoning—which, on the central topic of *implication* in logic, amount *solely* to various forms of the transitivity of $\rightarrow$—one sees that the *natural* way for ΔA to imply ΔB is *via* the subformula route; e.g., $A \rightarrow A \rightarrow . A \rightarrow B \rightarrow . A \rightarrow B$, as an instance of a transitivity *axiom*. (For ease in reading formulas, we employ the conventions of [1]; i.e., association to the left, and (sparing) use of dots for parentheses.) Similarly, $(A \rightarrow B \rightarrow . A \rightarrow B) \rightarrow . (C \rightarrow . A \rightarrow B) \rightarrow . C \rightarrow . A \rightarrow B$, as an instance of another transitivity axiom. Accordingly, applying the transitivity *rule* to these two axioms, it is sensible that ΔA should entail $\Delta(C \rightarrow . A \rightarrow B)$ in accordance with our subformula principle. What will take a little more work is the demonstration that the condition of the two forms of this principle is *necessary* as well as sufficient; i.e., that in the syllogistic systems of implication there are no *unnatural* ways to derive one identity from another. Accordingly, a demonstration that the subformula property does hold for identities in *P-W* and in S, together with some of the consequences that flow therefrom, will be the main point of this note.

1 We have got a little ahead of ourselves, since the reader without [4] at his elbow might wish to know which systems *P-W* and S *are*. For the moment, we view them as pure implicational logics. That is, the language L assumed here is one in which formulas A, B, C, etc., are built up from sentential variables p, q, r, etc., and a binary $\rightarrow$ connective. (Although we agree with [1] that its theory of implication is the *heart* of logic, we shall show in subsequent work how truth-functional tonsils, a quantificational appendix, and other familiar organs can be added to S. The theory of negation promises to be especially interesting since either a relevant negation like that of E and R or a connexive negation in the sense of Angell [2] and McCall [3] is compatible with the implicational insights of S.) S is axiomatized with the following two schemes (for which we use the combinatorial names associated therewith by Curry and Feys).

B $B\rightarrow C\rightarrow.A\rightarrow B\rightarrow.A\rightarrow C$ (prefixing)
B′ $A\rightarrow B\rightarrow.B\rightarrow C\rightarrow.A\rightarrow C$ (suffixing)

P-W has all the B and B′ axioms, together with the following axiom scheme.

I $A\rightarrow A$ (identity)

In [1], where Belnap's conjecture is set forth (and where, incidentally, *P-W* is confusingly renamed $T_{\rightarrow}$-W, *P-W* is supplied with the following (modus ponens) rule, which may also do duty for S.

→E. $A\rightarrow B\Rightarrow.A\Rightarrow B.$

The discerning reader, as we pointed out in [5], will already have seen what is wrong with the →E rule. Namely, on the *fruitful* confusion of logical → and metalogical ⇒, which we have all learned from Belnap and which is of enormous and continuing appeal to all left-thinking logicians, →E is itself of the I form, which is at least dicey from the *P-W* viewpoint, and, in view of the Powers property, downright naughty for S. Accordingly, left-thinking people will beam with delight to learn that →E may be replaced with the following rules, due to Dwyer, producing the same stock of theorems for either *P-W* or S.

Rule B. $B\rightarrow C\Rightarrow A\rightarrow B\rightarrow.A\rightarrow C$ (prefixing rule)
Rule B′. $A\rightarrow B\Rightarrow B\rightarrow C\rightarrow.A\rightarrow C$ (suffixing rule)
Rule BB. $B\rightarrow C\Rightarrow.A\rightarrow B\Rightarrow A\rightarrow C$ (transitivity rule)

So put, all the rules of *P-W* and S accord with first principles of S. (Of course we might also add, for *P-W*, an I rule—from A, infer A—though that would be redundant.) See [4] for further remarks about the Dwyer rules, which henceforth we take to be the *primitive* rules of our two systems. (But note, as further payoff, that these rules, unlike →E, *always* treat implications as *essentially relational*.)

