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# A Different Approach to Deontic Logic: Deontic Logic Viewed as a Variant of Dynamic Logic

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*1 Introduction* This article proposes a new setting for deontic logic, the logic of obligation, prohibition, and permissibility. Surveys of the several deontic logics that were devised in the past can be found in [12], [6], [1], [3]. For those familiar with the Dutch language, Soeteman's thesis [18] is also certainly worthwhile reading because of its well-considered comparative study of the various deontic systems in the literature. In our paper, deontic logic is reduced to a variant of so-called dynamic logic (e.g., [8], [9], [17]). The latter can be considered as a very weak modal logic resembling system $K$ with additional axioms for the behavior of the various actions, which are, by the way, strictly separated from assertions in the system. It will appear that this last property of the syntax will prevent us from asserting and proving in this logic many paradoxical and counterintuitive propositions that often crop up in the literature (see, e.g., [6], [10], [11], [18]). The philosophical idea behind separating actions and assertions is the simple observation that only assertions can be asserted and only actions can be acted or performed. So it is meaningless to state the obligation $O\phi$ of some proposition $\phi$, such as $OO\alpha$, where $\phi$ is taken to be the assertion stating that the action $\alpha$ is obligatory. Furthermore, of crucial importance is the consideration that an action may change the current situation (world) and an assertion does not. Furthermore, the fact that actions change situations implies some notion of passing of time. This obvious remark has, of course, been observed by other authors as well. Van Eck, for example, has given a deontic system in [19] where time is a central notion. However, there it is used in an entirely dif-

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ferent manner: accessibility relations between worlds are still defined within one time-slice and involve more perfect alternatives that could have existed at the same moment. In our approach the accessibility is defined between a world *before* a certain action $\alpha$ is performed and possible ones *after* $\alpha$ has been done, i.e., between worlds with different "time-stamps."

Dynamic Logic stems from the theory of computer science, where it is used to prove correctness properties of computer programs. If this seems odd, one has to realize that a computer program is in fact nothing but a sequence of actions of a certain kind. Although we shall follow the formalism of Dynamic Logic, we must also mention that Polish logicians such as Salwicki (see [16]) have occupied themselves with a similar logic, called Algorithmic Logic, to deal with program correctness. Some of this work dates from as long ago as 1959 ([21]).

To begin with, we shall give a formal syntax of the actions and assertions we allow in our approach. Next, we shall present a semantics for these, in a similar way as this is done in computer science and dynamic logic in particular.

We shall see that the modality of prohibition is less problematic to deal with than obligation, both formally and semantically. The manner in which the assertion $F\alpha$, it is forbidden to do action $\alpha$, is reduced to dynamic logic, is inspired by Anderson's proposal (in [2]) to reduce $O\alpha$ to an assertion in a Lewis-modal system, but will lack its most undesirable consequences (cf. [15]). A few 'paradoxes' remain derivable, such as, e.g., Ross's paradox: $O\alpha \supset O(\alpha \cup \beta)$. But it is argued quite convincingly in the literature (e.g., [4], [18]) that these are not real anomalies, and we propose a simple solution within our framework. As we shall see, the paradoxes that disappear in our approach are mostly of the kind involving a conditional under the *O*, *P* (permissibility) or *F*-symbol.

Finally we shall discuss several ways in which our basic system can be extended.

***2 Definition: Actions and assertions*** The central notion of our approach will be represented by a modal operator $[\alpha]$ associated with an action $\alpha$. The expression $[\alpha]\phi$ will mean the weakest precondition that is required to ensure that $\phi$ will hold after $\alpha$ has been done. So, if $[\alpha]\phi$ holds before $\alpha$ is done, $\phi$ will hold afterwards. Or, alternatively, $[\alpha]\phi$ means simply that if action $\alpha$ is done, $\phi$ will hold (afterwards). Hence $[\alpha]\phi$ is a more refined version of $\alpha \supset \phi$ in traditional deontic logic with the difference that now actions and assertions are separated, and a notion of time-lag is built in. $[\alpha]$ will be interpreted as a modal operator of the necessity ($\Box$) kind in a Kripke-structure induced by the performance of actions.

Formally, we introduce the sets *Act* of action expressions and *Ass* of assertions as follows. Let $A$ be a finite alphabet (i.e., set of symbols), denoting *elementary* or *atomic actions*. We use the letters $a, b, c$ with possible marks to range over $A$. The classes *Act* and *Ass* are the smallest sets satisfying the following clauses:

(i) $\underline{a} \in Act$ for every $a \in A$
(ii) constants $\underline{\varnothing} \in Act$, $\underline{U} \in Act$ pronounced as "*failure*" and "*whatever*", respectively.

For any $\alpha_1$, $\alpha_2 \in Act$ and $\phi \in Ass$:

(iii) The *sequential composition* $\alpha_1 ; \alpha_2 \in Act$, pronounced as "$\alpha_1$ followed by $\alpha_2$".
(iv) The *choice* $\alpha_1 \underline{\cup} \alpha_2 \in Act$, pronounced as "$\alpha_1$ or $\alpha_2$".
(v) The *joint* (or *simultaneous*) action $\alpha_1 \& \alpha_2 \in Act$, pronounced as "$\alpha_1$ together with $\alpha_2$".
(vi) The *conditional* action $\phi \rightarrow \alpha_1/\alpha_2 \in Act$, pronounced as "if $\phi$ then $\alpha_1$ else $\alpha_2$".
(vii) The *negated* action $\overline{\alpha_1} \in Act$, pronounced as "not-$\alpha_1$".

Note: The subclass of *Act* without conditional actions will be denoted by $Act_0$.

(1) *Ass* contains a fixed set of propositional letters.
(2) The special propositional letter $V \in Ass$.

For any $\phi_1$, $\phi_2 \in Ass$ and $\alpha \in Act$,

(3) $\phi_1 \vee \phi_2$, $\phi_1 \wedge \phi_2$, $\phi_1 \supset \phi_2$, $\phi_1 \equiv \phi_2$, $\neg\phi_1 \in Ass$
(4) $[\alpha]\phi_1$, $\langle\alpha\rangle\phi_1 \in Ass$.

**Notation** We use *true* and *false* as abbreviations of $\phi_0 \vee \neg\phi_0$ for some arbitrary, but fixed, $\phi_0 \in Ass$ and $\neg$*true*, respectively. Moreover, $F\alpha$ is an abbreviation of $[\alpha]V$, $O\alpha$ abbreviates $F\bar{\alpha}$, and $P\alpha$ stands for $\neg F\alpha$. In the sequel we let $p$ range over the least subset of *Ass* satisfying clauses (1) and (3), i.e., the sentences of Propositional Calculus.

***3 Informal semantics*** The semantics of the actions in *Act* given (informally) as follows. (For a *formal* semantics we refer to the appendix.)

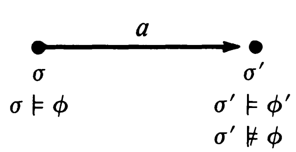
One has to imagine that one is in a state (world) $\sigma$, in which certain assertions hold. Then by doing an elementary action $a$ one moves to a next state $\sigma'$. In this state $\sigma'$ other assertions may hold than in $\sigma$, since $a$ might have changed something. For instance, if in $\sigma$ the proposition

$$\phi = \text{"vase } A \text{ has a blue color"}$$

is true and $a$ is the action "paint vase $A$ red", then the proposition $\phi$ clearly no longer holds in $\sigma'$, but

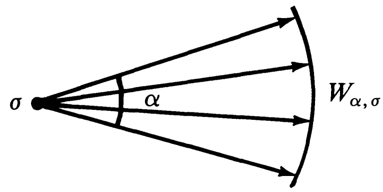
$$\phi' = \text{"vase } A \text{ is red"}$$

does. We picture this as follows



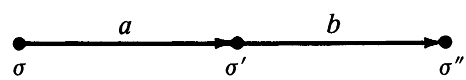

(We assume actions to terminate after a finite amount of time.)

