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Submodels and Definable Points in Models of Peano Arithmetic

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1 Introduction In this paper we consider some definable sets and elements in countable nonstandard models of Peano arithmetic (abbreviated by P). Definable elements and their properties were considered by Jensen and Ehrenfeucht [5] and McAloon [7]. We investigate other properties of these points, and relate them to intersections of submodels of countable nonstandard models of formal arithmetic. When in this paper we speak of nonstandard models of Peano arithmetic we assume that they are countable.

We now introduce some terminology and notation. By L_P we denote the language of P. By $\underline{M}$, $\underline{N}$, ... we denote models of L_P or simple expansions of this language, and by M, N, ... we denote their domains respectively. The $\underline{\omega}$ stands for the standard system of natural numbers. We shall abbreviate $a_0, \ldots, a_n \in M$ by $\vec{a} \in M$. If $a \in M$ then $\underline{a}$ denotes the name of a.

As usual, by $M \subseteq_e N$ ($M \prec_e N$, $M \subseteq_c N$, $M \prec_c N$) we denote respectively that N is an end extension (elementary end extension, cofinal extension, elementary cofinal extension) of M.

Let Γ be a set of formulas of a language L, and let $\underline{A}$, $\underline{B}$ be some models of L. A formula ϕ of L is a Γ-formula if $\phi \in \Gamma$. Assume $\underline{A} \subseteq \underline{B}$. Then $\underline{A} \subseteq_\Gamma \underline{B}$ iff for all Γ-formulas ϕ of L and all $\vec{a} \in A$, $\underline{A} \models \phi\vec{a}$ implies $\underline{B} \models \phi\vec{a}$. We write $\underline{A} \prec_\Gamma \underline{B}$ if "implies" is replaced by "iff" above. An element $a \in A$ is a Γ-element (in $\underline{A}$) iff a is defined by a Γ-formula in $\underline{A}$. In the case of P this is equivalent to $\underline{M} \models a = \mu x \phi x$, $\phi x \in \Gamma$. The set $T \cap \Gamma$ is sometimes denoted by T_Γ.

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*I presented some of my early results at the Logic Conference in Marseille, 1981 (Corollary 2.9.1). There I had a short but inspiring discussion on these matters with D. Marker, who informed me of a generalization belonging to him and A. Wilke (Corollary 2.7.2).

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If $\Pi(x,\vec{y}) \subseteq \Gamma$, $\vec{a} \in A$ and $\Pi(x,\vec{a})$ is finitely consistent over $\underline{A}$, then we call $\Pi(x,\vec{a})$ a Γ-type. In most cases Γ will be one of these sets: Σ_k^0, Π_k^0, Δ_k^0. We note the following facts concerning these sets:

Proposition 1.1 *Let $\underline{A}$, $\underline{B}$ be models of P. The following are equivalent*: (1) $\underline{A} \subseteq_{\Sigma_{k+1}^0} \underline{B}$; (2) $\underline{A} <_{\Sigma_k^0} \underline{B}$; (3) $\underline{A} <_{\Delta_{k+1}^0} \underline{B}$; (4) $\underline{A} \subseteq_{\Delta_{k+1}^0} \underline{B}$.

If $\underline{M}$ is a model of P and $S \subseteq M$, then S defines the least segment of $\underline{M}$ which contains S; this is $J_M(S) = \{x \in M\colon (\exists y \in S)\ x \leqslant y\}$. We omit the subscript M if there is no ambiguity. We recall the fundamental theorem on cofinal extensions.

