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The Expected Complexity of Analytic Tableaux Analyses in Propositional Calculus

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We subscribe to the philosophy that comparison of the expected complexity of algorithms may often provide a better indication of relative efficiency than a comparison of the worst case complexity (cf., e.g., Knuth [10]). Research into algorithms that are efficient on the average has burgeoned in recent years. Karp in [9] discusses such algorithms for several graph theoretic problems. For example Posá [13], Angluin and Valiant [1], and Karp [9] show there are polynomial time algorithms that almost surely find Hamiltonian circuits if they exist. In the survey [9] Karp remarks that no fast-average-time algorithms are known for such problems as constructing minimum colorings or maximum cliques. We close by mentioning a negative result of Chvátal. In [4] he shows that Tarjan-Trojanowski type algorithms for finding the stability of a graph run in average exponential time.

The purpose of our article is to illustrate techniques useful in determining the expected complexity of (a variant of) the analytic tableau algorithm for the satisfiability problem of propositional calculus. Before we proceed it is important to mention the work of Goldberg [5] on the Davis-Putnam procedure for testing propositional satisfiability. In particular he showed that when the uniform distribution is placed on the set of problem instances, the expected time of the Davis-Putnam procedure is $O(kr^2)$ where r is the number of clauses in the given set and k is the number of distinct atoms.

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The algorithm we consider is a variant of analytic tableaux as defined in Smullyan [16]. His definition was greatly influenced by Beth's method of semantic tableaux (see [3]). Our definition of a systematic tableau differs from the one given by Smullyan. For us, the systematic tableau for $T(\sigma \wedge \tau)$ consists of $T(\sigma)$ and its tableau followed at the end of each "live" branch by $T(\tau)$ and its tableau. As usual, branches are terminated ("die") whenever a contradiction occurs; i.e., as soon as $T(\sigma)$ and $F(\sigma)$ (or $T(\neg\sigma)$) occur in a branch for some σ. This variant notion of a systematic tableau is chosen because it is amenable to recursive analysis. We would not be surprised if it proves to have greater expected complexity than Smullyan's version.

The best measure we use for the complexity of an analytic tableau is the total number of atomic formula symbols among the formulas occurring in the tableau. Assuming the reading, etc., time of atomic formulas is bounded, this determines within a linear factor the running time of a program that writes the tableau, although perhaps not the running time of a program that generates the tableau. (The difficulty lies in determining bounds on the number of tests for contradictions.) To study this measure of complexity it is first necessary to study the number of branches (maximal linearly ordered subsets) in a tableau. This is also a reasonable measure of complexity as the running time of a program to generate the tableau is bounded by the number of branches times a low-degree polynomial in the length of the formula being analysed.

We primarily consider tableaux for the satisfiability of formulas whose connectives are $\wedge$, $\vee$, and $\neg$ and in which $\neg$'s apply only to atomic formulas. This guarantees that a contradiction terminating a branch involves an atomic formula. We define the *length* of a formula to be the number of $\wedge$'s and $\vee$'s in it. Any formula involving $\wedge$, $\vee$, $\rightarrow$, and $\neg$ can readily be put in such a form without greatly increasing the total number of symbols in it. (Of course, if $\leftrightarrow$ is also present the formula in the desired form may be much longer.)

More formally: Let α_k be a fixed set of k atomic formulas. Let $\mathscr{F}_n(k)$ be the set of all formulas (of the form described) whose atomic formulas are all in α_k. Let $\#_n(k)$ be the cardinality of $\mathscr{F}_n(k)$. Let $T_n(k)$ be the total number of branches among the systematic analyses (in the sense detailed above) of the formulas in $\mathscr{F}_n(k)$. Then $E_n(k) = T_n(k)/\#_n(k)$ is the expected number of branches for formulas in $\mathscr{F}_n(k)$.

As a formula of length n may contain as many as $n + 1$ different atomic formulas, in order to consider all essentially different atomic formulas we study $E_n(n + 1)$. We show

Main Theorem *There are constants c and d such that* $c\cdot(1.08)^n \leqslant E_n(n + 1) \leqslant d\cdot(1.125)^n$.

This and various further results are proved by combinatoric methods. First we find recursion equations for various sequences. Then we ascertain the asymptotic behavior of these sequences in one of two ways:

(1) We use the recursion equations to derive explicit formulas for the sequences and then apply Sterling's formula. We call this Method S.

(2) We use the recursion equations to derive identities satisfied by the generating functions of the sequences and determine the asymptotic behavior

from these identities. This latter technique was developed by Pólya in [12] to count hydrocarbons and has been extensively used ever since. Modern forms occur in Harary, Robinson, and Schwenk [6] and in the expository article of Bender [2] in which the technique is based on a special case of a theorem of Darboux. We, in fact, directly use Darboux's result because in our cases it is easier to use and because the methods of [2] and [6] do not cover all the cases we consider. We call this Method D.

Outline of the paper In Section 1 we determine the asymptotic behavior of $\#_n(k)$. Although this can be done by citing results for the closely related Catalan numbers (cf., e.g., [15]) we instead do this in ways that permit us to illustrate Methods S and D in a simple case.

In Section 2 we derive the overestimate of $E_n(n+1)$ by determining, using Method D, the asymptotic behavior of a certain overestimate $U_n(n+1)$ of $T_n(n+1)$. Deriving the recursion equations for $U_n(k)$ provides a good introduction to the techniques used in such manipulations.

In Sections 3 and 4 we derive the underestimate of $E_n(n+1)$. In Section 3 we find recursion equations for certain sequences related to $T_n(k)$ and in Section 4 we use Method S to obtain underestimates for these.

In Section 5 we further illustrate the power of Method D by determining the asymptotic behavior of $E_n(k)$ for k fixed.

In Section 6 we finally study the expected number of occurrences of atomic formulas symbols in formulas in the tableaux.

Many of these sections contain additional related results. We note that since τ is a tautology if and only if $\neg\tau$ is not satisfiable the results of this paper imply analogous ones for the tautology problem.

