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The Concept of n-Cylinder and its Relationship to Simple Sets

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1 Introduction The concept of 'n-cylinder' was originally defined [4] in order to construct noncylindrical decision problems using System functions, a kind of function defined by Cleave [1]. It is a generalization of Young's [6] concept of a semicylinder and it forms a link between a semicylinder and a cylinder. Its definition is as follows:

Definition A set P is an n-cylinder if and only if there is a recursive function f such that for all $x_1, x_2, \ldots, x_n$,

$$\{x_1, x_2, \ldots, x_n\} \subseteq P \Rightarrow f(x_1, x_2, \ldots, x_n) \in P - \{x_1, x_2, \ldots, x_n\}$$
$$\{x_1, x_2, \ldots, x_n\} \subseteq \bar{P} \Rightarrow f(x_1, x_2, \ldots, x_n) \in \bar{P} - \{x_1, x_2, \ldots, x_n\}.$$

This function f is called the n-cylinder function for P. It can be seen that a semicylinder is a 1-cylinder. In [5], properties of n-cylinders and their relationship to cylinders were explored, and subsequently it was shown that: (i) for each $n \geqslant 1$, the class of all $(n+1)$-cylinders is a proper subset of the class of all cylinders, and (ii) a set is a cylinder if and only if it is an n-cylinder for each $n \geqslant 1$. Thus we can deduce that as n tends to infinity, the class of all n-cylinders coincides with the class of all cylinders.

This article shows a major difference between the properties of n-cylinders and cylinders: for each $n \geqslant 1$, the class of all n-cylinders contains a simple set whereas it has been shown [3] that no cylinder can be simple. The existence of n-cylinders which are simple gives rise to the following question: "Can every one-one degree be represented by an n-cylinder for each $n \geqslant 1$?"

From the results obtained in [4] and [5] we conjecture that for every infinite recursively enumerable set $W_{e'}$ there is an n-cylinder $A_{n,e}$ for each $n \geqslant 1$ such that $W_e \underset{1-1}{\leqslant} A_{n,e}$ and $A_{n,e} \underset{n^*}{\leqslant} W_e$, where a set A is n^* reducible to a

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set B if A is many-one reducible to B via some recursive function f and for each x, $f^{-1}(x)$ has at most $(n+1)$ members. However, the question as to whether for each recursively enumerable set W_e there exists an n-cylinder $A_{n,e}$ for each $n \geqslant 1$ such that $W_e \underset{1-1}{\equiv} A_{n,e}$ still remains open.

In Section 3 of this paper we prove the following result (α) which shows the existence of n-cylinders which are simple. The preliminary definitions needed for this proof are given in Section 2. For the recursive function theory terminology used in this paper we refer to [3].

Result (α) For each $n \geqslant 1$, there is an n-cylinder A_n such that A_n is simple.

2 Definitional preliminaries The definition of System functions and the definitions in the theory of graphs given in this section have been obtained mainly from [1] and [2]. In Section 3, these graph theoretic concepts are employed in formulating certain algorithms.

Let $f: N \rightarrow P_w(N)$ where N is the set of all natural numbers and $P_w(N)$ is the set of all finite subsets of N.

For all $x \in N$, $f^{-1}(x) = \{y: x \in f(y)\}$.

By $y \in C_f x$ is meant: Either $y = x$ or $y \in f(x)$ or there exist $y_1, y_2, \ldots, y_n$ $(n \geqslant 1)$ such that $y_1 \in f(x)$, $y \in f(y_n)$ and for each $(1 \leqslant i \leqslant n-1)$, $y_{i+1} \in f(y_i)$.

By the expression $\bigvee_{i=1}^{n} K_i$ is meant: $K_1 \vee K_2 \vee K_3 \vee, \ldots, \vee K_n$.

A system function is a function $f: N \rightarrow P_w(N)$ such that there exist recursive functions a and b such that for all x, $f(x) = D_{a(x)}$ and $f^{-1}(x) = D_{b(x)}$ where D_n is the n^{th} finite set in some standard enumeration.

The class of all System functions will be denoted by $\mathfrak{S}$.