Gaifman's Splitting Theorem *If $\underline{M}$, $\underline{N}$ are models of P, and $\underline{M} \subseteq \underline{N}$, then $\underline{M} <_c J_N(M) \subseteq_e \underline{N}$.*

We have the following hierarchical refinement of Gaifman's Theorem:

Theorem 1.2 *Let $\underline{M}$, $\underline{N}$ be models of P, $\underline{M} <_c \underline{K} \subseteq_e \underline{N}$, and $\underline{M} <_{\Sigma_k^0} \underline{N}$. Then $\underline{K} <_{\Sigma_k^0} \underline{N}$.*

Proof: Let $\phi\vec{x}y$ be Σ_k^0, $\vec{a} \in K$, and assume $\underline{N} \models \exists y\phi\underline{\vec{a}}y$. We can choose $m \in M$ such that $\vec{a} < m$. By the Replacement Scheme in P there is a $b \in M$ such that

$$\underline{M} \models \forall\vec{x} < \underline{m}(\exists y\phi\vec{x}y \to \exists y < \underline{b}\,\phi\vec{x}y).$$

It is easily seen that $\forall\vec{x} < \underline{m}(\exists y\phi\vec{x}y \to \exists y < \underline{b}\phi\vec{x}y)$ is Δ_{k+1}^0, and as $\underline{M} <_{\Delta_{k+1}^0} \underline{N}$ we have

$$\underline{N} \models \forall\vec{x} < \underline{m}(\exists y\phi\vec{x}y \to \exists y < \underline{b}\,\phi\vec{x}y).$$

Therefore, $\underline{N} \models \exists y < \underline{b}\phi\underline{\vec{a}}y$, i.e., there is a $c \in K$ such that $\underline{N} \models \phi\underline{\vec{a}}\underline{c}$. By the Σ_k^0-version of the Tarski-Vaught Theorem we obtain $\underline{K} <_{\Sigma_k^0} \underline{N}$.

An extended version of A. Robinson's Overspill Lemma will be used throughout. This property might be considered as a partial saturation of nonstandard models of P (cf. [12], [8]).

Theorem 1.3 *For every $k \in \omega$, every nonstandard model $\underline{M}$ of P realizes every recursive Σ_k^0-type over $\underline{M}$.*

One of the main embeddability criteria is given by H. Friedman's Theorem. As usual, $SSy(\underline{M})$ denotes the standard system of $\underline{M}$, i.e., the collection of all sets of the form $\{x \in \omega\colon \underline{M} \models \phi x\vec{b}\}$ for $\vec{b} \in M$ and L_P formulas ϕ.

Friedman's Embeddability Theorem *Let $\underline{M}$, $\underline{N}$ be countable models of P. Then*:

(a) *$\underline{N}$ is embeddable into $\underline{M}$ iff $Th_\exists(\underline{M}) \subseteq Th_\exists(\underline{N})$ and $SSy(\underline{M}) \subseteq SSy(\underline{N})$.*
(b) *$\underline{N}$ is isomorphic to an initial segment of $\underline{M}$ iff $Th_{\Sigma_1^0}(\underline{N}) \subseteq Th_{\Sigma_1^0}(\underline{M})$ and $SSy(\underline{M}) = SSy(\underline{N})$.*

We observe that by a hierarchical refinement (cf. [12], p. 268) we obtain Σ_k^0 elementary embeddings in the theorem if Σ_1^0 and $\exists$ are replaced by Σ_k^0.

2 Definable elements In this section we shall consider definable elements in nonstandard models of P, and relate them to intersections of submodels (of a model $\underline{M}$ of P). This enables us to characterize those models of P which are Σ_k^0 elementary extensions of $\underline{\omega}$.

Lemma 2.1 *Let $\underline{M}$ be a model of P. If ϕx is Σ_{k+1}^0 and $\underline{M} \models \exists x \phi x$, then there is a Δ_{k+1}^0 element $d \in M$ such that $\underline{M} \models \phi \underline{d}$.*

Proof: Let $\phi x \doteq \exists y \psi xy$, ψxy is Π_k^0, and $\theta z \doteq \psi((z)_0, (z)_1)$. Then θz is Π_k^0, since

$$P \vdash \theta z \leftrightarrow (\exists u, v \leqslant z)(u = (z)_0 \wedge v = (z)_1 \wedge \psi uv).$$

As $\underline{M} \models \exists x \phi x$, we have $\underline{M} \models \exists z \theta z$, so let $b \in M$ be such that $\underline{M} \models b = \mu z \theta z$. Then b is Π_k^0, and $d = (b)_0$ is Δ_{k+1}^0, since

$$\alpha x \doteq \forall y (y = \mu z \psi((z)_0, (z)_1) \rightarrow x = (y)_0)$$
$$\beta x \doteq \exists y (y = \mu z \psi((z)_0, (z)_1) \wedge x = (y)_0)$$

define d, and $\alpha x, \beta x$ are Π_{k+1}^0, Σ_{k+1}^0, respectively. Obviously $P \vdash \alpha x \leftrightarrow \beta x$. Then $\underline{M} \models \phi \underline{d}$, since $\underline{M} \models \exists y \psi \underline{d} y$.