We now make a confession. Although we have presented *P-W* and S as two *distinct* systems, separated by the I axiom scheme, it is perhaps better to think of them as distinct *aspects* of the *same* system. We can make this point for readers of [4] in terms of the three-valued "worlds" semantics for *P-W* and S presented there. For note that in this semantics the rules for the evaluation of the truth-value of a formula at a "world" are *exactly the same* whether one thinks of the *arrows* in the formula as *P-W* arrows or as S arrows. What differ, rather, are the conditions on which an entailment shall be *valid* from the *P-W* and from the *S* viewpoints; specifically, there is a notion of *weak* validity, which verifies exactly the *P-W* entailments, and of *strong* validity, which does the same for the *S* theorems. From a syntactical viewpoint, we might think accordingly of an enclosing supersystem (for which we shall also reserve the name '*S*'), with two different assertion signs. Although we won't undertake that project here, we shall indulge ourselves in the following notational convention (given that all the things that we wish to assert here as theorems are in fact arrow-statements).

D< $A < B$ says that $A \to B$ is a theorem of S
D≤ $A \leqslant B$ says that $A \to B$ is a theorem of *P-W*.

Martin's theorem (foreshadowed in part by other authors cited) now says the following:

Fundamental theorem (for S and *P-W*) *Let A and B be any formulas of our implicational language. Then*

(1) *$A < B$ iff both $A \leqslant B$ and $A \neq B$*
(2) *$A \leqslant B$ iff either $A < B$ or $A = B$.*

$\neq$ and $=$ in (1) and (2) mean, of course, that the formulas in question are respectively distinct or identical as formulas. Note, too, that the Powers property for S is explicit in (1). For the Belnap property, suppose that $A \leqslant B$ and $B \leqslant A$. If A and B are distinct, then moreover $A < B$ and $B < A$, whence, by the transitivity rule for S, $A < A$, which is impossible by (1). So A *is* B, whence *P-W* has the Belnap property after all. (Among other things, note that this means that the *or* in (2) may be taken in the *exclusive* sense. So $<$, as introduced by D<, really means *less than*; and $\leqslant$ really means *less than* or *equal to*.)

What's the big deal, one might wonder, about a logical $<$ satisfying (1)? One could always cook one up from any entailment relation in the $\leqslant$ sense, just by taking (1) as a definition. In response, what must be said is that if one tries that tack with any of the entailment relations ordinarily offered for logical consumption, one does not come away with a $<$ that interacts fruitfully with the $\to$ *connectives* which stand in for implication in the *systematic* sense. That is, modern symbolic logicians have been wedded for so long to a logical principle that goes nowhere (the "archetypal" form of inference, as [1] calls $A \to A$), that they lack theoretically significant ways of doing without it. To be sure, one might always propose a very weak pair of logics, so bereft of deductive resources that (1) and (2) hold trivially. In this, however, we take solace from the fact that the fundamental theorem was a *hard* theorem, while nonetheless arising from the (pure implicational) residue of the Aristotelian *dictum de omni*. So S is responsive to the valid core of traditional logical insight, while it is *not* responsive to the modern palaver which turns the First Fallacy into a First Principle.

All of this takes us into philosophical questions. To tip a bit more of our own hand, we think that the truly valid arguments are arguments that *go somewhere*, producing, as Aristotle said, conclusions *other* than what was explicit in their premisses. Another view of logic—namely, that it consists of a collection of tautologies, which are literally truths conveying no information—has long been prevalent. Too long. For on this static view, logical truths are not like Aristotle's syllogisms (which he saw as *tools*), but like his God—just sitting around uselessly, waiting to be admired.

Our view that S may indeed have revolutionary significance for the philosophy of logic prompts a brief look behind at where it came from, with which we close this section. The basic claim, certainly, is that *pure transitivity principles* are at the heart of the Heart of Logic. In that sense, S came from Aristotle. More recently, the general project of picking subsystems of the classical propositional calculus, investigating their properties, and making various motivational claims for them, has long been with us; e.g., the work of people like C. A. Meredith and H. B. Curry certainly applies to S and *P-W*. But it was Anderson and Belnap, in their search for a minimal relevant logic, who spotted the central fruitful conjecture; moreover, it is the historical and traditional concerns to which these authors drew attention that seem to us the best reasons for attending to S. Also historically motivated, in a more directly Aristotelian way, was Angell's [2], whose pure implicational insights seem directly to have been the *P-W* ones. But the absolutely crucial insights, for both technical and philosophical reasons, grew out of the work of Powers summarized in [6]. In viewing the Belnap conjecture for *P-W* simply as "a logic problem," Powers saw that the key to the solution of this problem lay in the absolute separation of the I axiom from the other axioms of *P-W*. Philosophically, once Powers had suggested that the I axiom was *unnecessary* to get anything interesting out of *P-W*, many reasons occurred to us (as they will to the reader) why it was positively *undesirable* to have this axiom at all. For if the *sole* business of an axiom scheme is to churn out its own instances, none of which are of the slightest assistance in conducting any crucial arguments, one must certainly wonder whether that scheme has any sensible place in logic at all. As we do.