Let D be a digraph whose points are in N. By $x \in D$ is meant: x is a point of D. If $x \in D$ and $y \in D$, then $x \vec{\cdot} y$ (or $y \overleftarrow{\cdot} x$) is a directed line if and only if there is a line from x to y in D. By $x \rightarrow y$ (D) is meant: Either $x = y$ or $x \vec{\cdot} y$ is a directed line or there exist $v_1, v_2, \ldots, v_n$ $(n \geqslant 1)$ which are points of D such that $x = v_1$, $y = v_n$ and for each $i (1 \leqslant i \leqslant n-1)$, $v_i \vec{\cdot} v_{i+1}$ is a directed line.

By $x | y$ (D) is meant: It is not the case that $x \rightarrow y(D)$ or $y \rightarrow x(D)$.

3 Existence of simple sets which are n-cylinders We will now prove Result (α) stated in Section 1.

Proof of Result (α): We need to prove that for each $n \geqslant 1$, there is a simple set G_n such that $G_n \in K(n)$ where $K(n)$ is the class of all n-cylinders. This result follows from the following result:

Result (β) For each $g \in \mathfrak{S}$ and $n \geqslant 2$, let

$$A_n^g = \{(x_1, x_2, \ldots, x_n): W^g(x_1, x_2, \ldots, x_n)\}$$

where if n is a prime,

$$W^g(x_1, x_2, \ldots, x_n) \equiv \bigvee_{i=1}^{n-1} x_{i+1} \in C_g x_i V x_1 \in C_g x_n.$$

If n is not a prime,

$$W^g(x_1, x_2, \ldots, x_n) \equiv \bigvee_{j=1}^{m_1} \underline{W}^g(x_j^1) V \bigvee_{j=1}^{m_2} \underline{W}^g(x_j^2) V \ldots V \bigvee_{j=1}^{m_s} \underline{W}^g(x_j^s)$$

where $m_1, m_2, \ldots, m_s$ are all the divisors of n other than n (but including 1) and $a_1, a_2, \ldots, a_s$ are the respective quotients (i.e., for each $i(1 \leqslant i \leqslant s)$, $m_i a_i = n$) and

$$\underline{W}^g(x_r^k) = \bigvee_{t=1}^{a_k-1} x_{r+tm_k} \in C_g x_{r+(t-1)m_k} V x_r \in C_g x_{r+(a_k-1)m_k}.$$

Then:

(i) If $g \in \mathfrak{S}$, A_n^g is an m-cylinder for each $m < n$.

(ii) For each $n \geqslant 2$, there exists an $f \in \mathfrak{S}$ such that A_n^f is simple.

Proof of Result (β)(i): Let τ_n be a recursive function which maps N^n 1 − 1 and onto N. Let $\pi_1^n, \pi_2^n, \ldots, \pi_n^n$ be those recursive functions of one variable which yield inverse mappings to τ_n; i.e., for all x, $\tau_n(\pi_1^n(x), \pi_2^n(x), \ldots, \pi_n^n(x)) = x$. We need the following result (δ). (Its proof is trivial and we state only the result here.)

Result (δ) Given an n-tuple of numbers $x_1, x_2, \ldots, x_n$, define $\bar{x}_1 = (x_1, x_2, \ldots, x_n)$, $\bar{x}_n = (x_n, x_1, x_2, \ldots, x_{n-1})$ and for each $i(2 \leqslant i \leqslant n-1)$, $\bar{x}_i = (x_i, x_{i+1}, x_{i+2}, \ldots, x_n, x_1, x_2, \ldots, x_{i-1})$. Then

(i) If n is a prime, there exist p, q $(1 \leqslant p, q \leqslant n)$ such that $\bar{x}_p = \bar{x}_q$ only if $x_1 = x_2 = x_3 = , \ldots, = x_{n-1} = x_n$.

(ii) If n is not a prime there exist p, q $(1 \leqslant p, q \leqslant n)$ such that $\bar{x}_p = \bar{x}_q$ only if either $x_1 = x_2 = \ldots = x_{n-1} = x_n$ or there is a divisor m ($m \neq 1$ or n) of n where $m \cdot a = n$ such that for each $j(1 \leqslant j \leqslant m)$, $x_j = x_{j+m} = x_{j+2m} = \ldots = x_{j+(a-1)m}$.

For each $m < n$, construct a function h_m of m variables as follows: To compute $h_m(y_1, y_2, \ldots, y_m)$, first check whether the following condition (θ) holds.