Definition 2.2 $\Delta_k^M = \{x \in M: x \text{ is } \Delta_k^0 \text{ definable in } \underline{M}\}$.

In general, for any set of formulas Γ, Γ^M denotes the set of all Γ-definable elements in $\underline{M}$. Some of these numbers were considered in [5] and [7]. The following property of Δ_1^M elements is established in [5]: The code set of $Th(\underline{\omega}) \cap \Sigma_1^0$ belongs to $SSy(\underline{M})$ iff Δ_1^M is bounded below in $M - \omega$. We note that the code set S of $Th(\underline{\omega}) \cap \Sigma_1^0$ belongs to $SSy(\underline{M})$ iff every recursively enumerable subset of ω belongs to $SSy(\underline{M})$. First, if $S \in SSy(\underline{M})$, then for any recursively enumerable $A \subseteq \omega$ there is a Σ_1^0 formula ϕx such that $A = \{m \in \omega: \underline{\omega} \models \phi \underline{m}\}$, hence $A = \{m \in \omega: \ulcorner \phi \underline{m} \urcorner \in S\}$. As $SSy(\underline{M})$ is closed under relative recursion (cf. [10]), it follows that $A \in SSy(\underline{M})$. Further, if every recursively enumerable subset of ω belongs to $SSy(\underline{M})$, then $S \in SSy(\underline{M})$ since S is itself recursively enumerable. Therefore, we have the following corollary:

Corollary 2.2.1 *Every recursively enumerable subset of ω belongs to $SSy(\underline{M})$ iff $\Delta_1^M - \omega$ is bounded below.*

By Lemma 2.1 we have also

Corollary 2.2.2 *Δ_k^M is cofinal in the set of all Σ_k^0 elements of $\underline{M}$.*

We shall need the following lemma to describe intersections of some submodels of $\underline{M}$.

Lemma 2.3 *Let $\underline{M}$ be a model of P and $\lambda \in M$. If λ is not Δ_{k+1}^0-definable in $\underline{M}$, then there is a sequence $b_0, b_1, \ldots$ such that*

(a) *For each $n \in \omega$, λ is not Δ_{k+1}^0-definable in $\underline{M}$ with parameters in $\{b_0, \ldots, b_n\}$.*
(b) *For all Σ_{k+1}^0 formulas $\phi x_0 \ldots x_n$, $\underline{M} \models \phi \underline{a}_0 \ldots \underline{a}_n \rightarrow \phi \underline{b}_0 \ldots \underline{b}_n$, where $a_0, a_1, \ldots$ is an enumeration of the domain M, $a_0 = 0$.*

Proof: We prove (a) and (b) by induction on n. Define $b_0 = a_0$, and assume $b_0, \ldots, b_{n-1}$ have been constructed. Let

$$\begin{aligned}\Gamma w = \{&\phi \underline{a}_0 \ldots \underline{a}_n \to \phi \underline{b}_0 \ldots \underline{b}_{n-1} w \colon \phi x_0 \ldots x_n \text{ is } \Sigma^0_{k+1}\} \\ \cup\ \{&\lambda \neq \mu x \psi x \underline{b}_0 \ldots \underline{b}_{n-1} w \colon \psi \vec{x} y \text{ is a } \Delta^0_{k+1} \text{ formula of } L_P\}.\end{aligned}$$