We conclude in Section 7 with some open questions.

1 ***The number of formulas of length n*** $\#_n(k)$ satisfies the initial condition

(1.1a) $\#_0(k) = 2k$

and the recursion equation

(1.1b) $$\#_n(k) = 2 \sum_{i+j=n-1} \#_i(k)\#_j(k) \text{ for } n \geqslant 1.$$

(1.1b) is a consequence of the fact that in a formula of length n there is a main connective, which is either $\wedge$ or $\vee$, connecting subformulas in $\mathscr{F}_i(k)$ and $\mathscr{F}_j(k)$ respectively where $i + j = n - 1$.

The role of k can be isolated by defining

(1.2) $$\#_0 = 1 \text{ and } \#_n = 2 \sum_{i+j=n-1} \#_i\#_j \text{ for } n \geqslant 1$$

and observing by induction that

(1.3) $\#_n(k) = (2k)^{n+1}\#_n$.

$\#_n$ may be interpreted as the number of different ways of placing $\wedge$'s, $\vee$'s, and parentheses in a formula of length n. We call the placement of $\wedge$'s, $\vee$'s, and parentheses the *structure* of a formula.

(1.2) gives rise to an identity for the generating function of $\#_n$: (for any sequence f_n, the generating function is $f(x) = \sum_{n \geqslant 0} f_n x^n$. See, e.g., [14] for more details.)

(1.4) $\#(x) = 1 + 2x(\#(x))^2$.

As usual by $a_n \sim b_n$ (a_n asymptotic to b_n) we mean $\lim_{n \to +\infty} a_n/b_n = 1$.

(1.5) **Proposition** $\quad \#_n \sim \dfrac{8^n}{\sqrt{\pi}\, n^{3/2}}$.

Proof 1: Solving 1.4 for $\#(x)$ gives

(1.6) $$\#(x) = \frac{1 - (1 - 8x)^{1/2}}{4x}.$$

$\left(\text{Note the solution } \dfrac{1 + (1 - 8x)^{1/2}}{4x} \text{ is spurious since } \#(0) \text{ converges.}\right)$

Using the binomial expansion of $(1 - 8x)^{1/2}$ we obtain

(1.7) $$\#_n = \frac{(2n)!2^n}{n!(n+1)!}.$$

(Alternatively we could derive (1.7) by noting the fact that the number of ways of placing parentheses in a formula of length n is $\dfrac{(2n)!}{n!(n+1)!}$, which is called the n^{th} *Catalan number* and denoted c_n (cf., Riordan [15]). There are 2^n ways of choosing the $\wedge$'s and $\vee$'s. (1.5) can be derived as a corollary to asymptotic results for the Catalan numbers.)

Using Sterling's formula $\left(n! \sim \dfrac{\sqrt{2\pi}\, n^{n+1/2}}{e^n}\right)$ we have

$$\#_n \sim \frac{8^n}{\sqrt{\pi}\,(n+1)n^{1/2}} \sim \frac{8^n}{\sqrt{\pi}\, n^{3/2}}.$$

Proof 2: We use the following special case of a theorem of Darboux (see Bender [2], p. 498).

Theorem *Say* $f(z) = \sum_{n \geqslant 0} f_n z^n$ *is analytic near* 0 *and has only one singularity on its circle of convergence, say at* z_0. *Furthermore, say* $f(z)$ *can be written near* z_0 *as* $g(z) + \sum_{i=1}^{k} \left(1 - \frac{z}{z_0}\right)^{r_i} g_i(z)$, *where* g *is analytic near* z_0, $g_i(z)$ $1 \leqslant i \leqslant k$ *are analytic nonvanishing near* z_0, *and* $r_1 < \ldots < r_k$ *are rational numbers other than nonnegative integers.* (In this case we call $g(z) + \sum_{i=1}^{k} \left(1 - \frac{z}{z_0}\right)^{r_i} g_i(z)$ the *Darboux expansion* of $f(z)$ about z_0 and we say $f(z)$ is of *Darboux form* r_1 about z_0.) *Then*

$$f_n \sim \frac{g_1(z_0)}{n^{1+r_1}\Gamma(-r_1)z_0^n}$$

where Γ *is the classical gamma function.*

By (1.6) $\#(x)$ is of Darboux form $\frac{1}{2}$ about $x = \frac{1}{8}$. The proposition follows by Darboux's theorem.

2 ***The asymptotic overestimate*** To produce an overestimate we consider tableaux in which branches are not terminated when simple contradictions occur. We call such tableaux *overestimating*. We let $U_n(k)$ denote the total number of branches among the systematic overestimating tableaux for the formulas in $\mathscr{F}_n(k)$.

$U_n(k)$ satisfies the initial condition and recursion equation

(2.1a) $U_0(k) = 2k$

(2.1b) $U_n(k) = \sum_{i+j=n-1} (2\#_i(k)U_j(k) + U_i(k)U_j(k)).$

(2.1b) may be derived from

(2.2a) $U_{\sigma\vee\tau} = U_\sigma + U_\tau$ and
(2.2b) $U_{\sigma\wedge\tau} = U_\sigma \cdot U_\tau$

(where U_ϕ denotes the number of branches in the systematic overestimating tableau for ϕ) as follows:

$$\begin{aligned}
U_n(k) &= \sum_{i+j=n-1} \sum_{\substack{\sigma\in\mathscr{F}_i(k)\\ \tau\in\mathscr{F}_j(k)}} (U_{\sigma\vee\tau} + U_{\sigma\wedge\tau})\\
&= \sum_{i+j=n-1}\left(\sum_{\substack{\sigma\in\mathscr{F}_i(k)\\ \tau\in\mathscr{F}_j(k)}} U_\sigma + \sum_{\substack{\sigma\in\mathscr{F}_i(k)\\ \tau\in\mathscr{F}_j(k)}} U_\tau + \sum_{\substack{\sigma\in\mathscr{F}_i(k)\\ \tau\in\mathscr{F}_j(k)}} U_\sigma\cdot U_\tau\right)\\
&= \sum_{i+j=n-1}\left(\#_j(k)\sum_{\sigma\in\mathscr{F}_i(k)} U_\sigma + \#_i(k)\sum_{\tau\in\mathscr{F}_j(k)} U_\tau + \left(\sum_{\sigma\in\mathscr{F}_i(k)} U_\sigma\right)\left(\sum_{\tau\in\mathscr{F}_j(k)} U_\tau\right)\right)\\
&= \sum_{i+j=n-1} (\#_j(k)U_i(k) + \#_i(k)U_j(k) + U_i(k)U_j(k)).
\end{aligned}$$