In general, an action $\alpha \in Act$ may lead one into one of several possible states, due to the fact that we have a choice operator $\cup$ in our language. So $\alpha$ may map $\sigma$ into a *set* $W_{\alpha,\sigma}$ of states. In a picture:

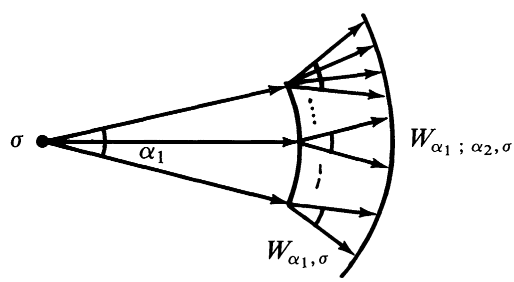



Now we can give the semantics of the various operators regarding *Act*:

(a) $\alpha_1 \,;\, \alpha_2$. This simply stands for doing $\alpha_1$ first and $\alpha_2$ next. For elementary actions $a, b$ we can picture it thus:

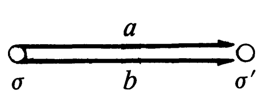



In general, however, we get

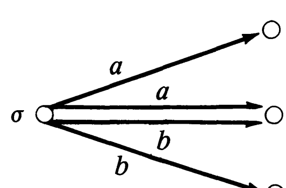



where $W_{\alpha_1 \,;\, \alpha_2,\sigma} = \cup_{\sigma' \in W_{\alpha_1,\sigma}} W_{\alpha_2,\sigma'}$

(b) $\alpha_1$ & $\alpha_2$: we perform $\alpha_1$ and $\alpha_2$ simultaneously. For elementary $a, b$:

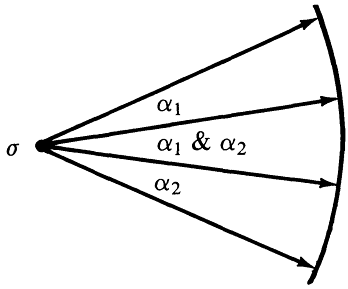



(c) $\alpha_1 \,\underline{\cup}\, \alpha_2$: we do either $\alpha_1$ or $\alpha_2$ (or both):




$a \cup b$

In general, $\alpha_1 \cup \alpha_2$ is pictured as




$$W_{\alpha_1 \underline{\cup} \alpha_2, \sigma} = W_{\alpha_1, \sigma} \cup W_{\alpha_2, \sigma} \cup W_{\alpha_1 \& \alpha_2, \sigma}$$

(This is slightly different from $\alpha_1 \cup \alpha_2$ in dynamic logic, where it is assumed that actions cannot occur in a really simultaneous manner.)

(d) $\phi \rightarrow \alpha_1/\alpha_2$: if $\phi$ holds in the state $\sigma$ you are in, then you perform action $\alpha_1$, and otherwise action $\alpha_2$. It must be emphasized that this conditional action $\phi \rightarrow \alpha_1/\alpha_2$ is something completely different from the conditional action $\alpha_1 \rightarrow \alpha_2$ in traditional deontic logic where *only* actions are involved. $\alpha_1 \rightarrow \alpha_2$, with intended meaning "if action $\alpha_1$ is done, $\alpha_2$ is done" is semantically more problematic and will be dealt with in Section 6.

In $\phi \rightarrow \alpha_1/\alpha_2$ the assertion $\phi$ is harmless in the sense that it does *not change* the current world. It is not an action but just an assertion about the current world, such as "the sun is shining (now)" or "it is (now in the current world) forbidden to do action $\alpha$".

Perhaps one wonders whether a *binary* conditional $\phi \rightarrow \alpha$, meaning if $\phi$ then do $\alpha$, can also be introduced as some special case of the ternary $\phi \rightarrow \alpha_1/\alpha_2$. The problem with this action $\phi \rightarrow \alpha$ is its status when $\phi$ does not hold: it still has to be an action, but which one is it then? We shall discuss this problem in Section 6 as well.

(e) $\bar{\alpha}$: *not*-$\alpha$. In this section the semantics of $\bar{\alpha}$ will not be specified fully in terms of the semantics of $\alpha$. We will not need this. It will be sufficient to consider $\bar{\alpha}$ as some action that satisfies the following axioms, which in our opinion must reasonably hold:

(I) $\bar{\bar{\alpha}} = \alpha$

(II) $\overline{\alpha_1 ; \alpha_2} = \overline{\alpha_1} \underline{\cup} (\alpha_1 ; \overline{\alpha_2})$

(III) $\overline{\alpha_1 \underline{\cup} \alpha_2} = \overline{\alpha_1} \& \overline{\alpha_2}$

(IV) $\overline{\alpha_1 \& \alpha_2} = \overline{\alpha_1} \underline{\cup} \overline{\alpha_2}$

(V) $\overline{\phi \rightarrow \alpha_1/\alpha_2} = \phi \rightarrow \overline{\alpha_1}/\overline{\alpha_2}$.

(In the appendix we present a model in which they hold, so these axioms are consistent with each other.)

Here we continue our informal discussion of the semantics. Observe that we do not require $\bar{\alpha}$ to be a set-complement of $\alpha$ in the sense that $\alpha \cup \bar{\alpha} = Act$ or anything of this kind. We do not impose this requirement because it implies that $\bar{\alpha} = Act\backslash\{\alpha\}$. This leads to trouble when we consider, e.g., $\bar{a} = \text{Act}\backslash\{a\}$ for elementary $a$: $\bar{a}$ now contains, for example, $ac$ for some elementary $c$ and this action $ac$ still involves doing action $a$ first. Whether one wants this or not cannot be determined a priori. Likewise we can argue about $\alpha$ & $\bar{\alpha}$. It is very reasonable to identify this joint action with the empty action $\varnothing$ (*'failure'*) which cannot be executed. In our model this corresponds to an empty set of successor states: $W_{\underline{\varnothing},\sigma} = \varnothing$ for every state $\sigma$. But what about $(\alpha;\alpha_1)$ & $(\bar{\alpha};\alpha_2)$? It seems very reasonable to identify this with $\varnothing$ as well, for we have to start with $\alpha$ *and* $\bar{\alpha}$ simultaneously. However, we shall leave this unspecified in this section, and we shall see that we do not need this exact specification to deal with many important theorems of Deontic Logic. Moreover, for the sublogic of prohibition, involving only $F$, $\bar{\alpha}$ will not be needed at all!

(f) $\underline{\varnothing}$. The action $\underline{\varnothing}$ ('failure') really denotes a complete nonaction in the sense that no action from the set of atomic actions is selected to be performed. This implies that the set of successor states after performing $\underline{\varnothing}$ is the empty set. In other words, $\underline{\varnothing}$ denotes an impossible action, since it "leads to nowhere".