(θ) There is an $i(1 \leqslant i \leqslant m)$ such that for some $j, k(1 \leqslant j, k \leqslant n)$, $\pi_j^n(y_i) = \pi_k^n(y_i)$ and there is an occurrence of $x_k \in C_g x_j$ in $W^g(x_1, x_2, \ldots, x_n)$.

If Condition (θ) holds, then find the least number $\tau_n(t, t, \ldots, t)$ such that there is no $i(1 \leqslant i \leqslant m)$ such that $y_i = \tau_n(t, t, \ldots, t)$. Set $h_m(y_1, y_2, \ldots, y_m) = \tau_n(t, t, \ldots, t)$.

If Condition (θ) does not hold, find the least number $\tau_n(z_1, z_2, \ldots, z_n)$ such that for some $p(1 \leqslant p \leqslant m)$,

$$\begin{aligned}\{z_1, z_1, \ldots, z_n\} &= \{\pi_1^n(y_p), \pi_2^n(y_p), \ldots, \pi_n^n(y_p)\},\\ (z_1, z_2, \ldots, z_n) &\in (\pi_2^n(y_p), \pi_3^n(y_p), \ldots, \pi_n^n(y_p), \pi_1^n(y_p)),\\ &\quad (\pi_3^n(y_p), \pi_4^n(y_p), \ldots, \pi_n^n(y_p), \pi_1^n(y_p), \pi_2^n(y_p)), \ldots,\\ &\quad (\pi_{n-1}^n(y_p), \pi_n^n(y_p), \pi_1^n(y_p), \pi_2^n(y_p), \ldots, \pi_{n-2}^n(y_p))\\ &\quad (\pi_n^n(y_p), \pi_1^n(y_p), \pi_2^n(y_p), \ldots, \pi_{n-1}^n(y_p)),\end{aligned}$$

and there is no $r(1 \leqslant r \leqslant m)$ such that $\tau_n(z_1, z_2, \ldots, z_n) = y_r$. As $m < n$

and as Condition (θ) does not hold, from Result (δ) and the definition of $W^g(x_1, x_2, \ldots, x_n)$, it can be seen that such a number $\tau_n(z_1, z_2, \ldots, z_n)$ exists.

Set $h_m(y_1, y_2, \ldots, y_m) = \tau_n(z_1, z_2, \ldots, z_n)$. It can be easily verified that h_m is an m-cylinder function for A_n^g for each $g \in \mathfrak{G}$.

Proof of Result (β)(ii):. We prove this result only for the case when $n = 2$. The essential points of our argument are clearly exhibited in this proof. (A similar argument can be applied for the case when $n > 2$.) The proof is divided into two parts. The first part consists of a programme in which labeled digraphs D^o, D^1, D^2, are constructed with the following properties:

(i) There exists a recursive function ρ such that for each m, $\rho(m)$ is the Gödel number of D^m.
(ii) For each m, D^{m+1} is an extension of D^m; i.e., all points of D^m are points of D^{m+1} and D^{m+1} contains as a point the least number which is not a point of D^m. Furthermore, if x, y are points of D^m, then there is a line from x to y in D^m if and only if there is a line from x to y in D^{m+1}.
(iii) For each m, m is a point of D^m.
(iv) Labels are taken from the infinite set $\{P^1_{e_i}, P^2_{e_i}: e \geqslant 0, i \geqslant 0\}$ of markers. In addition to these labels, markers of the form j_t^* or j_t^+ where j, $t \geqslant 0$ are used.

We also use the following two statements S1 and S2 in the programme.

S1 Introduce the labels $P^1_{e_0}$, $P^2_{e_0}$ to D^{m-1} and extend the resulting graph to $\hat{D}$.

S2 Introduce the labels $P^1_{e_i}$, $P^2_{e_i}$ to $\bar{D}$.