We prove that Γw is a type over $\underline{M}$. So choose formulas

$$\begin{aligned}\phi_i \underline{a}_0 \ldots \underline{a}_n &\to \phi_i \underline{b}_0 \ldots \underline{b}_{n-1} w, \quad i < r, \\ \lambda &\neq \mu x \psi_j x \underline{b}_0 \ldots \underline{b}_{n-1} w, \quad j < s\end{aligned}$$

from Γw. We may assume $\underline{M} \models \bigwedge_{i<r} \phi_i \underline{a}_0 \ldots \underline{a}_n$; thus

$$\underline{M} \models \exists x \bigwedge_{i<r} \phi_i \underline{a}_0 \ldots \underline{a}_{n-1} x.$$

The formula $\exists x \bigwedge_{i<r} \phi_i \underline{a}_0 \ldots \underline{a}_{n-1} x$ is Σ^0_{k+1}; thus by the inductive hypothesis

$$\underline{M} \models \exists x \bigwedge_{i<r} \phi_i \underline{b}_0 \ldots \underline{b}_{n-1} x.$$

By Lemma 2.1 it follows that there is a $\Delta^0_{k+1}(b_0, \ldots, b_{n-1})$ element d such that

$$\underline{M} \models \bigwedge_{i<r} \phi_i \underline{b}_0 \ldots \underline{b}_{n-1} \underline{d}.$$

Assume $\underline{M} \models \lambda = \mu x \psi_{j_0} x \underline{b}_0 \ldots \underline{b}_{n-1} \underline{d}$ for some $j_0 < s$, i.e., $\psi_{j_0} x \underline{b}_0 \ldots \underline{b}_{n-1} \underline{d}$ defines λ. Let $\theta x y_0 \ldots y_{n-1}$ be a Δ^0_{k+1} formula of L_P which defines d, i.e., $\underline{M} \models \underline{d} = \mu z \theta z \underline{b}_0 \ldots \underline{b}_{n-1}$. Then

$$\begin{aligned}\alpha x &\doteq \exists y (y = \mu z \theta \underline{b}_0 \ldots \underline{b}_{n-1} z \wedge \psi_{j_0} x \underline{b}_0 \ldots \underline{b}_{n-1} y) \\ \beta x &\doteq \forall y (y = \mu z \theta \underline{b}_0 \ldots \underline{b}_{n-1} z \to \psi_{j_0} x \underline{b}_0 \ldots \underline{b}_{n-1} y)\end{aligned}$$

define λ, and αx, βx are Σ^0_{k+1}, Π^0_{k+1}, respectively. As $P \vdash \alpha x \leftrightarrow \beta x$, it follows that λ is Δ^0_{k+1} definable in $\underline{M}$ with parameters in $\{b_0, \ldots, b_{n-1}\}$, contradicting our assumption on λ.

Let $b_n \in M$ realize the recursive Σ^0_{k+2} type Γw.

Definition 2.4 If $\underline{M}$ is a countable model of P, then

$$P^M_k = \cap \{N \colon \underline{N} <_{\Sigma^0_k} \underline{M}, \underline{N} \cong \underline{M}\}.$$

Theorem 2.5 $P^M_k = \Delta^M_{k+1}$.

Proof: First we prove $P^M_k \subseteq \Delta^M_{k+1}$. By Lemma 2.3 for each $\lambda \in M - \Delta^M_{k+1}$ there is a sequence $b_0, b_1, \ldots$ such that (a) and (b) of the lemma hold. Then $N = \{b_0, b_1, \ldots\}$ is a submodel of $\underline{M}$, and $\lambda \notin N$. Also $\underline{N} <_{\Sigma^0_k} \underline{M}$, since we have the following. Assume $\underline{N} \models \phi \underline{b}_0 \ldots \underline{b}_{n-1}$, where $\phi \vec{x}$ is Σ^0_{k+1}. As the mapping $f \colon a_i \mapsto b_i$ defines an isomorphism of $\underline{M}$ onto $\underline{N}$, it follows that $\underline{M} \models \phi \underline{a}_0 \ldots \underline{a}_{n-1}$. Then, by (b) of the lemma, we have $\underline{M} \models \phi \underline{b}_0 \ldots \underline{b}_{n-1}$. Therefore, $\underline{N} \subseteq_{\Sigma^0_{k+1}} \underline{M}$; i.e., $\underline{N} <_{\Sigma^0_k} \underline{M}$.