(g) $\underline{U}$. The action $\underline{U}$ ('whatever') is complementary to $\underline{\varnothing}$ in the sense that when $\underline{U}$ is performed, some set of atomic actions—which is chosen in a nondeterministic manner—is performed simultaneously.

Concerning the semantics of assertions we define for $\phi \in Ass$:

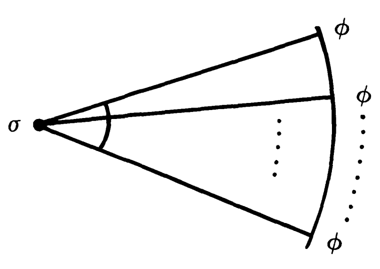
$$\sigma \models \phi \text{ iff } \phi \text{ holds in } \sigma$$

and

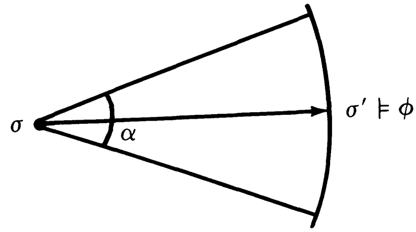
$$\models \phi \text{ iff } \phi \text{ holds for all } \sigma \in \Sigma, \text{ the universe of states.}$$

The semantics of the traditional connectives is as usual and we shall omit this here. For $[\alpha]\phi$, $\langle\alpha\rangle\phi$, and $V$ we have:

(a) $\sigma \models [\alpha]\phi$ iff $\forall\sigma' \in W_{\alpha,\sigma}$: $\sigma' \models \phi$. In a picture:




(b) $\langle\alpha\rangle\phi$ is the dual of $[\alpha]\phi$: $\sigma \models \langle\alpha\rangle\phi$ iff $\sigma \models \neg[\alpha]\neg\phi$, i.e., $\exists\sigma' \in W_{\alpha,\sigma}$: $\sigma' \models \phi$; i.e., there is some way by doing action $\alpha$ to achieve $\phi$.




(c) $V$ is a sentential constant denoting the so-called 'undesirable state-of-affairs', e.g., sanction, (liable to) punishment, trouble (with conscience, for example).

The abbreviations $F\alpha$, $O\alpha$, and $P\alpha$ are justified by:

(a) $\sigma \models F\alpha$ iff $\sigma \models [\alpha]V$; i.e., it is *forbidden* to do $\alpha$ in $\sigma$ iff one performs $\alpha$ in state $\sigma$ one gets into trouble.
(b) $\sigma \models O\alpha$ iff $\sigma \models F\bar{\alpha}$; i.e., it is *obligatory* to do $\alpha$ in $\sigma$ iff it is forbidden to do not-$\alpha$ in $\sigma$.
(c) $\sigma \models P\alpha$ iff $\sigma \models \neg F\alpha$; i.e., it is *permitted* to do $\alpha$ in $\sigma$ iff it is not forbidden to do $\alpha$ in $\sigma$.

The modal operator $[\alpha]$ is a very weak one: it is a derivative from system $K$. This is the smallest logical system that can be given an interpretation with Kripke-structures, with the additional property that $\models [\alpha]$ *true* (see [7]).

***4 The basic system PD$_e$L*** The system PD$_e$L of Propositional Deontic Logic is given by: ($\phi_1 \subset \phi_2$ stands for $\phi_2 \supset \phi_1$.)

**Axioms**

**(PC)** All tautologies of propositional calculus

**($\Box\supset$)** $\vdash [\alpha](\phi_1 \supset \phi_2) \supset ([\alpha]\phi_1 \supset [\alpha]\phi_2)$

**(;)** $\vdash [\alpha_1 ; \alpha_2]\phi \equiv [\alpha_1]([\alpha_2]\phi)$

**($\underline{\cup}$)** $\vdash [\alpha_1 \underline{\cup} \alpha_2]\phi \equiv [\alpha_1]\phi \wedge [\alpha_2]\phi$

**(&)** $\vdash [\alpha_1 \& \alpha_2]\phi \subset [\alpha_1]\phi \vee [\alpha_2]\phi$ (provided duration $(\alpha_1)$ = duration $(\alpha_2)$)[1]

**($\rightarrow$/)** $\vdash [\phi_1 \rightarrow \alpha_1/\alpha_2]\phi_2 \equiv (\phi_1 \supset [\alpha_1]\phi_2) \wedge (\neg\phi_1 \supset [\alpha_2]\phi_2)$

**($\Diamond$)** $\vdash \langle\alpha\rangle\phi \equiv \neg[\alpha]\neg\phi$

**($\bar{;}$)** $\vdash [\overline{\alpha_1 ; \alpha_2}]\phi \equiv [\bar{\alpha}_1]\phi \wedge [\alpha_1][\bar{\alpha}_2]\phi$

**($\bar{\underline{\cup}}$)** $\vdash [\overline{\alpha_1 \underline{\cup} \alpha_2}]\phi \subset [\bar{\alpha}_1]\phi \vee [\bar{\alpha}_2]\phi$ (provided duration $(\alpha_1)$ = duration $(\alpha_2)$)

**($\bar{\&}$)** $\vdash [\overline{\alpha_1 \& \alpha_2}]\phi \equiv [\bar{\alpha}_1]\phi \wedge [\bar{\alpha}_2]\phi$

**($\overline{\rightarrow}$/)** $\vdash [\overline{\phi_1 \rightarrow \alpha_1/\alpha_2}]\phi_2 \equiv (\phi_1 \supset [\bar{\alpha}_1]\phi_2) \wedge (\neg\phi_1 \supset [\bar{\alpha}_2]\phi_2)$

**($^{=}$)** $\vdash [\bar{\bar{\alpha}}]\phi \equiv [\alpha]\phi$

**($\varnothing$)** $\vdash [\varnothing]\phi$. ('failing').

**Rules**

**(MP)** $\dfrac{\vdash\phi,\ \vdash\phi\supset\psi}{\vdash\psi}$

**(N)** $\dfrac{\vdash\phi}{\vdash[\alpha]\phi}$.

The soundness of this system is given in the appendix. Note that we do *not* have

$$\models[\alpha_1\ \&\ \alpha_2]\phi\supset[\alpha_1]\phi\vee[\alpha_2]\phi,$$

as can be seen from the following example: take

$\alpha_1$ = painting the table red,
$\alpha_2$ = painting the table blue,
$\phi$ = *false*.

Now we have that $[\alpha_1\ \&\ \alpha_2]\phi$ is true, but neither $[\alpha_1]\phi$ nor $[\alpha_2]\phi$ is.