By S1 we mean the following: Find the least 2 numbers, say $a_1 < a_2$, not in D^{m-1} and introduce them as new points so that each point a_i ($i \in \{1,2\}$) forms a new component. Name a_i by $P^i_{e_o}$ for each i($i \in \{1,2\}$). Let the resulting graph be D_1^{m-1}. Then find the least four numbers $a_3 < a_4 < a_5 < a_6$ not in D_1^{m-1}. Adjoin these numbers as new points and join the lines $a_3 \dot{\rightarrow} a_1$, $a_3 \dot{\rightarrow} a_2$, $a_4 \dot{\leftrightarrow} a_3$, $a_1 \dot{\rightarrow} a_5$, $a_2 \dot{\rightarrow} a_5$, $a_5 \dot{\leftrightarrow} a_6$. Let the resulting graph be D_2^{m-1}. Let $b_1, b_2, \ldots, b_k$ be all the numbers which have beside them a symbol j_t^* where j, $t \geqslant 0$, and let $r_1, r_2, \ldots, r_s$ be all the numbers which have beside them a symbol p_q^+ where $p, q \geqslant 0$. Find the least $k + s$ numbers, say $y_1 < y_2 < \ldots < y_{k+s}$, not in D_2^{m-1}. Adjoin these numbers as new points and join the lines $y_1 \dot{\leftrightarrow} b_1$, $y_2 \dot{\leftrightarrow} b_2$, $\ldots$, $y_k \dot{\leftrightarrow} b_k$, $r_1 \dot{\leftrightarrow} y_{s+1}$, $r_2 \dot{\leftrightarrow} y_{k+2}$, $\ldots$, $r_s \dot{\leftrightarrow} y_{k+s}$. Erase the symbol beside each $b_i (1 \leqslant i \leqslant k)$ and palce it beside y_i. Similarly erase the symbol beside each $r_i (1 \leqslant i \leqslant s)$ and place it beside y_{k+i}. Then place the symbol e_o^* beside a_4 and the symbol e_o^+ beside a_6. Join the line $y_{k+s} \dot{\rightarrow} a_4$. The resulting graph is $\hat{D}$. (See Figure 1.)

By S2 we mean the following: Find the least two numbers, say $v_1 < v_2$, not in $\bar{D}$. Adjoin them as new points so that each point $v_j (j \in \{1,2\})$ forms a new component. Name $v_j (j \in \{1,2\})$ by $P^j_{e_i}$. Let the resulting graph be $\bar{D}_1$. Find the least four numbers say $v_3 < v_4 < v_5 < v_6$ not in $\bar{D}_1$. Adjoin them as new points and join the lines $v_3 \dot{\rightarrow} v_1$, $v_3 \dot{\rightarrow} v_2$, $v_4 \dot{\leftrightarrow} v_3$, $v_1 \dot{\rightarrow} v_5$, $v_2 \dot{\rightarrow} v_5$, $v_5 \dot{\leftrightarrow} v_6$. Find the largest number y which has beside it a symbol $p_q^+ (p, q \geqslant 0)$. Join the line $y \dot{\rightarrow} v_4$. Then