Now we prove $\Delta^M_{k+1} \subseteq P^M_k$. Let $\underline{N} <_{\Sigma^0_k} \underline{M}$, and $f \colon \underline{N} \cong \underline{M}$ be an isomorphism. If $a \in \Delta^M_{k+1}$ is defined, say, by a Δ^0_{k+1} formula ϕx, then $\underline{M} \models \underline{a} = \mu x \phi x$. Hence,

$\underline{N} \models f^{-1}\underline{a} = \mu x\phi x$, and as $y = \mu x\phi x$ is Δ^0_{k+1}, it follows that $\underline{M} \models f^{-1}\underline{a} = \mu x\phi x$; so $fa = a$. Thus $a \in N$, and therefore $\Delta^M_{k+1} \subseteq N$.

In [5] Δ^M_1 elements are called recursive numbers in $\underline{M}$. By above we have the following characterization of recursive numbers:

Corollary 2.5.1 $\quad \cap \{N: \underline{N} \subseteq \underline{M}, N \cong \underline{M}\} = \Delta^M_1$.

Theorem 2.5 enables us to find the intersection of those initial segments of $\underline{M}$ which are Σ^0_k embedded in $\underline{M}$. For that reason we introduce the following

Definition 2.6 $\quad Q^M_k = \cap \{\underline{K}: \underline{K} <_{e\Sigma^0_k} \underline{M}, \underline{K} \cong \underline{M}\}$.

Recall that $\Pi^M_k = \{x \in M: x$ is definable in M by a Π^0_k formula$\}$. This set is considered in [7] and [5] for the case $k = 1$.

Theorem 2.7 *If $\underline{M}$ is a countable model of P, then $Q^M_k \subseteq J(\Pi^M_k)$.*

Proof: The proof of this theorem which we shall present is a variant of the proof of Friedman's Embeddability Theorem.

Let $\lambda \in M$ be such that for all $x \in \pi^M_k$, $x < \lambda$, and let $a_0, a_1, \ldots, a_0 = 0$, be an enumeration of M. We shall define a new enumeration $e_0, e_1, \ldots$ of $\underline{M}$, and find a sequence $b_0, b_1, \ldots < \lambda$ such that the map $e_i \mapsto b_i$ defines an isomorphism $f: \underline{M} \cong \underline{N}$, $N = \{b_0, b_1, \ldots\}$ and $\underline{N} <_{e\Sigma^0_k} \underline{M}$. The construction is done by the use of the back and forth argument maintaining

(1) $\underline{M} \models \exists x\theta x\underline{e}_0 \ldots \underline{e}_n \rightarrow \exists x < \lambda\theta x\underline{b}_0 \ldots \underline{b}_n$, where $\theta x\vec{y}$ is an arbitrary Σ^0_{k+1} formula.

Define $e_0 = b_0 = a_0$. Suppose $e_0, \ldots, e_{m-1}, b_0, \ldots, b_{m-1}$ have been determined.

Step $m = 2n + 1$. Case $n = 0$. Then:

(1) $\underline{M} \models \exists x\theta x \rightarrow \exists x < \lambda\theta x$, where θx is an arbitrary Σ^0_{k+1} formula of L_P (with only one free variable x). So let θx be Σ^0_{k+1} and assume $\underline{M} \models \exists x\theta x$.

Then there is a Π^0_k formula ψxy such that

(2) $\underline{M} \models \forall x(\theta x \leftrightarrow \exists y\psi xy)$

(3) $\underline{M} \models \exists x\theta x \leftrightarrow \exists w(\exists x,y < w)\psi xy$.

As the formula $\phi w \doteq (\exists x,y < w)\psi xy$ is Π^0_k, and as $\underline{M} \models \exists w\phi w$, there is an element $c \in M$ such that $\underline{M} \models c = \mu x\phi x$; so $c \in \Pi^M_k$ and, by the choice of λ, c is a witness to $\underline{M} \models (\exists w < \lambda)\phi w$. Therefore, by (2) and (3) it follows that $\underline{M} \models (\exists x < \lambda)\theta x$.