We now give a list of theorems and derived rules of $PD_eL$ and most of their proofs:

**Theorems**

(1) $\vdash[\alpha]\,true$
(2) $\vdash\neg\langle\alpha\rangle\,false$
(3) $\vdash[\alpha](\phi_1\wedge\phi_2)\equiv[\alpha]\phi_1\wedge[\alpha]\phi_2$
(4) $\vdash\langle\alpha\rangle(\phi_1\vee\phi_2)\equiv\langle\alpha\rangle\phi_1\vee\langle\alpha\rangle\phi_2$
(5) $\vdash[\alpha](\phi_1\vee\phi_2)\subset[\alpha]\phi_1\vee[\alpha]\phi_2$
(6) $\vdash\langle\alpha_1\,;\,\alpha_2\rangle\phi\equiv\langle\alpha_1\rangle\langle\alpha_2\rangle\phi$
(7) $\vdash\langle\alpha_1\,\underline{\cup}\,\alpha_2\rangle\phi\equiv\langle\alpha_1\rangle\phi\vee\langle\alpha_2\rangle\phi$
(8) $\vdash\langle\alpha_1\ \&\ \alpha_2\rangle\phi\supset\langle\alpha_1\rangle\phi\wedge\langle\alpha_2\rangle\phi$ (provided duration $(\alpha_2)$ = duration $(\alpha_2)$)
(9) $\vdash F(\alpha_1\,;\,\alpha_2)\equiv[\alpha_1]F\alpha_2$
(10) $\vdash F(\alpha_1\,\underline{\cup}\,\alpha_2)\equiv F\alpha_1\wedge F\alpha_2$
(11) $\vdash F(\alpha_1\ \&\ \alpha_2)\subset F\alpha_1\vee F\alpha_2$ (provided duration $(\alpha_1)$ = duration $(\alpha_2)$)
(12) $\vdash F(\phi\rightarrow\alpha_1/\alpha_2)\equiv(\phi\supset F\alpha_1)\wedge(\neg\phi\supset F\alpha_2)$
(13) $\vdash F(\mathit{false}\rightarrow\alpha_1/\alpha_2)\equiv F\alpha_2$
(14) $\vdash F(\mathit{true}\rightarrow\alpha_1/\alpha_2)\equiv F\alpha_1$
(15) $\vdash F(F\alpha_1\rightarrow\alpha_1/\alpha_2)\equiv F\alpha_1\vee F\alpha_2$
(16) $\vdash F\alpha_1\supset F(\alpha_1\ \&\ \alpha_2)$ (where duration $(\alpha_1)$ = duration $(\alpha_2)$) Penitent's Paradox I
(17) $\vdash O(\alpha_1\,;\,\alpha_2)\equiv O\alpha_1\wedge[\alpha_1]O\alpha_2$
(18) $\vdash O(\alpha_1\,\underline{\cup}\,\alpha_2)\subset O\alpha_1\vee O\alpha_2$ (where duration $(\alpha_1)$ = duration $(\alpha_2)$)
(19) $\vdash O(\alpha_1\ \&\ \alpha_2)\equiv O\alpha_1\wedge O\alpha_2$
(20) $\vdash O(\phi\rightarrow\alpha_1/\alpha_2)\equiv(\phi\supset O\alpha_1)\wedge(\neg\phi\supset O\alpha_2)$
(21) $\vdash(O\alpha_1\wedge O\alpha_2)\supset O(\phi\rightarrow\alpha_1/\alpha_2)$ Derived Obligation I
(22) $\vdash O(O\alpha_1\rightarrow\alpha_1/\alpha_2)\equiv O\alpha_1\vee O\alpha_2$
(23) $\vdash O(\alpha_1\ \&\ \alpha_2)\supset O\alpha_1$ Derived Obligation II
(24) $\vdash O\alpha_1\supset O(\alpha_1\,\underline{\cup}\,\alpha_2)$ (where duration $(\alpha_1)$ = duration $(\alpha_2)$) Ross's Paradox

(25) $\vdash O\bar{\alpha}_1 \supset O(\overline{\alpha_1 \,\&\, \alpha_2})$ (where duration $(\alpha_1)$ = duration $(\alpha_2)$)
Penitent's Paradox II

(26) $\vdash P\alpha \equiv \langle\alpha\rangle \neg V$

(27) $\vdash P\alpha \equiv \neg O\bar{\alpha}$

(28) $\vdash P(\alpha_1\,;\,\alpha_2) \equiv \langle\alpha_1\rangle P\alpha_2$

(29) $\vdash P(\alpha_1 \uplus \alpha_2) \equiv P\alpha_1 \vee P\alpha_2$

(30) $\vdash P(\alpha_1 \,\&\, \alpha_2) \supset P\alpha_1 \wedge P\alpha_2$ (provided duration $(\alpha_1)$ = duration $(\alpha_2)$)

(31) $\vdash P(\phi \rightarrow \alpha_1/\alpha_2) \equiv (\phi \supset P\alpha_1) \wedge (\neg\phi \supset P\alpha_2)$

(32) $\vdash P(P\alpha_1 \rightarrow \alpha_1/\alpha_2) \equiv P\alpha_1 \vee P\alpha_2$

(33) $\vdash [\alpha_1 \uplus \alpha_2]\phi \equiv [\alpha_1 \uplus \alpha_2 \uplus (\alpha_1 \,\&\, \alpha_2)]\phi$

(34) $\dfrac{\vdash \phi \supset \psi}{\vdash [\alpha]\phi \supset [\alpha]\phi}$

(35) $\dfrac{\vdash [\alpha_1]\phi_1,\ \vdash [\alpha_2]\phi_2}{\vdash [\alpha_1 \,\&\, \alpha_2]\ (\phi_1 \wedge \phi_2)}$ (provided duration $(\alpha_1)$ = duration $(\alpha_2)$

(36) $\dfrac{\vdash [\alpha_1]\phi_1,\ \vdash [\alpha_2]\phi_2}{\vdash [\alpha_1 \uplus \alpha_2]\,(\phi_1 \vee \phi_2)}$.

Some proofs (cf. [7], [8], [9]):

(1) Directly from $\vdash$*true* and (N)

(2) $\neg\langle\alpha\rangle$ *false* $\equiv \neg\langle\alpha\rangle \neg$*true* $\equiv [\alpha]$ *true*

(3) a. $[\alpha](\phi_1 \wedge \phi_2) \supset [\alpha]\phi_1 \wedge [\alpha]\phi_2$.
We use rule (34) to be derived below: observe

$$\phi_1 \wedge \phi_2 \supset \phi_1 \vdash_{(34)} [\alpha](\phi_1 \wedge \phi_2) \supset [\alpha]\phi_1$$

and

$$\phi_1 \wedge \phi_2 \supset \phi_2 \vdash_{(34)} [\alpha](\phi_1 \wedge \phi_2) \supset [\alpha]\phi_2.$$

Therefore we have that

$$\vdash([\alpha](\phi_1 \wedge \phi_2) \supset [\alpha]\phi_1) \wedge ([\alpha](\phi_1 \wedge \phi_2) \supset [\alpha]\phi_2).$$

Consequently, also

$$\vdash(\neg[\alpha](\phi_1 \wedge \phi_2) \vee [\alpha]\phi_1) \wedge (\neg[\alpha](\phi_1 \wedge \phi_2) \vee [\alpha]\phi_2)$$

which implies

$$\vdash\neg[\alpha](\phi_1 \wedge \phi_2) \vee ([\alpha]\phi_1 \wedge [\alpha]\phi_2)$$

i.e.,

$$\vdash[\alpha](\phi_1 \wedge \phi_2) \supset ([\alpha]\phi_1 \wedge [\alpha]\phi_2).$$

b. $[\alpha]\phi_1 \wedge [\alpha]\phi_2 \supset [\alpha](\phi_1 \wedge \phi_2)$.