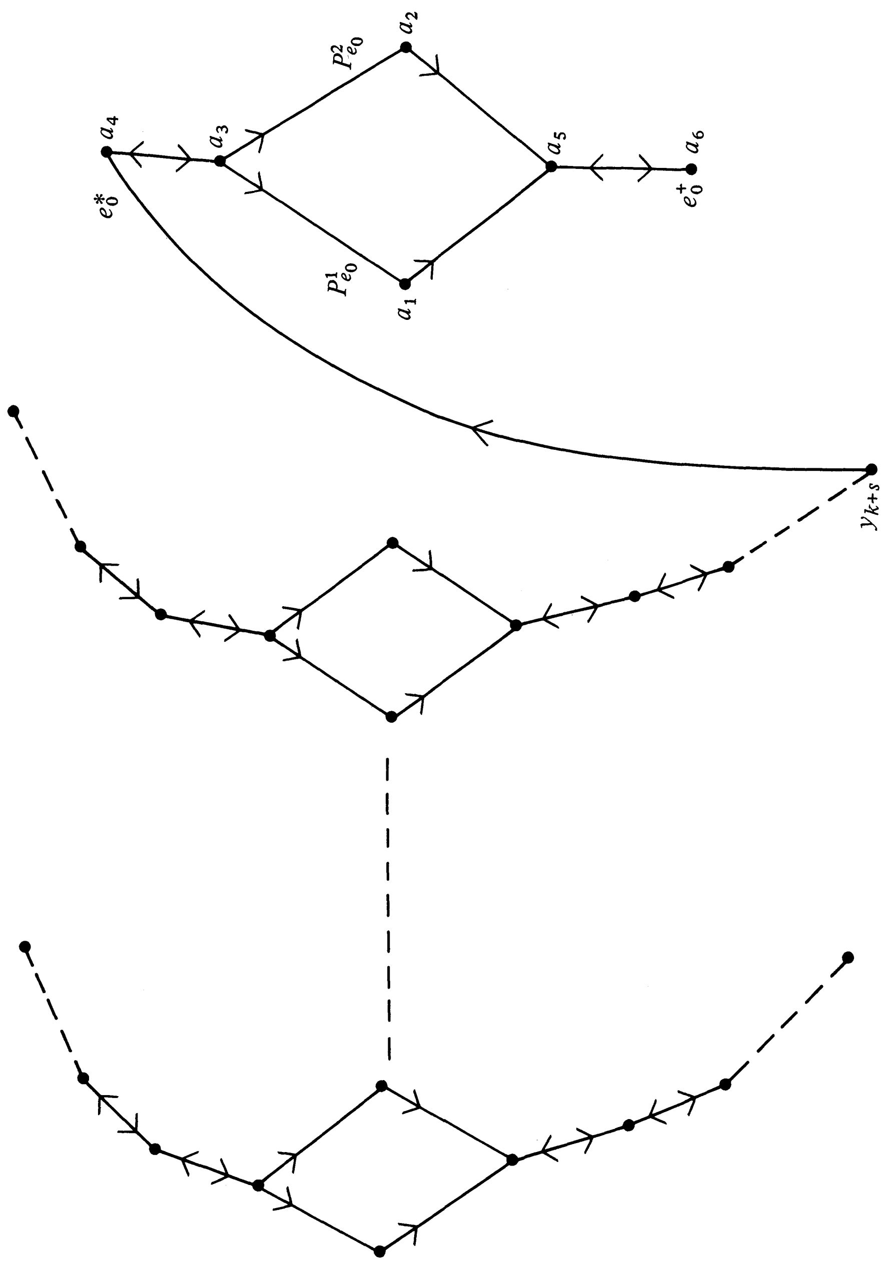

Figure 1

place the symbol e_i^* beside v_4 and the symbol e_i^+ beside v_6. (See Figure 2.) The second part of the proof consists of four lemmas by means of which it will be proved that there is an $f \in \mathfrak{S}$ such that every infinite recursively enumerable set W_e intersects A_2^f.

I Programme Construct the digraph D^m and a list $Z_m (m \geqslant 0)$ as follows:

Stage 0. D^o consists of the points 0, 1, 2, 3, 4, 5 and the lines $2 \to 0$, $2 \to 1$, $2 \leftrightarrow 3$, $1 \to 4$, $2 \to 4$, $4 \leftrightarrow 5$. 0, 1 are labeled $P^1_{0_0}$, $P^2_{0_0}$, respectively. The symbol 0_0^* is placed beside 3 and the symbol 0_0^+ is placed beside 6. $Z_0 = \phi$.

Stage $m (m \geqslant 1)$, Step 1. Introduce the labels $P^1_{m_0}$, $P^2_{m_0}$ to D^{m-1} and extend the resulting graph to $\hat{D}$.

Step 2. Find the least number $e \leqslant m$ such that there exist numbers x_1, x_2, z all $\leqslant m$ satisfying R where R is the conjunction of the following conditions R_1, R_2, and R_3, where

$R_1 \equiv T(e, \tau(x_1, x_2), z)$ where T is the Kleene's T-predicate and τ is a recursive function which maps N^2 1–1 and onto N
$R_2 \equiv \tau(x_1, x_2) > 2e$
$R_3 \equiv e \notin Z_{m-1}$.

If there does not exist such an e, set $D^m = \hat{D}$ and $Z_m = Z_{m-1}$. If there exists such an e, define:

$$e^m = (\mu e)(\exists z, x_1, x_2 \text{ all } \leqslant m) R(e, x_1, x_2, z, m)$$
$$x_1^m = (\mu x_1)(\exists z, x_2 \text{ both } \leqslant m) R(e^m, x_1, x_2, z, m)$$
$$x_2^m = (\mu x_2)(\exists z \leqslant m) R(e^m, x_1^m, x_2, z, m).$$

For convenience, let e^m, x_1^m, x_2^m be e, x_1, x_2, respectively. The application of Step 3 to e will be called an 'attack' on e.