2 In this section, we return to our announced task—namely, showing that the subformula property holds for the (identity) formulas ΔA. As our initial remarks indicated, this was a failed lemma in one of our unsuccessful attacks on the fundamental theorem. So, forgetting that the fundamental theorem has been proved, let us show how to derive it from the subformula property. Leaving out a step that was covered in [6], and again in [5], it will suffice for this purpose to show that the subformula property implies both of the Powers and Belnap properties.

Observation 1 *Suppose that the subformula property holds for S. Then, for all A, $A \rightarrow A$ is unprovable in S (i.e., S has the Powers property).*

Proof: Trivial. For suppose that $A < A$. Then, by application of the prefixing rule and the definition DΔ, $\Delta A < \Delta A$. But A is not a proper subformula of itself, violating the subformula property for S.

Observation 2 *Suppose that the subformula property holds for P-W. Then, for all A,B, if $A \leqslant B$ and $B \leqslant A$, then $A = B$ (i.e., P-W satisfies Belnap).*

Proof: Assume that both $A \rightarrow B$ and $B \rightarrow A$ are theorems of *P-W*. Then, by the prefixing rule, $\Delta A \leqslant A \rightarrow B$ from the former, and, by suffixing, $A \rightarrow B \leqslant \Delta B$

from the latter. So, by transitivity, $\Delta A \leqslant \Delta B$. On the obvious symmetry, $\Delta B \leqslant \Delta A$ as well. By the subformula property for *P-W*, each of A,B is a subformula of the other. So A *is* B, ending the proof.

Note that the arguments for Observations 1 and 2 are independent, and that each depends on only the form of the subformula property appropriate to the system in question. So each of the central conjectures could have been established from the matching subformula property, while again we refer the reader to [5] and [6] to finish off the proof of the fundamental theorem (from either).

Let us now attack the subformula property directly (relying on [4] for the fundamental theorem). We begin with some observations that are now trivial.

Observation 3 *If the subformula property holds for S, then it holds for P-W.*

Proof: Let A and B be formulas of *P-W*. We may assume that $\Delta A < \Delta B$ iff A is a *proper* subformula of B. We must show that $\Delta A \leqslant \Delta B$ iff A is a subformula of B. From right to left this is true by the I axiom of *P-W* if $A = B$, and is otherwise true on the assumption. Conversely, if $\Delta A \leqslant \Delta B$, then by (2) of the fundamental theorem, either $\Delta A < \Delta B$, in which case A is a subformula of B on assumption; or ΔA *is* ΔB, in which case A *is* B, ending the proof of the observation.

Observation 4 *Suppose that A is a proper subformula of B. Then* $\Delta A < \Delta B$.

Proof: By a straightforward induction on the depth of nesting of A as a constituent of B, using the prefixing, suffixing, and transitivity axioms as suggested in initial illustrative remarks. Enough said.

We have now reduced our problem with the subformula property to (what promised all along to be) the hard case–namely, showing that if $\Delta A < \Delta B$, then A is indeed a proper subformula of B. To do this, we think of proofs as arranged in tree form, with axioms at the tips. And we define the *rank* of an S-proof, as the length of its longest branch. By the *rank* of a Theorem A of S, we mean the *least* rank of all the S-proofs of A. As an example, here is a proof of our *favourite* formula $(p\to .q\to r)\to .s\to q\to .p\to .s\to r$ (which we found in Prior's [8]):

$$\text{(BB rule)}\ \dfrac{\text{(B}'\text{ rule)}\ \dfrac{s\to q < q\to r\to .s\to r \qquad \text{(B}'\text{ axiom)}}{(q\to r\to .s\to r)\to .p\to .s\to r < s\to q\to .p\to .s\to r} \quad p\to .q\to r < (q\to r\to .s\to r)\to .p\to .s\to r}{p\to .q\to r < s\to q\to .p\to .s\to r}\ \text{(B}'\text{ axiom)}$$

Evidently we are allowed any order of application of B, B′, and BB rules. In fact, we may always apply the (two-premiss) BB rules *last*; i.e., let us call a proof (tree) *normal* provided that, on all branches of that tree, no application of a BB rule precedes any application of a B rule or a B′ rule. (Note that our illustrative proof is in fact normal.) Although it is not needed for the task at hand, the following lemma is interesting enough to deserve mention.