$U_n = \dfrac{U_n(k)}{(2k)^{n+1}}$ satisfies

(2.3) $U_0 = 1$ and $U_n = \sum_{i+j=n-1} (2\#_iU_j + U_iU_j).$

(2.4) **Proposition** $\quad U_n \sim \dfrac{3\sqrt{3}\,9^n}{2\sqrt{\pi}\,n^{3/2}}.$

Proof: The proof is similar to (1.5). It uses the generating function identity

(2.5) $U(x) = 1 + 2x\#(x)U(x) + x(U(x))^2$

and (1.6) to determine the Darboux form of $U(x)$ about $x = \frac{1}{9}$. Computation shows

$$\text{(2.6)}\quad U(x) = \frac{\dfrac{1+(1-8x)^{1/2}}{2} - \left(\dfrac{1-12x+(1-8x)^{1/2}}{2}\right)^{1/2}}{2x}$$

$$= \frac{1+(1-8x)^{1/2}}{4x} - (1-9x)^{1/2} \frac{\left(\dfrac{8x}{(1-8x)^{1/2}+(12x-1)}\right)^{1/2}}{2x}.$$

Darboux's theorem yields the result.

As $\dfrac{U_n(n+1)}{\#_n(n+1)} = \dfrac{U_n}{\#_n}$,

(2.7) Theorem $\qquad E_n(n+1) \leqslant \dfrac{U_n(n+1)}{\#_n(n+1)} \sim \dfrac{3\sqrt{3}}{2}\left(\dfrac{9}{8}\right)^n.$

Remarks: (1) the above overestimate is independent of k. For k fixed we do better in Section 5 by obtaining the precise asymptotic behavior of $E_n(k)$.

(2) Precisely the same overestimate applies in many other cases. We give these examples:

(a) The expected number of branches for Smullyan systematic tableaux.

Let U_σ^S denote the number of branches in Smullyan's overestimating systematic tableau for σ. The result follows by

Proposition $\qquad U_\sigma^S = U_\sigma$.

Proof: More generally let $U^S_{\sigma_1,\ldots,\sigma_n}$ denote the number of branches in the Smullyan overestimating tableau for the tree headed $\begin{matrix} T(\sigma_1) \\ \vdots \\ T(\sigma_n)\end{matrix}$. These satisfy the initial conditions and recursion equations

$$U^S_{\alpha,\sigma_1,\ldots,\sigma_n} = \begin{cases} U^S_{\sigma_1,\ldots,\sigma_n} & \text{if } n \geqslant 1 \\ 1 & \text{if } n = 0 \end{cases} \qquad \text{for } \alpha \in \mathscr{F}_0(k)$$

$$U^S_{\sigma\vee\tau,\sigma_1,\ldots,\sigma_n} = U^S_{\sigma_1,\ldots,\sigma_n,\sigma} + U^S_{\sigma_1,\ldots,\sigma_n,\tau}$$

$$U^S_{\sigma\wedge\tau,\sigma_1,\ldots,\sigma_n} = U^S_{\sigma_1,\ldots,\sigma_n,\sigma,\tau}.$$

The proof is completed by observing that $U_{\sigma_1,\ldots,\sigma_n} = \prod_{i=1}^{n} U_{\sigma_i}$ satisfies the same equations.

(b) The expected number of branches for systematic tableaux of formulas involving $\neg$, $\wedge$, $\vee$, and $\rightarrow$ in which $\neg\neg$'s are eliminated but $\neg$'s are not pushed in. The length of such a formula is the number of binary connectives.

(c) The expected number of conjuncts that occur when a formula of length n is put in conjunctive normal form.

(3) Using the same techniques we can determine the asymptotic behavior of the variance and higher moments of the overestimate. (We are indebted to R. W. Robinson for some suggestions.) We sketch this for the variance:

Let

$$M_n(k) = \sum_{\phi \in \mathscr{T}_n(k)} U_\phi^2$$

and

$$V_n(k) = \sum_{\phi \in \mathscr{T}_n(k)} \left(U_\phi - \frac{U_n(k)}{\#_n(k)}\right)^2.$$

Then the variance

$$\frac{V_n(k)}{\#_n(k)} = \frac{M_n(k)}{\#_n(k)} - \left(\frac{U_n(k)}{\#_n(k)}\right)^2.$$

By (2.7)

$$\left(\frac{U_n(k)}{\#_n(k)}\right)^2 \sim \frac{27}{4}\left(\frac{81}{64}\right)^n.$$

It remains to determine the asymptotic behavior of $\frac{M_n(k)}{\#_n(k)}$. Using (2.2) and reasoning as for (2.1) we can show

$$M_0(k) = 2k$$

and

$$M_n(k) = \sum_{i+j=n-1} (M_i(k)M_j(k) + 2\#_i(k)M_j(k) + 2U_i(k)U_j(k)).$$

So

$$M_n = \frac{M_n(k)}{(2k)^{n+1}} \text{ satisfies } M_0 = 1,$$

$$M_n = \sum_{i+j=n-1} (M_iM_j + 2\#_iM_j + 2U_iU_j).$$

Using the resulting generating function identity and (1.6) and (2.6) we can show $M(x)$ has its only singularity on its circle of convergence at $x_0 = \frac{-13 + 44\sqrt{2}}{529}$ and has Darboux form $\frac{1}{2}$ about x_0. As a consequence $\frac{V_n}{\#_n} \sim c \cdot (1.3433107\ldots)^n$ for some constant c; i.e., the growth rate of the standard deviation $\sqrt{\frac{V_n}{\#_n}}$ is $1.1590128\ldots$ as opposed to 1.125 for the mean.