$\phi_1 \supset (\phi_2 \supset (\phi_1 \wedge \phi_2)) \vdash_{(34)} [\alpha]\phi_1 \supset [\alpha](\phi_2 \supset (\phi_1 \wedge \phi_2)) \vdash$ (by $\Box\supset$ and MP)
$[\alpha]\phi_1 \supset ([\alpha]\phi_2 \supset [\alpha](\phi_1 \wedge \phi_2)) \vdash ([\alpha]\phi_1 \wedge [\alpha]\phi_2) \supset$
$[\alpha](\phi_1 \wedge \phi_2)$

(4) $\langle\alpha\rangle(\phi_1 \vee \phi_2) \equiv \neg[\alpha]\neg(\phi_1 \vee \phi_2) \equiv \neg[\alpha](\neg\phi_1 \wedge \neg\phi_2) \equiv$
$\neg([\alpha]\neg\phi_1 \wedge [\alpha]\neg\phi_2) \equiv \neg[\alpha]\neg\phi_1 \vee \neg[\alpha]\neg\phi_2 \equiv \langle\alpha\rangle\phi_1 \vee \langle\alpha\rangle\phi_2$

(5) $\phi_1 \supset \phi_1 \vee \phi_2 \vdash_{(34)} [\alpha]\phi_1 \supset [\alpha](\phi_1 \vee \phi_2)$
$\phi_2 \supset \phi_1 \vee \phi_2 \vdash_{(34)} [\alpha]\phi_2 \supset [\alpha](\phi_1 \vee \phi_2)$
Hence, $\vdash [\alpha]\phi_1 \vee [\alpha]\phi_2 \supset [\alpha](\phi_1 \vee \phi_2)$

(6) $\langle\alpha_1 ; \alpha_2\rangle\phi \equiv \neg[\alpha_1 ; \alpha_2]\neg\phi \equiv \neg[\alpha_1][\alpha_2]\neg\phi \equiv \neg[\alpha_1]\neg\langle\alpha_2\rangle\phi \equiv$
$\langle\alpha_1\rangle\langle\alpha_2\rangle\phi$

(7) $\langle\alpha_1 \underline{\cup} \alpha_2\rangle\phi \equiv \neg[\alpha_1 \underline{\cup} \alpha_2]\neg\phi \equiv \neg([\alpha_1]\neg\phi \wedge [\alpha_2]\neg\phi) \equiv$
$\neg[\alpha_1]\neg\phi \vee \neg[\alpha_2]\neg\phi \equiv \langle\alpha_1\rangle\phi \vee \langle\alpha_2\rangle\phi$

(8) $\langle\alpha_1 \,\&\, \alpha_2\rangle\phi \equiv \neg[\alpha_1 \,\&\, \alpha_2]\neg\phi \supset \neg([\alpha_1]\neg\phi \vee [\alpha_2]\neg\phi) \equiv$
$\neg[\alpha_1]\neg\phi \wedge \neg[\alpha_2]\neg\phi \equiv \langle\alpha_1\rangle\phi \wedge \langle\alpha_2\rangle\phi$

(9) $F(\alpha_1 ; \alpha_2) \equiv [\alpha_1 ; \alpha_2]V \equiv [\alpha_1][\alpha_2]V \equiv [\alpha_1]F\alpha_2$

(10) $F(\alpha_1 \underline{\cup} \alpha_2) \equiv [\alpha_1 \underline{\cup} \alpha_2]V \equiv [\alpha_1]V \wedge [\alpha_2]V \equiv F\alpha_1 \wedge F\alpha_2$

(13)–(15) Directly from (12).

(16) By (11)

(17) $O(\alpha_1 ; \alpha_2) \equiv F(\overline{\alpha_1 ; \alpha_2}) \equiv F(\bar{\alpha}_1 \underline{\cup} (\alpha_1 ; \bar{\alpha}_2)) \equiv F\bar{\alpha}_1 \wedge F(\alpha_1 ; \bar{\alpha}_2) \equiv$
$F\bar{\alpha}_1 \wedge [\alpha_1]F\bar{\alpha}_2 \equiv O\alpha_1 \wedge [\alpha_1]O\alpha_2$

(20) $O(\phi \to \alpha_1/\alpha_2) \equiv F(\overline{\phi \to \alpha_1/\alpha_2}) \equiv F(\phi \to \overline{\alpha_1}/\overline{\alpha_2}) \equiv$
$(\phi \supset [\overline{\alpha_1}]V) \wedge (\neg\phi \supset [\overline{\alpha_2}]V) \equiv (\phi \supset O\alpha_1) \wedge (\neg\phi \supset O\alpha_2)$.

(25) Directly from (16)

(26) $P\alpha \equiv \neg F\alpha \equiv \neg[\alpha]V \equiv \langle\alpha\rangle\neg V$

(27) $P\alpha \equiv \neg F\alpha \equiv \neg F\bar{\bar{\alpha}} \equiv \neg O\bar{\alpha}$

(30) $P(\alpha_1 \,\&\, \alpha_2) \equiv \neg F(\alpha_1 \,\&\, \alpha_2) \supset \neg(F\alpha_1 \vee F\alpha_2) \equiv$
$\neg F\alpha_1 \wedge \neg F\alpha_2 \equiv P\alpha_1 \wedge P\alpha_2$

(31) $P(\phi \to \alpha_1/\alpha_2) \equiv \neg F(\phi \to \alpha_1/\alpha_2) \equiv \neg(\phi \supset [\alpha_1]V) \vee \neg(\neg\phi \supset [\alpha_2]V) \equiv$
$(\phi \wedge \neg[\alpha_1]V) \vee (\neg\phi \wedge \neg[\alpha_2]V) \equiv (\phi \wedge P\alpha_1) \vee (\neg\phi \wedge P\alpha_2) \equiv$
$(\phi \supset P\alpha_1) \wedge (\neg\phi \supset P\alpha_2)$.

(33) $[\alpha_1 \underline{\cup} \alpha_2 \underline{\cup} (\alpha_1 \,\&\, \alpha_2)]\phi \equiv [\alpha_1]\phi \wedge [\alpha_2]\phi \wedge [\alpha_1 \,\&\, \alpha_2]\phi \equiv$
(since by (&): $[\alpha_1]\phi \wedge [\alpha_2]\phi \supset [\alpha_1 \,\&\, \alpha_2]\phi$:)
$[\alpha_1]\phi \wedge [\alpha_2]\phi \equiv [\alpha_1 \underline{\cup} \alpha_2]\phi$

(34) $\phi \supset \psi \vdash_{(N)} [\alpha](\phi \supset \psi) \vdash$
(by ($\Box\supset$) and (MP))
$[\alpha]\phi \supset [\alpha]\psi$

(35) $[\alpha_1]\phi_1 \vdash [\alpha_1]\phi_1 \vee [\alpha_2]\phi_1 \vdash$
(by (&) and (MP))
$[\alpha_1 \& \alpha_2]\phi_1$.
Likewise, $[\alpha_2]\phi_2 \vdash [\alpha_1 \& \alpha_2]\phi_2$.
Finally, $[\alpha_1 \& \alpha_2]\phi_1 \wedge [\alpha_1 \& \alpha_2]\phi_2 \vdash$
(by (3))
$[\alpha_1 \& \alpha_2](\phi_1 \wedge \phi_2)$.

(36) $[\alpha_1]\phi_1 \vdash [\alpha_1](\phi_1 \vee \phi_2)$ (by (5) and (MP)).
Likewise $[\alpha_2]\phi_2 \vdash [\alpha_2](\phi_1 \vee \phi_2)$
Finally, $[\alpha_1](\phi_1 \vee \phi_2) \wedge [\alpha_2](\phi_1 \wedge \phi_2) \vdash$
(by ($\underline{\cup}$))
$[\alpha_1 \underline{\cup} \alpha_2](\phi_1 \vee \phi_2)$.