Case $n > 0$. Let e_m be a_i with the least index i such that $a_i \neq e_0, \ldots, e_{m-1}$. To determine b_m consider

$$\Gamma w = \{w < \underline{\lambda}\} \cup \{\exists x\theta xe_0 \ldots e_m \rightarrow \exists x < \lambda\theta xb_0 \ldots b_{m-1}w: \theta x\vec{y} \text{ is } \Sigma^0_{k+1}\}.$$

Obviously, Γw is a recursive set of Σ^0_{k+2} formulas of L_P. We show that Γw is a type over $\underline{M}$.

So let

$$\exists x_i \theta_i x_i \underline{e}_0 \ldots \underline{e}_m \to \exists x_i < \lambda \theta_i x_i \underline{b}_0 \ldots b_{m-1} w, \quad i < s,$$

be from Γw (we may assume that for $i \neq j$ the variables x_i and x_j are different). We show that all these formulas are realized in $\underline{M}$ together with $w < \lambda$. We may assume $\underline{M} \models \bigwedge_{i<s} \exists x_i \theta_i x_i \underline{e}_0 \ldots \underline{e}_m$. Thus

$$\underline{M} \models \exists y \bigwedge_{i<s} \exists x_i \theta_i x_i \underline{e}_0 \ldots \underline{e}_{m-1} y;$$

i.e.,

$$\underline{M} \models \exists z (\exists y < z)(\exists x_0 \ldots x_{s-1} < z) \bigwedge_{i<s} \theta_i x_i \underline{e}_0 \ldots \underline{e}_{m-1} y.$$

As the above sentence is Σ^0_{k+1}, we have by the inductive hypothesis

$$\underline{M} \models (\exists z < \underline{\lambda})(\exists y < z)(\exists x_0 \ldots x_{s-1} < z) \bigwedge_{i<s} \theta_i x_i b_0 \ldots b_{m-1} y.$$

Therefore,

$$\underline{M} \models (\exists y < \lambda) \bigwedge_{i<s} (\exists x_i < \lambda) \theta_i x_i b_0 \ldots b_{m-1} y;$$

i.e., Γw is finitely consistent. Therefore, by Theorem 1.3 Γw is realized in $\underline{M}$ by some b_m.

Step $m = 2n + 2$. We distinguish two cases:

Case 1. There is no $c < \lambda$ such that $c \neq e_0, \ldots, e_{m-1}$ and $c < b_i$ for some $i < m$. Then we proceed to the next step, taking $e_m = e_{m-1}$, $b_m = b_{m-1}$.

Case 2. There is a $c < \lambda$ such that $c \neq e_0, \ldots, e_{m-1}$ and $c < b_i$ for some $i < m$. Then b_m is chosen to be the first such c in the enumeration $a_0, a_1, \ldots$.

To find e_m we consider the following recursive set of Σ^0_{k+2} formulas

$$\Gamma w = \{w < \underline{e}_i\} \cup \{\exists x \theta x \underline{e}_0 \ldots \underline{e}_{m-1} w \to (\exists x < \lambda) \theta x \underline{b}_0 \ldots \underline{b}_m : \theta x \vec{y} \text{ is } \Sigma^0_{k+1}\}.$$

We show that Γw is a type over $\underline{M}$. Assume Γw is not consistent; so there is a j such that

$$\underline{M} \models \neg \exists w (w < \underline{e}_i \wedge \bigwedge_{r<j} (\exists x \psi_r x_r \underline{e}_0 \ldots \underline{e}_{m-1} w \to (\exists x < \lambda) \psi_r x_r \underline{b}_0 \ldots \underline{b}_m))$$

where $\psi_r x \vec{y}$ are Σ^0_{k+1} formulas (we may assume that the variables $x_0, \ldots, x_{j-1}$ are different). Let $Y \subseteq \{0, 1, \ldots, j-1\}$ be such that

$$r \in Y \text{ implies } \underline{M} \models (\forall x_r < \lambda) \neg \psi_r x_r \underline{b}_0 \ldots \underline{b}_m$$
$$r \notin Y \text{ implies } \underline{M} \models (\exists x_r < \lambda) \psi_r x_r \underline{b}_0 \ldots \underline{b}_m.$$