3 ***Recursion equations for $T_n(k)$*** There are two difficulties with the recursion equations for $T_n(k)$ which prevent us from using Darboux's Theorem to discuss the asymptotic behavior of $T_n(n + 1)$. The first is that in an inductive definition of analytic tableaux we must decide whether a branch splitting off part of the way down has a contradiction not only in itself but also with entries above it in the tableau. As a result we cannot directly give recursion equations for $T_n(k)$. Instead we proceed as follows.

Notation: P, Q, and R denote consistent subsets of $\mathscr{F}_0(k)$ (the set of k atomic formulas and their negations).

Given P we consider tableaux in which branches are terminated not only if they contain a contradiction but also if they contain a contradiction with P; i.e., the branch contains $T(\alpha)$ for some α with $\sim\alpha \in P$ (where $\sim\beta = \neg\beta$ and $\sim(\neg\beta) = \beta$ for β atomic). We call such a tableau a *tableau with input P.* Let $T_\phi(P,C)$ denote the number of branches in the tableau for ϕ with input P which terminate in a contradiction. Let $T_n(P,C,k) = \sum_{\phi \in \mathscr{F}_n(k)} T_\phi(P,C)$.

In a tableau with input P we say a live branch B has output Q if $Q = P \cup \{\alpha \in \mathscr{F}_0(k) | T(\alpha) \in B\}$. Let $T_\phi(P,Q)$ denote the number of branches in the tableau for ϕ with input P which have output Q. Let $T_n(P,Q,k) = \sum_{\phi \in \mathscr{F}_n(k)} T_\phi(P,Q)$.

Thus $T_n(k) = T_n(\Phi,C,k) + \sum_Q T_n(\Phi,Q,k)$. The initial conditions are

$$(3.1) \quad T_0(P,Q,k) = \begin{cases} |P| & \text{if } Q = P \text{ or } Q = C \\ 1 & \text{if } Q \supseteq P, \quad |Q - P| = 1 \\ 0 & \text{otherwise} \end{cases}$$

(Here $|X|$ denotes the cardinality of X.)

Using

$$(3.2) \quad \begin{aligned} T_{\sigma\vee\tau}(P,Q) &= T_\sigma(P,Q) + T_\tau(P,Q) \\ T_{\sigma\wedge\tau}(P,Q) &= \sum_{P\subseteq R\subseteq Q} T_\sigma(P,R)T_\tau(R,Q) \\ T_{\sigma\vee\tau}(P,C) &= T_\sigma(P,C) + T_\tau(P,C) \\ T_{\sigma\wedge\tau}(P,C) &= T_\sigma(P,C) + \sum_{P\subseteq R} T_\sigma(P,R)T_\tau(R,C) \end{aligned}$$

we can derive (reasoning as for (2.1b))

$$(3.3) \quad T_n(P,Q,k) = \sum_{i+j=n-1} \left(2\#_i(k)T_j(P,Q,k) + \sum_{r=p}^{q} \sum_{\substack{P\subseteq R\subseteq Q \\ |R|=r}} T_i(P,R,k)T_j(R,Q,k) \right)$$

$$T_n(P,C,k) = \sum_{i+j=n-1} \left(3\#_i(k)T_j(P,C,k) + \sum_{r=p}^{k} \sum_{\substack{P\subseteq R \\ |R|=r}} T_i(P,R,k)T_j(R,C,k) \right)$$

Lemma *$T_n(P,Q,k)$ and $T_n(P,C,k)$ depend only on $|P|$ and $|Q|$; i.e.,*

(3.4a) *if $|P| = |P'|$, $|Q| = |Q'|$, $P \subseteq Q$, $P' \subseteq Q'$, then $T_n(P,Q,k) = T_n(P',Q',k)$;*

(3.4b) *if $|P| = |P'|$, then $T_n(P,C,k) = T_n(P',C,k)$.*

Proof: Let f be a 1-1 map of $\mathscr{F}_0(k)$ onto $\mathscr{F}_0(k)$ mapping P onto P', Q onto Q', and satisfying $f(\sim\alpha) = \sim f(\alpha)$ for all α. Then f can be extended in an obvious way to a 1-1 map of $\mathscr{F}_n(k)$ onto $\mathscr{F}_n(k)$. Now (a) follows from the observation that $T_\phi(P,Q) = T_{f(\phi)}(P',Q')$. The proof of (b) is similar.

As a consequence we can let $T_n(p,q,k)$ denote $\sum_{\substack{P\subseteq Q\\ |Q|=q}} T_n(P,Q,k)$ and $T_n(p,C,k)$ denote $T_n(P,C,k)$ for any P with $|P| = p$. We note

$$(3.5)\quad T_n(k) = T_n(0,C,k) + \sum_{q=0}^{k} T_n(0,q,k).$$

Facts:

(3.6a) Given P with $|P| = p$ and given $q \geqslant p$ there are $\binom{k-p}{q-p} 2^{q-p}$ Q's with $Q \supseteq P$ and $|Q| = q$.

(3.6b) Given P,Q with $|P| = p$, $|Q| = q$, and $P \subseteq Q$ and given r with $p \leqslant r \leqslant q$ there are $\binom{q-p}{r-p}$ R's with $P \subseteq R \subseteq Q$ and $|R| = r$.

We indicate the proof of (a): to select Q we must choose one element of each of $q - p$ pairs chosen from the $k - p$ pairs of atomic formulas and their negations not "represented" by P.