Step 3, Case 1. There do not exist $j (j \leqslant m)$ and $k (k \geqslant 0)$ such that x_1, x_2 are labeled $P^u_{j_k}$, $P^v_{j_k}$, respectively, where $1 \leqslant u, v \leqslant 2$ and $u \neq v$. Then set $D^m = \hat{D}$ and $Z_m = Z_{m-1} \cup \{e\}$.

Case 2. Case 1 does not hold.

(i) If $j \leqslant e$, set $D^m = \hat{D}$, $Z_m = Z_{m-1}$.

(ii) $j > e$. Suppose the symbols j_k^*, j_k^+ are placed beside y_1, y_2, respectively. Join the line $y_2 \to y_1$. Delete $P^1_{j_k}$, $P^2_{j_k}$. Let the resulting graph be $\bar{D}$. Introduce the labels $P^1_{j_{k+1}}$, $P^2_{j_{k+1}}$ to $\bar{D}$. The resulting graph is D^m Set $Z_m = Z_{m-1} \cup \{e\}$.

This ends the programme.

Set $D = \bigcup_{m=0}^{\infty} D^m$ where $D^i \cup D^j = D^j$ if $j \geqslant i$
$= D^i$ if $j < i$.

Clearly all points incident with x in D are lines of D^{x+1}. Define:

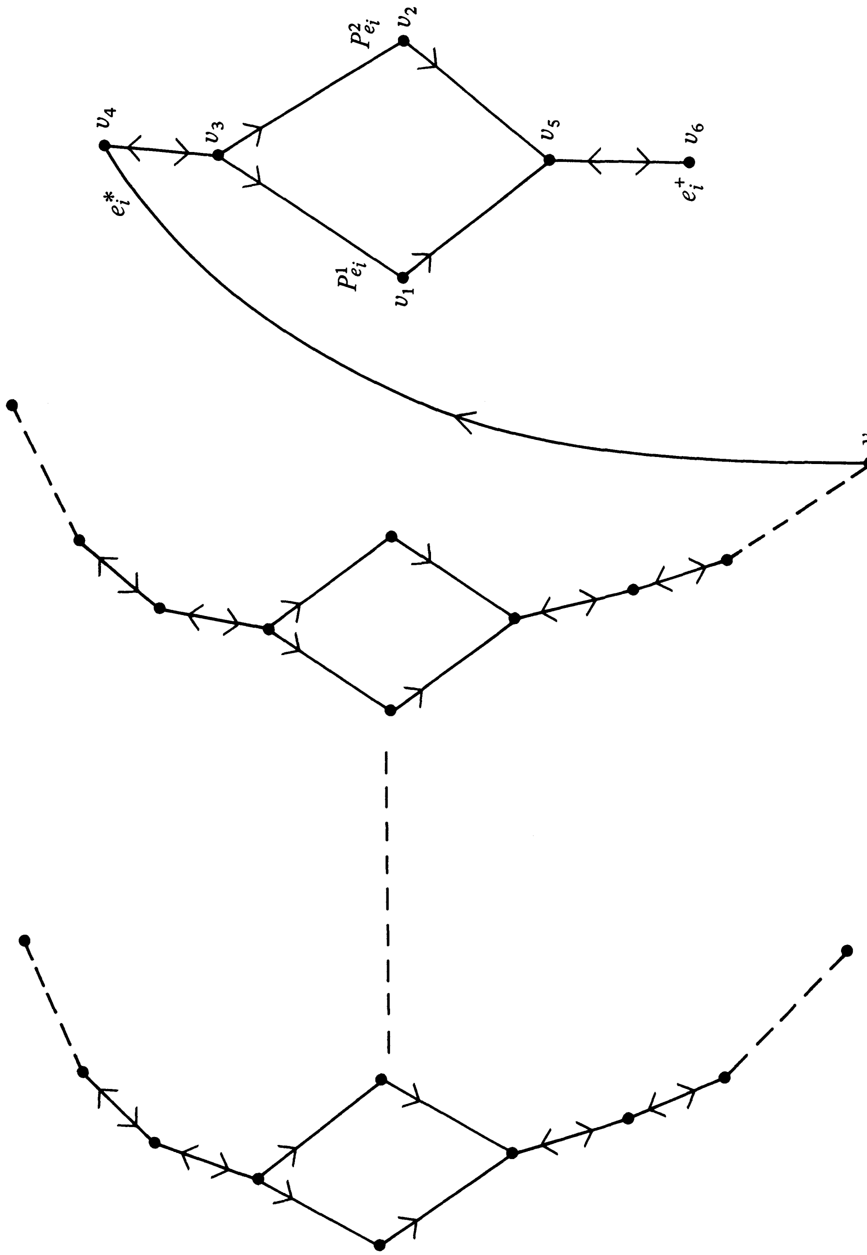

Figure 2

$$f(x) = \{y: x \to y\ (D)\} = \{y: x \to y\ (D^{x+1})\}$$
$$f^{-1}(x) = \{y: y \to x\ (D)\} = \{y: y \to x\ (D^{x+1})\}.$$

