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The Validity of Disjunctive Syllogism Is Not So Easily Proved

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1 This note is prompted by John Burgess's "Relevance: A Fallacy?" [2], which offers an argument in favour of the deductive validity of the argument form Disjunctive Syllogism, *DS* (*A*,not-*A* or *B*/∴*B*). The kind of argument he gives is not so unusual, and can be encountered around the literature (e.g., [3], p. 666) and not infrequently in the verbal pronouncements of philosophers. The bones of the reply I will give to Burgess can also be found in a number of places and as long ago as 1972 (e.g., [4]-[6]), though I do not think it has been systematically developed anywhere. Since Burgess's argument is representative of a widespread kind of mistake about relevant logics, it is worthwhile to try to say clearly what is wrong with it.

Burgess disclaims any attempt to discuss the extensive literature on relevant logics other than Anderson and Belnap's 1975 "masterwork" *Entailment.* Hence, his argument is best viewed as a piece of internal criticism of that book. However, he makes several remarks which imply fairly clearly that his sights are on more general targets, and are intended to apply to all "self-styled" relevant logicians. Let me therefore concede straight away that in my view Anderson and Belnap's discussion of *DS* in *Entailment* is inadequate. It would be rash, however, to draw the conclusion that there is no hope offered within the broad programme loosely classifiable as "relevantist" for shoring up their rejection of *DS*. Indeed, in view of the well-known Lewis proofs of the irrelevant principle of *Ex Falso Quodlibet,* there had better be.

Burgess says that the issue as far as he is concerned is whether relevant logics "are in better agreement with common sense than classical logic", and

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not whether some other intuitively comprehensible modeling for the relevant logics can be found elsewhere. So be it. Beyond the mild caveat that common sense might be strained somewhat by our investigation, let me agree with this demarcation of the area of contention. It is worth noting, though, that while Anderson and Belnap obviously thought their logics were in better agreement with common sense logic (or natural logic, or natural language, or some such thing), that might not be the only reason that could be advanced for adopting a relevant rather than a classical logic. Instead, one might appeal to pragmatic criteria such as overall simplicity of the foundations of mathematics or science as grounds for a *reconstruction* of natural logic along relevant lines.

One final piece of clarification. In all the standard relevant logics, a distinction is made between extensional, truth functional 'or', '$\vee$', and intensional 'or', '+'. The extensional form of *DS* ($A, \sim A \vee B / \therefore B$), is not generally valid in these logics (even in their purely truth functional, "zero degree" fragments), whereas the intensional form ($A, \sim A + B / \therefore B$) is. Burgess wants to show that certain valid natural language examples of *DS* have to be understood as of the extensional kind, so that extensional *DS* must be deductively valid. He represents the relevance position as having to hold that valid natural language examples of *DS* involve appeal to the (valid) intensional form of *DS*, either by virtue of direct translation of 'or' into '+' or because in such cases the crucial premiss using '+' is always available. I think that questions about intensional disjunction cloud the issue here, something for which Anderson and Belnap are at least partly to blame. I will be concerned to show where Burgess goes astray by agreeing that his examples use 'or' extensionally but arguing that they do not show that extensional *DS* is deductively valid; so I want to set aside questions about '+' altogether. Throughout this paper, then, 'or' is taken as '$\vee$'.

2 I begin by offering an explication of the useful intuitive idea of a *deductive situation*. Human beings are often in the position of deducing sentences from other sentences. Disputes as to the validity of a deduction from certain premisses can, I propose, be thought of as disputes as to the exact nature of the deductive situation containing those premisses. To make this more precise, we can introduce the idea of an L-theory (relative to a logic L). Consider a language $\underline{L}$ closed under conjunctions ($\wedge$), disjunctions ($\vee$), implications or entailments ($\rightarrow$), and negations ($\sim$). A *logic* in $\underline{L}$ is a subset L of $\underline{L}$ closed under the rule of uniform substitution. Now we can define the notion of an *L-theory relative to a logic L* (e.g., a *PC*-theory, or an *S5*-theory, or an *E*-theory). An L-theory is a set of sentences containing all the consequences of all the members of the theory which the logic L says are consequences. More formally, a subset X of $\underline{L}$ is an *L-theory* iff: (1) if $A \in X$ and $\vdash_L A \rightarrow B$ then $B \in X$, and (2) if $A \in X$ and $B \in X$ then $A \wedge B \in X$. (The second of these requirements is reasonable, but does not in general follow from the first, so needs independent specification.) I propose to explicate the idea of a deductive situation by identifying it with that of an L-theory where L is "natural" or "common sense" or "correct" logic.