From (3.1), (3.3), and (3.6) we may conclude:

(3.7) **Theorem**

$$T_0(p,q,k) = \begin{cases} p & \text{if } q = p \text{ or } C \\ 2k - 2p & \text{if } q = p + 1 \\ 0 & \text{if } p + 1 < q \leqslant k \end{cases}$$

$$T_n(p,q,k) = \sum_{i+j=n-1} \left(2\#_i(k)T_j(p,q,k) + \sum_{r=p}^{q} T_i(p,r,k)T_j(r,q,k)\right)$$

$$T_n(p,C,k) = \sum_{i+j=n-1} \left(3\#_i(k)T_j(p,C,k) + \sum_{r=p}^{k} T_i(p,r,k)T_j(r,C,k)\right).$$

It is at this point that the second difficulty preventing application of Darboux's Theorem arises. Our standard ploy for eliminating the role of k by defining, say $S_n(p,q,k) = \dfrac{T_n(p,q,k)}{(2k)^{n+1}}$, does not work. This can be seen from the equations

$$(3.8)\quad S_0(p,q,k) = \begin{cases} \dfrac{p}{2k} & \text{if } q = p \text{ or } C \\[2ex] \dfrac{k-p}{k} & \text{if } q = p + 1 \\[2ex] 0 & \text{if } p + 1 < q \leqslant k \end{cases}$$

$$S_n(p,q,k) = \sum_{i+j=n-1} \left(2\#_i S_j(p,q,k) + \sum_{r=p}^{q} S_i(p,r,k)S_j(r,q,k)\right)$$

$$S_n(p,C,k) = \sum_{i+j=n-1} \left(3\#_i S_j(p,C,k) + \sum_{r=p}^{k} S_i(p,r,k)S_j(r,C,k)\right)$$

$$S_n(k) = S_n(0,C,k) + \sum_{r=0}^{k} S_n(0,r,k).$$

4 The asymptotic underestimate If we define

(4.1) $R_0(p,p+1,k) = \dfrac{k-p}{k}$, $R_0(p,q,k) = 0$ for $q > p+1$

$$R_n(p,q,k) = \sum_{i+j=n-1} \left(2\#_i R_j(p,q,k) + \sum_{r=p+1}^{q-1} R_i(p,r,k)R_j(r,q,k)\right)$$

then as these omit positive terms from the corresponding equations for $S_n(p,q,k)$ we may show by induction on n and $q - p$ that $R_n(p,q,k) \leqslant S_n(p,q,k)$ for $p < q$. And so we have an underestimate of the S_n's.

The generating function identities for $R(x,p,q,k)$ imply

$$R(x,p,p+1,k) = \frac{k-p}{k}(1 - 2x\#(x))^{-1}$$

and

$$R(x,p,q,k) = \left(\sum_{r=p+1}^{q-1} xR(x,p,r,k)R(x,r,q,k)\right)(1 - 2x\#(x))^{-1}.$$

Noting by (1.4) that $(1 - 2x\#(x))^{-1} = \#(x)$ and using the inductive definition of Catalan numbers $\left(c_0 = 1, c_n = \sum_{i+j=n-1} c_i c_j\right)$ we can conclude by induction

(4.2a) $$R(x,p,p+r,k) = c_{r-1}\left(\frac{k-p}{k}\right)\left(\frac{k-(p+1)}{k}\right)\dots$$
$$\left(\frac{k-(p+r-1)}{k}\right) x^{r-1}(\#(x))^{2r-1}$$

and in particular

(4.2b) $$R(x,0,r,k) = c_{r-1}\frac{k!}{(k-r)!k^r}x^{r-1}(\#(x))^{2r-1}.$$

By problem 2a in Riordan [14], p. 153,

(4.3) $$(c(x))^{2r-1} = \sum_{n\geqslant 0} \frac{2r-1}{2n+2r-1}\binom{2n+2r-1}{n}x^n.$$

As $\#(x) = c(2x)$ we may conclude from (4.2b) and (4.3) that

(4.4a) $$R_n(0,r,k) = \frac{1}{2n+1}\binom{2n+1}{n+r}\binom{2r-1}{r}\frac{k!}{(k-r)!k^r}2^{n-(r-1)}$$

and specifically

(4.4b) $$R_n(0,r,n+1) = \frac{1}{2n+1}\binom{2n+1}{n+r}\binom{2r-1}{r}\frac{(n+1)!}{(n+1-r)!(n+1)^r}2^{n-(r-1)}.$$

Theorem *There is a constant c such that* $S_n(n+1) \geqslant c\,\dfrac{(8y_0)^n}{n^2}$ *where* y_0 *is the maximum value of* $f(x) = \left(\left(\frac{e}{2}\right)^x (1-x)^{2(1-x)}(1+x)^{(1-x)}\right)^{-1}$ *for* $0 < x < 1$. $\Bigl($*By calculus the maximum value occurs at* $x_0 = \dfrac{5-\sqrt{17}}{4}$ *and* $y_0 = 1.0805169\ldots\Bigr)$.

Motivation: Let $0 < x < 1$. Let $r_n = xn$ and pretend it is an integer. We examine the asymptotic behavior of the underestimate $R_n(0,r_n,n+1)$. It is

$$0\left(\frac{1}{n}\binom{2n}{n+r}\binom{2r}{r}\frac{n!}{(n-r)!n^r}\,2^{n-r}\right).$$

By Sterling's formula this is

$$0\left(\frac{(8f(x))^n}{n^2}\right).$$

Proof: We underestimate $S_n(n+1)$ by $R_n(0,r_n,n+1)$ where r_n is the greatest integer $\leqslant x_0 n$. Reasoning similarly as in the motivation

$$R_n(0,r_n,n+1) = 0\left(\frac{8^n\left(f\left(\frac{r_n}{n}\right)\right)^n}{n^2}\right).$$

As $x_0 - \frac{1}{n} < \frac{r_n}{n} < x_0$ and f is increasing on $(0,x_0)$, asymptotically

$$R_n(0,r_n,n+1) \geqslant c\,\frac{8^n\left(f\left(x_0-\frac{1}{n}\right)\right)^n}{n^2}.$$

Since $\left(f\left(x_0-\frac{1}{n}\right)\right)^n$ is asymptotic to a constant times $(f(x_0))^n$ we are done.