***5 Remarks on the basic system and the derived theorems: Paradoxes and pseudo-paradoxes*** Among the theorems of the system we find very familiar ones, both evident truths and more controversial assertions. The desirable theorems include assertions such as:

(9) Maintaining that if $\alpha_1 ; \alpha_2$ is forbidden, it is forbidden to do $\alpha_2$ if $\alpha_1$ has been done already.
(10) $\alpha_1 \underline{\cup} \alpha_2$ is forbidden if both are forbidden.
(17) $\alpha_1 ; \alpha_2$ is obligatory iff $\alpha_1$ is obligatory and $\alpha_2$ is obligatory once $\alpha_1$ has been done.
(19) $\alpha_1 \& \alpha_2$ is obligatory iff both $\alpha_1$ and $\alpha_2$ are.
(26,27) It is permitted to do $\alpha$ iff there is a way to do $\alpha$ that avoids 'trouble' iff it is not obligated to do not-$\alpha$.

The following are also interesting:

(12) "If $\phi$ then do $\alpha_1$ else $\alpha_2$" is forbidden iff $\alpha_1$ is forbidden when $\phi$ holds and $\alpha_2$ is forbidden when $\phi$ does not hold.
(14) It is forbidden to do $true \to \alpha_1/\alpha_2$ iff it is forbidden to do $\alpha_1$. This is indeed to be expected, since $true \to \alpha_1/\alpha_2$ is not really a conditional action.
(15) It is forbidden to do: "$\alpha_1$ if $\alpha_1$ is forbidden or $\alpha_2$ otherwise" iff it is either forbidden to do $\alpha_1$ or it is forbidden to do $\alpha_2$.

Likewise for obligation and permission.

Well-known 'paradoxes' are also among our theorems, such as Ross's (24) and the paradoxes of derived obligation (21), (25). However, as is also argued in, for instance, [4], these are not real anomalies. For example, (21) is perfectly reasonable in asserting that if both $\alpha_1$ and $\alpha_2$ are obligatory, then also the conditional action "if $\phi$ then $\alpha_1$ else $\alpha_2$" resulting in doing either $\alpha_1$ or $\alpha_2$ (depending on the truth value of $\phi$) is obligatory. We shall return to Ross's Paradox presently, and propose a possible solution to this as well.

What is important to note, though, is that really undesirable assertions such as "Ought implies Can": $O\alpha \supset P\alpha$ and even worse $O\neg p \supset O(p \to \alpha)$, $O\alpha \supset OO\alpha$, $OO\alpha \supset \alpha$, $(Op \wedge (p \supset Oq)) \supset Oq$, and $(p \supset q) \supset O(p \supset q)$ (cf. [10], [4], [15]) are either false or nonsensical (not even well-formed) in our system.

Also, in some cases the paradox just vanishes. For example, consider the following version of the Chisholm paradox, stated in classical deontic logic as

$$Op \wedge O(p \rightarrow q) \wedge (\neg p \supset O\neg q) \wedge \neg p$$

with intended meaning that the following assertions hold:

1. It is obligatory to do $p$.
2. It is obligatory to do $q$ if $p$ has been done.
3. If $p$ has not been done it is obligatory to do not-$q$.
4. $p$ is not done.

Although its meaning is perfectly clear, it is problematical in classical deontic logic, since it involves an obligation (3) in case some other obligation is violated. However, the intention of it is represented in our system without any problem by the assertion

$$O\alpha_1 \wedge [\alpha_1]O\alpha_2 \wedge [\overline{\alpha_1}]O\overline{\alpha_2}$$

which implies, e.g., the assertion

$$O(\alpha_1 \,;\, \alpha_2) \wedge [\overline{\alpha_1}](V \wedge O\overline{\alpha_2}) \text{ or}$$
$$O(\alpha_1 \,;\, \alpha_2) \wedge [\overline{\alpha_1}](V \wedge F\alpha_2):$$

it is in principle obligatory to do $\alpha_1 \,;\, \alpha_2$, but if $\alpha_1$ is not done, then, besides already being liable to punishment for not doing $\alpha_1$, one is also forbidden to do $\alpha_2$. So instead of arriving at an inconsistency we get a meaningful assertion in this case.

The so-called Paradox of Free Choice Permission can be dealt with as well. Although we have as a theorem $P(\alpha_1 \cup \alpha_2) \equiv P\alpha_1 \vee P\alpha_2$ (just as in the standard system (cf. [6])), some authors argue that it would be far more reasonable to have $P(\alpha_1 \cup \alpha_2) \equiv P\alpha_1 \wedge P\alpha_2$, as follows: If one is permitted to do either $\alpha_1$ or $\alpha_2$, one may choose between $\alpha_1$ and $\alpha_2$, and therefore both $\alpha_1$ and $\alpha_2$ must be permitted. But, since this is not compatible with the standard system, it has remained a paradox in the literature (see, e.g., [20], [14]).

However, in our system this paradox can be resolved by realizing exactly what is meant by permission. By defining $P\alpha \equiv \neg F\alpha \equiv \langle\alpha\rangle\neg V$ we have taken a special (and useful) interpretation of permissibility: $\alpha$ is permitted iff in some way $\alpha$ can be done without getting into trouble (punishment). In common use of language, however, "it is permitted to do $\alpha$" means more: besides the possibility of doing $\alpha$ without being punished, it also suggests that the choice of how to perform $\alpha$ is left to the actor (cf. [20]). But then this more complicated notion of permission is captured exactly by defining $P_F\alpha \equiv P\alpha \wedge [\alpha]\neg V \equiv \langle\alpha\rangle\neg V \wedge [\alpha]\neg V$; i.e., it is possible to do $\alpha$ without getting into trouble and every way of doing $\alpha$ is allowed. This free choice permission $P_F$ now has the following reasonable property: $P_F(\alpha_1 \cup \alpha_2) \equiv (P_F\alpha_1 \wedge [\alpha_2]\neg V) \vee (P_F\alpha_2 \wedge [\alpha_1]\neg V)$, meaning: every way of doing $\alpha_1$ or $\alpha_2$ is allowed and, moreover, there is at least one way to do either $\alpha_1$ or $\alpha_2$.

Next in this section we propose a solution to Ross's Paradox: $O\alpha_1 \supset O(\alpha_1 \cup \alpha_2)$. That this assertion in a traditional form such as "one is obligated to post the letter implies that one is obligated to post the letter or burn it" is felt as paradoxical, results from the following interpretation. When one is obliged to perform an action $\alpha$, it is suggested that the way in which $\alpha$ is done is left to the actor. So when $O(\alpha_1 \cup \alpha_2)$ is true, it must be permitted to do either $\alpha_1$ or $\alpha_2$. This suggests the following definition of a more common use of obligation $O'$ by using the notion of free choice permissibility that was discussed above: $O'\alpha \equiv O\alpha \wedge P_F\alpha \equiv [\bar{\alpha}]V \wedge [\alpha]\neg V \wedge \langle\alpha\rangle\neg V$. For this $O'$ it holds no longer that $O'\alpha_1 \supset O'(\alpha_1 \cup \alpha_2)$, since it does not hold that $P_F\alpha_1 \supset P_F(\alpha_1 \cup \alpha_2)$.

We claim that the original notion $O$ of obligation can still be fruitfully used as a useful abstraction from the more involved natural language notion of obligation (just as the material implication in propositional logic is a useful simplification of the implication used in natural language). When really needed, the latter can be simulated in the system by $O'$, but only if the full context is known.

