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Alphabetical Order

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We define an *alphabet* ξ to be a finite or countably infinite (von Neumann) ordinal greater than 1. ξ^* is the set of all finite sequences of members of ξ. We use 'a','b','c',... as variables over members of ξ,'x','y','z',... as variables over members of ξ^*, and use juxtaposition to denote concatenation in the obvious way: e.g., if $x{:}m \to \xi$ and $y{:}n \to \xi$, then xay is the finite sequence $z{:}(m + 1 + n) \to \xi$ such that $z(i) = x(i)$ if $i < m$, $z(m) = a$, and $z(i) = y(j)$ if $i = m + 1 + j < m + 1 + n$. $\varnothing$ is the empty finite sequence, which has domain 0.

Alphabetical order on ξ^* is the relation that holds between x and y iff either for some $z \neq \varnothing$, $y = xz$ or for some a,b,z,z',z'', with $a < b$, $x = zaz'$ and $y = zbz''$. Alphabetical order on ξ^* is clearly a countable linear order in which every element x has the immediate successor $x0$. But it is not a well ordering: consider $\{\ldots,001,01,1\}$. Some further peculiarities of this order are stated in the last paragraph of this note.

We shall characterize the order type α of alphabetical order on ξ^*, which turns out not to depend on ξ. In fact, we have the following theorem.

Theorem *Let η be the order type of less-than on the rationals. Then $\alpha = \omega(1 + \eta)$.*

Proof: Let ξ^{*-} be the set of all nonempty sequences in ξ^* whose last element is some member of ξ other than 0. First note that R, the restriction to ξ^{*-} of alphabetical order on ξ^*, is dense (if $yRyzc$, then $c \neq 0$ and $yRyz0cRyzc$; if $zaz'Rzbz''$, then $zaz'Rzaz'1Rzbz''$) and lacks endpoints (if $c \neq 0$, then $z0cRzcRzcc$). By a famous theorem of Cantor's, since R is also a countable linear order, R is isomorphic to less-than on the rationals. Now observe that $\varnothing$ is the alphabetically earliest member of ξ^*, $x0$ is the immediate alphabetical successor of x, and every element of ξ^* is either 0^m for some $m \geq 0$ or $x0^m$ for

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some x in ξ^{*-} and some $m \geq 0$. Alphabetical order on ξ^* may therefore be characterized as follows: first come all sequences 0^m, $m \geq 0$. Then come all sequences $x0^m$, with x in ξ^{*-} and $m \geq 0$: if $x0^m$ and $y0^n$ are two sequences with x and y in ξ^{*-}, then $x0^m$ precedes $y0^n$ iff either x R-precedes y or both $x = y$ and $m < n$. These later sequences thus have the order-type $\omega\eta$. So alphabetical order on ξ^* is isomorphic to the ordering obtained from less-than on the rationals by first replacing each rational by an ω-sequence and then prefixing another ω-sequence to the result. That is to say, $\alpha = \omega + \omega\eta = \omega(1 + \eta)$.

Thus alphabetical ordering on $\{0,1\}^*$ is isomorphic to that on $\{0,1,2\}^*$; an isomorphism f is given by: $f0^n = 0^n$, $n \geq 0$; $f00x = 0fx$, x a sequence containing at least one 1; and $f01x = 1fx$ and $f1x = 2fx$, x an arbitrary sequence.

Since $\eta = \eta + 1 + \eta$, $\alpha = \omega + \omega\eta = 1 + \omega + \omega\eta = 1 + \omega(1 + \eta) = 1 + \omega(1 + \eta + 1 + \eta) = 1 + \omega(1 + \eta)2 = 1 + \alpha 2$. Similarly, $\alpha = 1 + \alpha n$, for all finite n. And since $\eta = (\eta + 1)\omega$, $\alpha = \omega(1 + \eta) = \omega(1 + (\eta + 1)\omega) = \omega + \omega(\eta + 1)\omega = 1 + \omega + (\omega\eta + \omega)\omega = 1 + (\omega + \omega\eta)\omega = 1 + \alpha\omega$.

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