Remark: With considerably more effort using classical techniques for determining the asymptotic behavior of sums (see Bender [2], Section 3) one can slightly improve the theorem by showing

$$\sum_{r=1}^{n+1} R_n(0,r,n+1) = 0\left(\frac{(8f(x_0))^n}{n^{3/2}}\right).$$

As an immediate corollary we have:

Theorem *There is a constant c such that* $E_n(n+1) \geqslant c\,\dfrac{(y_0)^n}{n^{1/2}}$.

Remarks: (1) It is natural to ask what is the average number of branches among the satisfiable formulas of length n. We can obtain a similar underestimate to that above (differing only in the power of n). Among the satisfiable formulas of length n are $(\phi \vee \alpha_n)$ where α_n is $n+1^{\text{st}}$ atomic formula and ϕ has length $n-1$ and doesn't involve α_n. The total number of branches in the tableau for any such formula is one more than the number of branches for ϕ. Let $s(n)$ be the number of satisfiable formulas of length n. The expected number of branches for satisfiable formulas of length n is

$$\geqslant \frac{T_{n-1}(n)}{s(n)} \geqslant \frac{T_{n-1}(n)}{\#_n(n+1)} = \frac{(2n)^{n-1}S_{n-1}(n)}{(2n+2)^n \#_n}$$

$$\geqslant \frac{(2n)^{n-1}}{(2n+2)^n}\, c\, \frac{(8y_0)^{n-1}n^{-2}}{8^n n^{-3/2}} = \frac{c}{2n^{3/2}}\left(\frac{n}{n+1}\right)^n \frac{1}{8y_0} y_0^n \sim d\,\frac{y_0^n}{n^{3/2}}$$

for some d.

(2) An obvious improvement in analytic tableaux consists of giving some fixed procedure for creating a systematic tableau one branch at a time and stopping when either some live branch is found witnessing the satisfiability of the formula, or every branch has died demonstrating the formula is invalid. We observe that for any such procedure the expected number of branches must also satisfy an underestimate similar to that above $\left(\text{in this case } \frac{c \cdot y_0^n}{n^{5/2}} \text{ for some } c\right)$. This is essentially because: (i) there are lots of invalid formulas for which the procedure gains nothing and (ii) there is a defect in our notion of systematic tableau in that it fails to recognize efficiently obviously invalid formulas.

Specifically any formula of length n of the form $(\phi \wedge (\alpha_n \wedge \neg\alpha_n))$ where α_n is $n + 1^{\text{st}}$ atomic formula and ϕ has length $n - 2$ and does not involve the n^{th} or $n + 1^{\text{st}}$ atomic formula is invalid. Because of the manner in which we handle $\wedge$'s, the number of branches in the tableau for such a formula is the same as the number of branches for ϕ. We can now reason as in the previous remark.

*5 **Asymptotic behavior of $E_n(k)$, k fixed*** As k is fixed we usually omit its mention.

Using the generating function identities implied by (3.8) we can show

$$\text{(5.1a)} \quad S(x,p,p) = \frac{p}{2k} + 2x\#(x)S(x,p,p) + x(S(x,p,p))^2$$

and hence

$$\text{(5.1b)} \quad S(x,p,p) = \frac{\dfrac{1+(1-8x)^{1/2}}{2} - \left(\dfrac{1 - 4\left(1+\frac{p}{k}\right)x + (1-8x)^{1/2}}{2}\right)^{1/2}}{2x}$$

$$\text{(5.2)} \quad S(x,p,p+1) = \frac{k-p}{k} D(x,p,p+1) \text{ for } p+1 \leqslant k$$

$$\text{(5.3)} \quad S(x,p,q) = \left(\sum_{r=p+1}^{q-1} xS(x,p,r)S(x,r,q)\right) D(x,p,q) \text{ for } p+1 < q \leqslant k$$

and

(5.4) $$S(x,p,C) = \left(\frac{p}{2k} + \sum_{r=p+1}^{k} xS(x,p,r)S(x,r,C)\right) D(x,p,C) \text{ for } p \leqslant k$$

where

(5.5) $$D(x,p,q) = (1 - 2x\#(x) - xS(x,p,p) - xS(x,q,q))^{-1}$$

and

(5.6) $$D(x,p,C) = (1 - 3x\#(x) - xS(x,p,p))^{-1}.$$

Using these we can show that $S(x,p,q)$ for $q \geqslant p$ or $q = C$ all have their only singularity on their circle of convergence at $x = \frac{1}{8}$. Furthermore, we can show that $S(x,0,C)$ has Darboux form $-\frac{1}{4}$, $S(x,0,k)$ has Darboux form $\frac{1}{4}$, and $S(x,0,p)$ for $p < k$ has Darboux form $\frac{1}{2}$. (The $(1 - 8x)^{-1/4}$ term in $S(x,0,C)$ arises from a $(1 - 8x)^{-1/4}$ term in $D(x,k,C)$. The $(1 - 8x)^{1/4}$ term in $S(x,0,k)$ arises from $(1 - 8x)^{1/4}$ terms in $S(x,k,k)$ and in $D(x,p,k)$ for $p < k$.)

By Darboux's theorem and (1.5) we have:

Theorem *For k fixed*

(a) *the expected number of branches is asymptotic to $c(k)n^{3/4}$ for some $c(k)$.*
(b) *Of these branches asymptotically almost all die.*
(c) *The expected number which do not die is asymptotic to $c'(k)n^{1/4}$ for some $c'(k)$.*
(d) *Of these asymptotically almost all have an output of k formulas.*
(e) *Of those which have a consistent output of $<k$ formulas, for each $p < k$ asymptotically a constant number (depending on p and k) have output of size p.*

Remarks: (1) The qualitative aspects of this theorem (i.e., that the expected number of branches is polynomially bounded and (b) and (d)) are not surprising. The significance lies in the fact that intuitions have been made precise. (2) Both Bender in [2], Theorem 5, and Harary, Robinson, and Schwenk in [6] present procedures that allow one to determine in many cases the asymptotic behavior of a sequence f_n from an analytic identity satisfied by its generating function $f(x)$. The analytic identities are usually obtained from generating function identities. These procedures could have been used to find the asymptotic behavior of many of our sequences including $S_n(p,p,k)$ and $D_n(p,C,k)$ for $p < k$. Among the reasons we did not use these procedures are: (i) direct use of Darboux's theorem, on which both procedures rely, is easier and (ii) the procedures are not applicable in all cases; e.g., $S_n(k,k,k)$ and $D_n(k,C,k)$.