We will say that an L-theory X is a *DS-theory* (written $DS(X)$) iff if $A \in X$ and $\sim A \vee B \in X$ then $B \in X$. Suppose, as Burgess believes, that *DS* is a universally valid principle of common sense or natural logic. Then it must be that every deductive situation is a *DS*-theory: whichever natural logic L is, it must always be that B is in every deductive situation which contains both A and $\sim A \vee B$. Furthermore, if Burgess is wrong and Anderson and Belnap are right and DS is not universally valid, then for some A, B, it must be that B fails to be in some deductive situation containing A and $\sim A \vee B$. (Any reasonable completeness theorem for L will deliver that result.) And that is what any putative counterexample to DS must evidently achieve: to produce a deductive situation in which A and $\sim A \vee B$ hold (belong) but B does not.

3 In the terminology of this paper, Burgess's strategy is to produce examples of deductive situations, claim that the examples are not special in any way which vitiates the generality of his argument, and claim that the situations are DS(-theories). This is to show that the validity of DS is required for common sense thinking. Now it is important for such a strategy that the deductive situation be correctly identified, for it might be that *further* information about the deductive situation is covertly imported which is sufficient to ensure that the deductive situation is DS. We would then be dealing, in effect, with a larger deductive situation and so the demonstration that B holds in it does nothing to show that the universal validity of DS is what is solely responsible for B's holding. It is this error which I claim Burgess has made.

The position I propose is that although DS is not universally valid, it is an acceptable mode of reasoning under certain circumstances. The situation seems to be like this. Many relevance people feel suspicious of DS because it seems to break down in what might be called "abnormal" deductive situations, particularly inconsistent situations. It is not infrequently claimed by relevance logicians that theories such as naïve set theory, classical pre-Cauchy calculus, the Böhr theory of the atom, quantum theory, natural language with its own truth predicate, and Peano arithmetic are or might well be non-DS. If this claim is correct, then some logic for which DS fails is a better model of natural logic than classical logic is. On the other hand, DS does seem to be a natural mode of inference in 'normal' deductive situations, the kind encountered every day. These two intuitions about DS can be reconciled if we can give an account of deductive validity according to which DS holds only in normal situations, and that is what I claim.

Some more definitions. An L-theory X is *consistent* iff for no A are both A and $\sim A$ in X. X is *trivial* iff X is the whole language $\underline{L}$. X is *nonprime with respect to* $A \vee B$ iff $A \vee B \in X$ but $A \notin X$ and $B \notin X$. X is *nonprime* iff X is nonprime with respect to some disjunction. X is *prime* iff it is not nonprime. The failure of primeness is no mystery, even for truth-functional disjunction. Consider Peano arithmetic formulated with a base of classical logic (i.e., classical Peano arithmetic, PA). Let G be its Gödel sentence. Then certainly $\vdash_{PA} G \vee \sim G$. But, by Gödel's first Incompleteness Theorem, if PA is consistent, neither $\vdash_{PA} G$ nor $\vdash_{PA} \sim G$. Hence if PA is consistent it is nonprime (with respect to $G \vee \sim G$).

Now some things are known about conditions under which L-theories are, and fail to be DS:

(1) Any inconsistent but nontrivial L-theory fails to be DS (under very weak assumptions about the logic L). Reason: If X is inconsistent, then for

some A, $A \in X$ and $\sim A \in X$. Since $\sim A \in X$, if $\vdash_L \sim A \rightarrow (\sim A \vee B)$, we have $\sim A \vee B \in X$, for arbitrary B. If *DS* held for X, we could deduce that $B \in X$, for arbitrary B, i.e., X would be trivial. Ex hypothesi, X is not trivial, so *DS* fails for X. In particular, all the usual relevant logics have inconsistent and nontrivial theories.

(2) If X is nonprime with respect to any disjunction $\sim A \vee B$ while $A \in X$, then *DS* fails for X. Reason: While $A \in X$, and $\sim A \vee B \in X$, the failure of primeness ensures that $B \notin X$.

(3) If, on the other hand, X is consistent and prime, then *DS* holds for X. Reason: Let $A \in X$, $\sim A \vee B \in X$. Since X is prime, at least one of $\sim A \in X$ and $B \in X$. But X is consistent, so $\sim A \notin X$. Hence $B \in X$.

(4) For certain choices of logic L, such as classical logic and intuitionism, *DS* holds for all L-theories.

A point to note about the proof under (3) that consistency and primeness implies *DS* is that it seems to appeal to a metalinguistic principle of *DS* as it were. However, it is not being claimed that *DS* is never legitimate. On the contrary, in normal well-behaved situations *DS* is to be expected to hold, and there does not seem to be anything untoward about the metalinguistic situation here. For example, we might formalize the metatheory and prove it to be consistent and prime. The foregoing considerations, then, enable us to conclude that a necessary and sufficient condition for a nontrivial deductive situation to be *DS* is that it be consistent and, for all subsets $\{A, \sim A \vee B\}$, prime with respect to $\sim A \vee B$. A special case sufficient for *DS* is where the deductive situation is consistent and complete (as Kripke's possible worlds are), since it is easy to show that, given De Morgan's Laws, consistency and completeness imply primeness.