Then $f \in S$ and $A_2^f = \{(x_1, x_2): (x_1 \to x_2 \vee x_2 \to x_1)(D)\}$.

II Definition A label L is *fixed* at a stage numbered H if it is assigned to the same point at all stages numbered $n \geqslant H$.

Lemma 1 *For each $e (e \geqslant 0)$, there is a unique number t_e and a Stage M_e such that $P^1_{et_e}$, $P^2_{et_e}$ are fixed at Stage M_e.*

Proof: For each e, P^1_{eo}, P^2_{eo} are introduced at Stage e. P^1_{ei}, P^2_{ei} $(i \geqslant 0)$ are deleted and P^1_{ei+1}, P^2_{ei+1} are introduced via Case 2(ii) of a stage numbered m only if a number $k < e$ is attacked at Stage M. Furthermore this number k will never be attacked at a Stage $n > m$. As there exist only finitely many numbers less than e, and at any stage in the programme as there exists only one number, say q, such that P^1_{eq}, P^2_{eq} are assigned to points, there exists a unique number say t and a Stage M such that P^1_{et}, P^2_{et} are fixed at Stage M. Set $t_e = t$ and $M_e = M$.

Lemma 2 *A_2^f is recursively enumerable.*

Proof: $A_2^f = \{(x_1, x_2): (x_1 \to x_2 \vee x_2 \to x_1)(D)\}$. From the programme it can be seen that for any number $m \geqslant 0$, and for any pair of points (x, y), if $x \to y\ (D^m)$, then $x \to y\ (D)$. Construct a list L as follows:

Stage n^* $(n \geqslant 0)$. For any points x, y, if $x \to y\ (D^n)$, then place (x, y) and (y, x) in L.

This list gives an effective enumeration of the set A_2^f. Therefore A_2^f is recursively enumerable.

Lemma 3 *$\overline{A_2^f}$ is infinite.*

Proof: By Lemma 1, for each e, there is a unique number t_e and a Stage M_e such that the labels $P^1_{et_e}$, $P^2_{et_e}$ are fixed at Stage M_e. From the programme it can be seen that $\overline{A_2^f} = \{(x_1, x_2): x_1, x_2$ are labeled either $P^1_{et_e}$, $P^2_{et_e}$, respectively, or $P^2_{et_e}$, $P^1_{et_e}$, respectively, for some $e \geqslant 0\}$. Clearly $\overline{A_2^f}$ is infinite.

Lemma 4 *Every infinite recursively enumerable set intersects A_2^f.*

Proof: If $W_e = \{x: (\exists y) T(e, x, y)\}$ is infinite, then there exist infinitely many numbers x belonging to W_e such that $x > 2e$. Furthermore, as there exist only finitely many numbers less than e, there is a Stage m at which e will be attacked and Case 2(i) will not occur at Step 3 of Stage m. Then there exist x_1, x_2 such that $\tau(x_1, x_2) \in W_e$ and at Stage m it will be ensured that $(x_1 \to x_2 \vee x_2 \to x_1)(D^m)$ holds. As D is an extension of D^m, $(x_1 \to x_2 \vee x_2 \to x_1)(D^m) \Rightarrow (x_1 \to x_2 \vee x_2 \to x_1)(D)$; i.e., $(x_1, x_2) \in A_2^f$. We have shown that A_2^f is recursively enumerable, $\overline{A_2^f}$ is infinite and every infinite recursively enumerable set intersects A_2^f. Therefore A_2^f is simple. This proves Result (β)(ii) for the case when $n = 2$.

Note that for the case when $n > 2$, the proof is similar to the above proof except that labels $P^1_{et}, P^2_{et}, \ldots, P^n_{et}$ $(e \geqslant 0, t \geqslant 0)$ will have to be used.

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