Both procedures require that if $f(x)$ has its only singularity on its circle of convergence at $x = x_0$, then $f(x_0)$ converges and if $F(x,f(x)) = 0$ is the analytic identity satisfied by $f(x)$ near $(x_0,f(x_0))$, then $F(x_0,f(x_0)) = 0$, $\frac{\partial F}{\partial y}(x_0,f(x_0)) = 0$ and $\frac{\partial^2 F}{\partial y^2}(x_0,f(x_0)) \neq 0$. In the case of $D_n(k,C,k)$, $D(\frac{1}{8},k,C,k)$ diverges. For $S_n(k,k,k)$, F, $\frac{\partial F}{\partial y}$, $\frac{\partial^2 F}{\partial y^2}$, $\frac{\partial^3 F}{\partial y^3}$ all vanish at $(x_0,f(x_0))$ and only $\frac{\partial^4 F}{\partial y^4}$ is $\neq 0$ there. We are able to handle $D_n(k,C,k)$ and $S_n(k,k,k)$, for by using (5.1a) and (5.6) we can explicitly obtain the Darboux expansions of $D(x,k,C,k)$ and $S(x,k,k,k)$.

The case of $S_n(k,k,k)$ raises the following question: Say $f(x) = \sum_{n\geqslant 0} f_n x^n$ where $f_n > 0$ has radius of convergence x_0 with f having its only singularity on the circle of convergence at $x = x_0$. Say $f(x_0)$ converges. Let $y_0 = f(x_0)$. Suppose $f(x)$ satisfies an analytic identity $F(x,f(x)) = 0$ where $F(x,y)$ is analytic near (x_0,y_0) and $\frac{\partial^i F}{\partial y^i}(x_0,y_0) = 0$ for $0 \leqslant i < k$ and $\frac{\partial^k F}{\partial y^k}(x_0,y_0) \neq 0$ $(k \geqslant 2)$. Then what can one conclude about the asymptotic behavior of f_n?

Generalizing on Bender's proof of Theorem 5 in [2], by the Weierstrass Preparation theorem (see [7], p. 144) $F(x,y) = U(x,y)P(x,y)$ where $U(x,y)$ is analytic nonvanishing near (x_0,y_0) and $P(x,y)$, a so-called Weierstrass polynomial, is of the form $(y - y_0)^k + \sum_{i=0}^{k-1} g_i(x)(y - y_0)^i$ where $g_i(x)$, $0 \leqslant i < k$, are analytic near x_0, vanishing at x_0. Hence $F(x,f(x)) = 0$ implies $P(x,f(x)) = 0$. Thus $f(x)$ has a fractional power series expansion about x_0 which may be put in the form $f(x) - f(x_0) = \sum_{i=0}^{l-1} f_i(x)(x - x_0)^{r_i}$ where $f_i(x)$, $0 \leqslant i < l$, are analytic near x_0, r_i are rational with $r_0 < r_1 < \ldots < r_{l-1}$. By following the beginning of the procedure in Walker [17], p. 98, to find the above fractional power series expansion one can show: (i) $r_0 \geqslant \frac{1}{k}$; (ii) (in lowest form) the denominator of r_0 is at most k; and (iii) $r_0 = \frac{1}{k}$ if and only if $\frac{\partial P}{\partial x}(x_0,y_0) \neq 0$ (or equivalently $\frac{\partial F}{\partial x}(x_0,y_0) \neq 0$). Thus if $\frac{\partial F}{\partial x}(x_0,y_0) \neq 0$ by Darboux's theorem

$$f_n \sim c\,\frac{1}{n^{1+1/k}x_0^n}$$

for some constant c. Furthermore, by plugging the fractional power series expansion for $y = f(x)$ about x_0 into the Taylor series expansion of $F(x,y)$ about (x_0,y_0) and looking at the coefficient of $(x - x_0)$ in the result one can show that

$$c = \left|\frac{\left(x_0 k!\,\frac{\partial F}{\partial x}(x_0,y_0)\left(\frac{\partial^k F}{\partial y^k}(x_0,y_0)\right)^{-1}\right)^{1/k}}{k\Gamma(1 - 1/k)}\right|.$$

Moreover, if $\frac{\partial F}{\partial x}(x_0,y_0) = 0$ by Darboux's theorem

$$f_n \sim c\,\frac{1}{n^{1+r}x_0^n}\ \text{for some constant } c$$

where $r > 1/k$ (we know some r_i is nonintegral as f is not analytic near x_0). We have completely analyzed this latter case [11].

6 On the expected number of occurrences of atomic formulas symbols Let $\widetilde{E}_n(k)$ denote the expected total number of occurrences of atomic formulas symbols in formulas in tableaux of members of $\mathcal{F}_n(k)$. $\widetilde{E}_n(n+1)$ has the same underestimate as $E_n(n+1)$. An overestimate is given by

Theorem *Asymptotically* $\widetilde{E}_n(n+1) \leqslant 18\sqrt{3}\,(\frac{9}{8})^n$.

Remark: It is worthwhile noting this is just twelve times the asymptotic overestimate of $E_n(n+1)$.

Proof: We consider overestimating tableaux. Let A_ϕ be the total number of occurrences of atomic formulas symbols in formulas in the overestimating tableau for ϕ. Let $A_n(k) = \sum_{\phi \in \mathcal{F}_n(k)} A_\phi$. The initial condition is

(6.1) $A_0(k) = 2k$.