Observe that $Y \neq \phi$. Let $Y = \{r_0, \ldots, r_s\}$. Then

$$\underline{M} \models (\forall w < \underline{e}_i) \bigvee_{r \in Y} \exists x_r \psi_r x_r \underline{e}_0 \ldots \underline{e}_{m-1} w.$$

By Replacement Scheme in P we have

$$\underline{M} \models \exists z(\forall w < \underline{e}_i)(\exists x_{r_0}, \ldots, x_{r_s} < z) \bigvee_{r \in Y} \psi_r x_r \underline{e}_0 \ldots \underline{e}_{m-1} w.$$

As the above sentence is Σ^0_{k+1}, by the induction hypothesis we have

$$\underline{M} \models (\exists z < \underline{\lambda})(\forall w < \underline{b}_i)(\exists x_{r_0}, \ldots, x_{r_s} < z) \bigvee_{r \in Y} \psi_r x_r \underline{b}_0 \ldots \underline{b}_{m-1} w.$$

Hence, as $b_m < b_i$, we have for some $r \in Y$

$$\underline{M} \models (\exists x_r < \underline{\lambda}) \psi_r x_r \underline{b}_0 \ldots \underline{b}_m$$

contradicting our choice of Y. Therefore, Γw is a recursive Σ^0_{k+2} type; thus it is realized by some $\underline{e}_m \in M$.

Let $N = \{b_0, b_1, \ldots\}$. Then $\underline{N} \subseteq_e \underline{M}$, and $f(e_i) = b_i$ defines an isomorphism from $\underline{M}$ onto $\underline{N}$. We show that $\underline{N} <_{\Sigma^0_k} \underline{M}$. Let $\phi x_0 \ldots x_n$ be a Σ^0_{k+1} formula of L_P, and assume $\underline{N} \models \phi \underline{b}_0 \ldots \underline{b}_n$. Then for some Π^0_k formula ψ, $\phi \doteq \exists x \psi x$; hence $\underline{N} \models \exists x \psi x \underline{b}_0 \ldots \underline{b}_n$; i.e., $\underline{N} \models \exists x \psi x f \underline{e}_0 \ldots f \underline{e}_n$. So $\underline{M} \models \exists x \psi x \underline{e}_0 \ldots \underline{e}_n$. By (1) it follows that $\underline{M} \models (\exists x < \underline{\lambda}) \psi x \underline{b}_0 \ldots \underline{b}_n$. Therefore $\underline{N} \subseteq_{\Sigma^0_{k+1}} \underline{M}$. As for all $x \in N$, $x < \lambda$, we have $Q^M_k \subseteq J(\Pi^M_k)$.

Another proof of the last theorem is possible. For that it suffices to prove:

Theorem 2.8 *If $\lambda \in M$ and for all $x \in \pi^M_k$ we have $x < \lambda$, then there is a model $\underline{M}_1$ such that $\underline{M}_1 \cong \underline{M}$, $\underline{M}_1 <_{\Sigma^0_k} \underline{M}$, and $x < \lambda$ for all $x \in M_1$.*

The proof of this theorem is given at the odd step of the proof of Theorem 2.7. So let $\lambda \in M$, such that for all $x \in \pi^M_k$ we have $x < \lambda$, and $\underline{M}_1 <_{\Sigma^0_k} \underline{M}$, $\underline{M}_1 \cong \underline{M}$. Let $\underline{M}_2$ be such that $\underline{M}_1 <_c \underline{M}_2 \subseteq_e \underline{M}$. Such an $\underline{M}_2$ exists by Gaifman's Splitting Theorem. Therefore, by Theorem 1.2 it follows that $\underline{M}_2 <_{\Sigma^0_k} \underline{M}$. As $Th(\underline{M}_2) = Th(\underline{M})$ and as $SSy(\underline{M}) = SSy(\underline{M}_2)$, we may apply the hierarchical refinement of Friedman's Embeddability Theorem, i.e., there is $\underline{N} <_{e\Sigma^0_k} \underline{M}_2$ such that $\underline{N} \cong \underline{M}$. Then $\underline{N} <_{e\Sigma^0_k} \underline{M}$ and $x < \lambda$ for all $x \in N$.