Finally, a few words about axiom

**(F)** $\vdash [\varnothing]\phi$ ('failing').

If one is asked to do an impossible action, e.g., $\alpha \mathbin{\&} \bar{\alpha}$, then there are no successor states and consequently every assertion $\phi$ is true in every successor state (since there are none). Although this axiom (F) seems to be not very meaningful, it now becomes possible to prove, e.g.,

**(DA)** $\vdash O(\alpha \cup \bar{\alpha})$ ('do anything');

i.e., it is obligatory to do something, which is sometimes a convenient property.

*Proof:* $[\varnothing]V \equiv [\alpha \mathbin{\&} \bar{\alpha}]V \equiv [\overline{\alpha \cup \bar{\alpha}}]V \equiv O(\alpha \cup \bar{\alpha})$.

***6 Possible extensions of the basic system $PD_eL$*** It is possible to use the basic system as a platform for further extension in several ways:

***6.1 Identities regarding joint and negated actions*** First, it is possible to give a more precise specification of the joint and negated actions. This can be done by giving identities such as the following (cf. the semantic model in the appendix):

$$\alpha_1 \mathbin{\&} (\alpha_2 \cup \alpha_3) = (\alpha_1 \mathbin{\&} \alpha_2) \cup (\alpha_1 \mathbin{\&} \alpha_3)$$
$$\alpha_1 \cup (\alpha_2 \mathbin{\&} \alpha_3) = (\alpha_1 \cup \alpha_2) \mathbin{\&} (\alpha_1 \cup \alpha_3)$$
$$\alpha \mathbin{\&} \bar{\alpha} = \varnothing \text{ and } \varnothing \cup \alpha = \alpha$$
$$\alpha \mathbin{\&} \alpha = \alpha \cup \alpha = \alpha$$
$$\alpha_1 \mathbin{\&} (\alpha_1 \,;\, \alpha_2) = \alpha_1 \,;\, \alpha_2.$$

The last identity can be explained informally by an example: "opening the door" together with "opening the door and then leaving" is the same as "opening the door and then leaving."

Furthermore, if one introduces a notion of *duration* of an action (for example, the maximal number of elementary actions that are sequentially composed), we can give the following more general identity

$$\alpha_1 \mathbin{\&} (\alpha_2 \,;\, \alpha_3) = (\alpha_1 \mathbin{\&} \alpha_2);\alpha_3,$$

provided that duration $(\alpha_1)$ = duration $(\alpha_2)$. For example, "knocking on the door for a minute" together with "whistling for a minute and then entering" is the same as "knocking on the door (for a minute) together with whistling for a minute, and then entering".

These semantical identities can be added as axioms yielding additional theorems. For instance, $\vdash O(\alpha_1 \underline{\cup} \alpha_2) \wedge F\alpha_1 \supset O\alpha_2$.

*Proof:* $O(\alpha_1 \underline{\cup} \alpha_2) \wedge F\alpha_1 \equiv O(\alpha_1 \underline{\cup} \alpha_2) \wedge O\overline{\alpha_1} \equiv O((\alpha_1 \underline{\cup} \alpha_2) \,\&\, \overline{\alpha_1}) \equiv O((\alpha_1 \,\&\, \overline{\alpha_1}) \underline{\cup} (\alpha_2 \,\&\, \overline{\alpha_1})) \equiv O(\varnothing \underline{\cup} (\alpha_2 \,\&\, \overline{\alpha_1})) \equiv O(\alpha_2 \,\&\, \overline{\alpha_1}) \equiv O\alpha_2 \wedge O\overline{\alpha_1} \equiv O\alpha_2 \wedge F\alpha_1$. So $O(\alpha_1 \underline{\cup} \alpha_2) \wedge F\alpha_1 \supset O\alpha_2 \wedge F\alpha_1 \supset O\alpha_2$.

***6.2 Conditional actions $\alpha_1 \to \alpha_2$*** Secondly, we can extend our language of actions with, for example, a conditional action $\alpha_1 \to \alpha_2$, meaning that if $\alpha_1$ is done, $\alpha_2$ will be done afterwards. We can introduce $\alpha_1 \to \alpha_2$ in the system by means of the definition

$$\alpha_1 \to \alpha_2 = \bar{\alpha}_1 \underline{\cup} (\alpha_1 \,;\, \alpha_2)$$

asserting that $\alpha_1 \to \alpha_2$ means that either $\alpha_1$ is not done or $\alpha_1$ is done followed by $\alpha_2$. In combination with the first extension involving additional semantical identities, we can now (im)prove the analoga of the problematical

$$O(A \to B) \equiv A \to OB \text{ and}$$
$$O(A \to B) \to (OA \to OB)$$

of traditional deontic logic:

(37) $\vdash O(\alpha_1 \to \alpha_2) \equiv [\alpha_1] O\alpha_2$

and

(38) $\vdash O(\alpha_1 \to \alpha_2) \supset (O\alpha_1 \supset O(\alpha_1 \,;\, \alpha_2))$.

*Proof:*
(37) $O(\alpha_1 \to \alpha_2) \equiv O(\bar{\alpha}_1 \underline{\cup} (\alpha_1 \,;\, \alpha_2)) \equiv [\overline{\bar{\alpha}_1 \underline{\cup} (\alpha_1 \,;\, \alpha_2)}]V \equiv$
$[\bar{\bar{\alpha}}_1 \,\&\, \overline{\alpha_1 \,;\, \alpha_2}]V \equiv [\alpha_1 \,\&\, (\bar{\alpha}_1 \underline{\cup} \alpha_1 \,;\, \bar{\alpha}_2)]V \equiv$
$[(\alpha_1 \,\&\, \bar{\alpha}_1) \underline{\cup} (\alpha_1 \,\&\, (\alpha_1 \,;\, \bar{\alpha}_2))]V \equiv [\alpha_1 \,\&\, (\alpha_1 \,;\, \bar{\alpha}_2)]V \equiv$
$[\alpha_1 \,;\, \bar{\alpha}_2]V \equiv [\alpha_1] O\alpha_2$

(38) $O(\alpha_1 \to \alpha_2) \wedge O\alpha_1 \equiv [\alpha_1] O\alpha_2 \wedge O\alpha_1 \equiv O(\alpha_1 \,;\, \alpha_2)$.

Regarding prohibition and permissibility of $\alpha_1 \to \alpha_2$ we obtain

(39) $\vdash F(\alpha_1 \to \alpha_2) \equiv O\alpha_1 \wedge F(\alpha_1 \,;\, \alpha_2)$

(40) $\vdash P(\alpha_1 \to \alpha_2) \equiv O\alpha_1 \supset P(\alpha_1 \,;\, \alpha_2)$.

*Proof:*
(39) $F(\alpha_1 \to \alpha_2) \equiv [\overline{\alpha_1} \underline{\cup} (\alpha_1 \,;\, \alpha_2)]V \equiv [\overline{\alpha_1}]V \wedge [\alpha_1][\alpha_2]V \equiv$
$O\alpha_1 \wedge F(\alpha_1 \,;\, \alpha_2)$

(40) $P(\alpha_1 \to \alpha_2) \equiv \neg F(\alpha_1 \to \alpha_2) \equiv \neg(O\alpha_1 \wedge F(\alpha_1 \,;\, \alpha_2)) \equiv$
$\neg O\alpha_1 \vee \neg F(\alpha_1 \,;\, \alpha_2) = O\alpha_1 \supset P(\alpha_1 \,;\, \alpha_2)$.