Normal form lemma for S-proofs *Every theorem has a normal proof.*

Proof: Show by (strong) induction on n that each theorem A of S of rank n has a normal proof of rank n. The reader who takes the time may find the details of the proof rather pretty.

We now get on with the hard case.

Main lemma *Suppose that $\Delta A < \Delta B$. Then A is a proper subformula of B.*

Proof: By induction on the rank n of the S-theorem in question. If $n = 1$, then $A \rightarrow A \rightarrow . B \rightarrow B$ is either a prefixing or a suffixing axiom. If it is a B axiom, then the formula B must be of the form $C \rightarrow A$; if a B′ axiom, then it is of the form $A \rightarrow C$. In either case, the main lemma holds.

Suppose now that our formula has rank $n > 1$. Suppose that it came by the B rule; i.e., from $A < B$, by prefixing C to both sides. But then $C = A$ and $C = B$, whence A is the same formula as B. But by the fundamental theorem, nothing of the form $A \rightarrow A$, including $\Delta A \rightarrow \Delta A$, is a theorem of S. So our theorem did not come by the B rule. For similar reasons, which the reader may verify, it did not come by the B′ rule either. Accordingly, since $n > 1$, it must have come by the transitivity rule BB. Schematically the situation is

$$\begin{array}{ll} \text{Rank} \leqslant n-1 & \dfrac{\Delta A < M \qquad M < \Delta B}{\Delta A < \Delta B} \text{ BB rule} \\ \text{Rank} = n & \end{array}$$

We now ask, "What is the form of the formula M?" The question seems odd, but let us observe that M is at least a theorem of P-W. (For ΔA is, as an identity, as is $\Delta A \rightarrow M$, evidently; whence, since the class of P-W theorems is closed under $\rightarrow$E, as noted above, M is a P-W theorem.) Is M also an S-theorem? No, because the class of S-theorems is also closed under $\rightarrow$E, while by the Powers property the identity ΔB *cannot* be a theorem of S. But there is only one sort of formula, by the fundamental theorem, which is a P-W theorem without being an S-theorem; namely, an identity ΔC. But then, on our inductive hypothesis, A is a proper subformula of C, and C is a proper subformula of B. By transitivity of the proper subformula relation, A is a proper subformula of B. This completes the inductive argument, and the proof of the main lemma.[2]

Our main business is now concluded.

Corollary to the fundamental theorem *$A \rightarrow A \rightarrow . B \rightarrow B$ is a theorem of P-W iff A is a subformula of B; it is moreover a theorem of S iff A is a proper subformula of B.*

Proof: By Observations 3 and 4, and the main lemma. End of Proof.

NOTES

1. An axiomatic formulation of S (and of P-W) will be found in a page or two.

2. The air of simplicity in this argument is illusory. For, while the list of Equivalents to the Principle of Belnap is growing, the hard part is to get an easy proof of any of them. Still, that there is a well-defined subset of S on which entailment mirrors the *proper part* relation remains fascinating; it would be fun to find some deeper reason why this is the case. And please do not chide us, dear reader, for having noted the equivalence of the Powers, Belnap, and Subformula Principles, when, if we are to take these principles with utmost seriousness, each of them is truly equivalent only to itself. For, when we are arguing *about* logic, we have a *topic*, just as when we seek to set out the truths appropriate to any subject. That is, metalogic is *not* logic, but a *theory about* logic; and we have no objections, in a special theory, to certain propositions being *metalogically* equivalent, any more than we would have objections to pairs of propositions being physically equivalent, or mathematically equivalent, in that attenuated sense of equivalence which special assumptions bring to the disciplines that are ruled by them.

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R. K. Meyer
Department of Philosophy
The Research School of Social Sciences
Australian National University
P.O. Box 4
Canberra ACT 2600

E. P. Martin
University of Queensland
and
University of Melbourne