Corollary 2.7.1 $Q_k = J(P_k) = J(\Delta^M_{k+1}) = J(\Pi^M_k)$.

Proof: By Theorems 2.5 and 2.7 we have

$$\Pi^M_k \subseteq \Delta^M_{k+1} \subseteq Q_k \subseteq J(\Pi^M_k) \subseteq J(\Delta^M_{k+1}).$$

Corollary 2.7.2 (D. Marker, A. Wilkie) $\cap \{K: \underline{K} \subseteq_e \underline{M}, \underline{K} \cong \underline{M}\} = J(\Sigma^M_0)$.

Proof: We have $\Sigma^M_0 = \Pi^M_0$, so by Theorem 2.7 $Q_0 \subseteq J(\Sigma^M_0)$. But Σ^0_0 definable elements are preserved under embeddings $f: \underline{M} \to \underline{K}$.

Corollary 2.7.3 *For all $k \in \omega$, $\Pi^M_k \subseteq_c \Sigma^M_{k+1}$.*

Proof: By Corollaries 2.2.1 and 2.7.1.

Since $J(\Pi^M_k)$ is a proper subset of $\underline{M}$ whenever $\underline{M}$ is a model of P (observe that $\{x > \mu y \phi y: \phi y \text{ is } \Pi^0_k\}$ is a recursive Σ^0_{k+2} type over $\underline{M}$), we have also:

Corollary 2.7.4 *For every $k \in \omega$ there is a $J <_{e\Sigma_k^0} \underline{M}$ such that $J \cong \underline{M}$ and $J \neq \underline{M}$.*

Now we are able to characterize Σ_k^0 extensions of natural numbers.

Theorem 2.9 *The following are equivalent for all $k \in \omega$ and all models $\underline{M}$ of P:*

(a) $\underline{\omega} <_{\Sigma_{k+1}^0} \underline{M}$; (b) $\Delta_{k+1}^M = \omega$; (c) $\Pi_k^M = \omega$; (d) $P_k = \omega$; (e) $Q_k = \omega$.

Proof: According to the theorems above, it suffices to prove the equivalence (a) $\leftrightarrow$ (b):

(a) $\rightarrow$ (b) If $\underline{\omega} <_{\Sigma_{k+1}^0} \underline{M}$ and $a \in \Delta_{k+1}^M$, then there is a Δ_{k+1}^0 formula ϕx such that $\underline{M} \models a = \mu x \phi x$. Thus $M \models \exists x \phi x$. Hence $\underline{\omega} \models \exists x \phi x$. So for some $n \in \omega$, $\underline{M} \models \phi \underline{n}$, i.e., $a \leqslant n$.

(b) $\rightarrow$ (a) Assume $\Delta_{k+1}^M = \omega$. Let ϕ be any Σ_{k+1}^0 sentence and assume there is a Π_k^0 formula θx such that $\underline{M} \models \phi \leftrightarrow \exists x \theta x$. Thus, for some $b \in M$, $\underline{M} \models b = \mu x \theta x$. Hence $b \in \Delta_{k+1}^M$, i.e., $b \in \omega$. Therefore, we have proved: for any Σ_{k+1}^0 formula ϕx, if $\underline{M} \models \exists x \phi x$, then $\underline{M} \models \phi \underline{n}$ for some $n \in \omega$.

By a hierarchical refinement of the Tarski-Vaught Lemma it follows that $\underline{\omega} <_{\Sigma_{k+1}^0} \underline{M}$.

Corollary 2.9.1 $\underline{\omega} <_{\Sigma_1^0} \underline{M}$ *iff* $\cap \{N: \underline{N} \subseteq_e \underline{M}, \underline{N} \cong \underline{M}\} = \omega$.

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