4 This preamble enables me to make my main point against Burgess. If the deductive situations he describes patently contain *extra* information sufficient to guarantee that *DS* holds of them no matter what logic L is involved—such as the information that they are consistent and prime—then his argument *cannot* show that the (universal) validity of *DS* is required by those situations. I claim that this is what has happened. To see this, let us look at the examples Burgess gives. I simplify drastically for brevity.

In the first example, we are presented with a deck of cards and the information that a certain card in question is not both card A and card B, and that it is card A. Burgess claims that it is legitimate to conclude that it is not card B. Clearly this can be recast as an example of *DS*. But, unfortunately for relevant logic:

> Had Wyberg been a relevantist, unwilling to make a deductive step not licenced by the Anderson-Belnap systems E and R, he would have been unable to eliminate the queen of clubs from his calculations, and would have lost the game. A relevantist would fare badly in this game and others, and in game-like situations in social life, diplomacy, and other areas—unless, of course, he betrayed in practice the relevantistic principles he espoused in theory. ([2], p. 100)

However, now we are in a position to see that the deductive situation as Burgess presents it—a deck of ordinary playing cards, not card A or not card B, etc.—is *certainly consistent and prime.* It would be quite absurd to say that the situation is one where we have both card A and not card A. Equally, if it is either not card A or not card B, then at least one of those options obtains. So of course *DS* may be legitimately used. Relevant logicians are still worth employing as wargamers. The assumption of consistency and primeness here is so obvious as to be invisible. That is why it must be regarded with suspicion as possibly an operative factor in the situation. If it is, then nothing follows about the validity of *DS*.

In the second example, we are presented with a hypothetical discovery in number theory of the form $(n)(A(n) \vee B(n))$, and invited to conclude from a proof of $\sim A(1)$, that $B(1)$. (Again, to recast as an instance of *DS*, instantiate and use double negation.) What could be more harmless? Quite a bit: the assumption of consistency and primeness is, again, present. But here it is, instructively, much less obviously true. Suppose that number theory is inconsistent, and in particular that $A(1) \vee B(1)$ holds because $A(1)$ holds. Do we really want to conclude, if we come into possession of a proof of $\sim A(1)$, that $B(1)$? Of course, if it is *already* believed that classical logic is true, so that *DS* holds of number theory, then we will be prepared to conclude that $B(1)$, by the principle that everything can be deduced from a contradiction. But that begs the question. Again, suppose that arithmetic failed to be prime at $A(1) \vee B(1)$. Then from a proof of $\sim A(1)$ it would be quite illegitimate to conclude $B(1)$. But this is not what Burgess is supposing to be the case; in fact he quite explicitly supposes that a proof of $B(1)$ exists. The extra information Burgess needs to make his case for *DS* work is clearly present. But the presence of the extra information destroys his case.

One final quick example Burgess gives is that of someone once told that A or B but cannot remember which. Finally, he establishes $\sim A$, and so concludes that B.

> Such examples . . . show that, as far as negation, conjunction and disjunction are concerned, 'classical' logic . . . is far closer to common sense and accepted mathematical practice than is the 'relevant' logic of Anderson and Belnap. ([2], p. 102)

Certainly here the presupposition of consistency and primeness is less obvious. But it is there all the same, I submit, in virtue of the "presupposition of normality". Consistency and primeness are normal, nice, well-behaved. People are not ordinarily confronted with inconsistent or nonprime situations, so find moves like *DS* natural to make. If the situation were abnormal, say a mathematical one where primeness were in doubt, then it would be a more dubious move to deduce B. Thus, again, the kind of argument one often hears informally: '*DS* must hold. Look, if I know that today is Monday or Tuesday, and I know that it isn't Monday, I must conclude that it is Tuesday'. But what more normal a deductive situation can one imagine?

A final point against Burgess. He accuses the relevance programme of confusing logical implication with reasoning or inferring, a distinction of Harman's. I claim, to the contrary, that he is guilty of precisely that confusion.

In those terms, the issue is whether *DS* captures a logical implication, with Anderson and Belnap denying it and Burgess claiming it. In the light of the previous discussion, however, the examples he provides are nothing that a relevant logician need deny to be *useful reasoning or inferring*. It is quite proper to import extra facts about the deductive situation in order to extract all the useful information out of it. In thinking that he has raised a difficulty for the relevantist position, Burgess shows that he has precisely not appreciated the difference between usefully reasoning and universally valid deduction.

5 A consequence of the position of this paper is that the claim of the relevance programme, that *DS* is not universally valid, entails the claim that not all nontrivial deductive situations are consistent and prime. In order to show that to be incorrect, one must plainly adopt a different strategy from Burgess's. What must be considered, instead, are putative examples of nontrivial inconsistent deductive situations. Clearly these will be decidedly of the unusual type. But if a rule such as *DS* is to be valid, then it needs to hold in all deductive situations, not just normal ones. It is precisely the relevantist claim that abnormal, unusual situations where *DS* fails need to be taken into account. It is pointless to dispute this by concentrating on conditions in which it is known that *DS* holds.

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