Using (1.1b) and

(6.2) $$A_{\sigma \vee \tau} = n + 1 + A_\sigma + A_\tau \text{ and}$$
$$A_{\sigma \wedge \tau} = n + 1 + A_\sigma + U_\sigma A_\tau$$

where U_σ is as in Section 2 and n is the length of $\sigma \vee \tau$, $\sigma \wedge \tau$, one obtains the recursion equation

(6.3) $$A_n(k) = (n+1)\#_n(k) + \sum_{i+j=n-1} (3\#_i(k)A_j(k) + U_i(k)A_j(k)).$$

For $A_n = \dfrac{A_n(k)}{(2k)^{n+1}}$ we have

(6.4) $$A_0 = 1,\ A_n = (n+1)\#_n + \sum_{i+j=n-1} (3\#_i A_j + U_i A_j).$$

The generating function identity is

(6.5) $$A(x) = \frac{d}{dx}(x\#(x)) + 3x\#(x)A(x) + xU(x)A(x).$$

Using (6.5) (and (1.6), (2.6)) we can show the only singularity on the circle of convergence is at $x = \frac{1}{9}$ and we can determine the Darboux expansion of $A(x)$ about $x = \frac{1}{9}$. The theorem follows by Method D and (1.5).

For k fixed we have

Theorem $\widetilde{E}_n(k) \sim c(k)n^{5/4}$ *for some constant* $c(k)$.

Proof: Let $A_\phi(P)$ be the total number of occurrences of atomic formulas symbols in formulas in the tableau for ϕ with input P. Let $\widetilde{A}_n(P,k) = \sum_{\phi \in \mathcal{F}_n(k)} A_\phi(P)$. As in the proof of (3.4), this depends only on $p = |P|$. Hence we denote it $\widetilde{A}_n(p,k)$.

The initial condition is

(6.6) $\widetilde{A}_0(p,k) = 2k$.

Using (1.1b) and

(6.7) $$A_{\sigma \vee \tau}(P) = n + 1 + A_\sigma(P) + A_\tau(P)$$
$$A_{\sigma \wedge \tau}(P) = n + 1 + A_\sigma(P) + \sum_{P \subseteq Q} T_\sigma(P,Q)A_\tau(Q)$$

where $T_\sigma(P,Q)$ is as in Section 3 and n is the length of $\sigma \vee \tau$, $\sigma \wedge \tau$, one obtains the recursion equation

$$(6.8)\quad \widetilde{A}_n(p,k) = (n+1)\#_n(k) + \sum_{i+j=n-1} \left(3\#_i(k)\widetilde{A}_j(p,k) + \sum_{q=p}^{k} T_i(p,q,k)\widetilde{A}_j(q,k)\right).$$

For $A_n(p,k) = \dfrac{\widetilde{A}_n(p,k)}{(2k)^{n+1}}$ we have

$$(6.9)\quad \begin{aligned} &A_0(p,k) = 1 \\ &A_n(p,k) = (n+1)\#_n + \sum_{i+j=n-1} \left(3\#_i A_j(p,k) + \sum_{q=p}^{k} S_i(p,q,k)A_j(q,k)\right) \end{aligned}$$

and

$$(6.10)\quad A(x,p,k) = \frac{d}{dx}(x\#(x)) + 3x\#(x)A(x) + \sum_{q=p}^{k} S(x,p,q,k)A(x,q,k).$$

By (6.10) and the work needed to fill in the details in Section 5 we may conclude by decreasing induction on p that $A(x,p,k)$ has its only singularity on its circle of convergence at $x = \frac{1}{8}$ and is of Darboux form $-\frac{3}{4}$ about $x = \frac{1}{8}$. (The $(1 - 8x)^{-3/4}$ term in $A(x,0,k)$ arises from a $(1 - 8x)^{-3/4}$ term in $A(x,k,k) = \frac{d}{dx}(x\#(x))D(x,k,C,k)$ which in turn arises as $\frac{d}{dx}(x\#(x)) = (1 - 8x)^{-1/2}$ and as already noted $D(x,k,C,k)$ has a $(1 - 8x)^{-1/4}$ term.) Method D and (1.5) now yield the result.

Remarks: (1) In a similar (but slightly easier manner) one can analyze the expected number of nodes in tableaux. For $k = n + 1$ the expected number of nodes in an overestimating tableau for a formula of length n is asymptotic to $9\sqrt{3}\,(\frac{9}{8})^n$. For k fixed the expected number of nodes is asymptotic to $c \cdot n^{3/4}$.

(2) For fixed k the expected running time of a program to generate the tableau of a formula of length n is linearly bounded by $\widetilde{E}_n(k)$ and hence linearly bounded by $n^{5/4}$. This is true as tests for contradictions need only be done at nodes where the entry is a member of $\mathscr{F}_0(k)$ and at each such occasion it suffices to make a maximum of (the constant) k comparisons. By remark (1) the number of tests for contradictions is linearly bounded by $n^{3/4}$.

7 *Open questions and further discussion*

1. Can one obtain better asymptotic over and underestimates? One possible way to obtain a better underestimate is as follows: By induction one can show $Q_n(p,p + r,k) \leqslant S_n(p,p + r,k)$ where $Q_n(p,p + r,k)$ is determined by

$$Q(x,p,p+r,k) = c_{r-1}\left(\frac{k-p}{k}\right)\cdots$$
$$\left(\frac{k-(p+r-1)}{k}\right) x^{r-1}Q(x,p+1,k)(Q(x,p+2))^2$$
$$\ldots (Q(x,p+r,k))^2$$

where $Q(x,q,k) = (1 - 2x\#(x) - xS(x,q,q,k))^{-1} = \frac{2k}{q} S(x,q,q,k)$. It remains to study

$$Q(x,0,r,k) = c_{r-1}\binom{k}{r}\frac{2^{2r-1}k^{r-1}}{r!} x^{r-1}S(x,1,1,k)(S(x,2,2,k))^2 \ldots (S(x,k,k,k))^2.$$

2. Can one obtain asymptotic underestimates for Smullyan's systematic tableaux? Such results could be nice as these tableaux do not have the defects noted in Section 4.

3. Can one obtain similar analyses for other distributions of formulas than the one inherent in our work?

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