Perhaps these results are somewhat surprising, but it indicates that the intended meaning of $\alpha_1 \to \alpha_2$ is to be considered carefully. In our view it means that either $\alpha_1 \,;\, \alpha_2$ is done or the "escape route" $\overline{\alpha_1}$! So if $\alpha_1 \to \alpha_2$ is forbidden, the escape $\overline{\alpha_1}$ is forbidden as well! Otherwise there would be no difference between $\alpha_1 \to \alpha_2$ and $\alpha_1 ; \alpha_2$! And if $\alpha_1 \to \alpha_2$ is permitted, it is permitted to do $\overline{\alpha_1}$ or it is permitted to do $\alpha_1 ; \alpha_2$. In other words, if one is obligated to do $\alpha_1$ (and not to use the escape route) then $\alpha_1 ; \alpha_2$ must be permitted.

So although $\alpha_1 \to \alpha_2$ and $\alpha_1 ; \alpha_2$ are closely connected, we now have the following discriminating theorems about them:

(37) $\vdash O(\alpha_1 \to \alpha_2) \equiv [\alpha_1]O\alpha_2$
(17) $\vdash O(\alpha_1 ; \alpha_2) \equiv O\alpha_1 \wedge [\alpha_1]O\alpha_2$
(39) $\vdash F(\alpha_1 \to \alpha_2) \equiv O\alpha_1 \wedge F(\alpha_1 ; \alpha_2)$
(9) $\vdash F(\alpha_1 ; \alpha_2) \equiv [\alpha_1]F\alpha_2$
(40) $\vdash P(\alpha_1 \to \alpha_2) \equiv O\alpha_1 \supset P(\alpha_1 ; \alpha_2)$
(28) $\vdash P(\alpha_1 ; \alpha_2) \equiv \langle\alpha_1\rangle P\alpha_2$.

Next we might introduce a binary conditional action $\phi \to \alpha$, meaning if $\phi$ holds then do $\alpha$. As we mentioned before, the problem is what this *action* comprises when $\phi$ does not hold, and this problem on the level of actions is grossly underestimated in the literature. Three obvious possibilities arise:

(1) $\phi \to \alpha$ *fails* (i.e., $= \underline{\emptyset}$) if $\neg\phi$ holds,
(2) $\phi \to \alpha$ when $\neg\phi$ is an *idle* or *dummy* action $\underline{d}$ yielding an identical successor state (i.e., $W_{d,\sigma} = \{\sigma\}$), or
(3) we *do not care* what $\phi \to \alpha$ is when $\neg\phi$ holds, i.e., in that case $\phi \to \alpha$ is the universal action $\underline{U} = \bigcup_{\alpha \in Act'} \alpha$, so every action in $Act'$ will do, where $Act' = \{\alpha \in Act \mid \text{duration } (\alpha) = 1\}$.

So we can introduce three varieties of $\phi \to \alpha$:

$\phi \to_1 \alpha = \phi \to \alpha/\underline{\emptyset}$
$\phi \to_2 \alpha = \phi \to \alpha/\underline{d}$
$\phi \to_3 \alpha = \phi \to \alpha/\underline{U}$.

To make a choice we have to consider the properties of $\emptyset$, $d$ and $U$: clearly the following facts hold:

(41) $F\underline{\emptyset} \equiv [\underline{\emptyset}]V$

and as we shall see later on, we can identify $[\emptyset]V$ with *true*, so

(42) $F\underline{\emptyset} \equiv [\underline{\emptyset}]V \equiv true$
(43) $F\underline{d} \equiv [\underline{d}]V \equiv V$
(44) $F\underline{U} \equiv [\underline{U}]V \equiv \bigwedge_{\alpha \in Act'} [\alpha]V$.

((44) expresses that every action is forbidden.)

(45) $P\underline{\emptyset} \equiv \langle\underline{\emptyset}\rangle\neg V \equiv \neg[\underline{\emptyset}]V \equiv false$
(46) $P\underline{d} \equiv \langle\underline{d}\rangle\neg V \equiv \neg V$
(47) $P\underline{U} \equiv \langle\underline{U}\rangle\neg V \equiv \bigvee_{\alpha \in Act'} \langle\alpha\rangle\neg V$.

((47) expresses that some action is permitted.)

Furthermore, since it is reasonable to have $\overline{\underline{\emptyset}} = \underline{U}$ and $\overline{\underline{U}} = \underline{\emptyset}$, we obtain

(48) $O\underline{\emptyset} \equiv [\overline{\underline{\emptyset}}]V \equiv [\underline{U}]V \equiv F\underline{U}$.
(49) $O\underline{d} \equiv [\overline{\underline{d}}]V$.

(50) $O\underline{U} \equiv [\bar{\underline{U}}]V \equiv [\varnothing]V \equiv true.$

((50) expresses that it is obligatory to do *something* (albeit a dummy action).)

Now, using (12), (20), and (31) we find that

(51) $F(\phi \rightarrow_1 \alpha) \equiv F(\phi \rightarrow \alpha/\underline{\varnothing}) \equiv (\phi \supset F\alpha) \wedge (\neg\phi \supset F\underline{\varnothing}) \equiv (\phi \supset F\alpha)$

(52) $F(\phi \rightarrow_2 \alpha) \equiv F(\phi \rightarrow \alpha/\underline{d}) \equiv \phi \supset F\alpha) \wedge (\neg\phi \supset V)$

(53) $F(\phi \rightarrow_3 \alpha) \equiv F(\phi \rightarrow \alpha/\underline{U}) \equiv (\phi \supset F\alpha) \wedge (\neg\phi \supset F\underline{U})$

(54) $P(\phi \rightarrow_1 \alpha) \equiv P(\phi \rightarrow \alpha/\varnothing) \equiv (\phi \supset P\alpha) \wedge (\neg\phi \supset P\underline{\varnothing}) \equiv$
$(\phi \supset P\alpha) \wedge (\neg\phi \supset false) \equiv (\phi \supset P\alpha) \wedge \phi \equiv \phi \wedge P\alpha$

(55) $P(\phi \rightarrow_2 \alpha) \equiv P(\phi \rightarrow \alpha/d) \equiv (\phi \supset P\alpha) \wedge (\neg\phi \supset \neg V)$

(56) $P(\phi \rightarrow_3 \alpha) \equiv P(\phi \rightarrow \alpha/\underline{U}) \equiv (\phi \supset P\alpha) \wedge (\neg\phi \supset P\underline{U})$

(57) $O(\phi \rightarrow_1 \alpha) \equiv O(\phi \rightarrow \alpha/\underline{\varnothing}) \equiv (\phi \supset O\alpha) \wedge (\neg\phi \supset O\underline{\varnothing}) \equiv$
$(\phi \supset O\alpha) \wedge (\neg\phi \supset F\underline{U})$

(58) $O(\phi \rightarrow_2 \alpha) \equiv O(\phi \rightarrow \alpha/\underline{d}) \equiv (\phi \supset O\alpha) \wedge (\neg\phi \supset O\underline{d})$

(59) $O(\phi \rightarrow_3 \alpha) \equiv O(\phi \rightarrow \alpha/\underline{U}) \equiv (\phi \supset O\alpha) \wedge (\neg\phi \supset O\underline{U}) \equiv$
$(\phi \supset O\alpha) \wedge (\neg\phi \supset true) \equiv \phi \supset O\alpha.$

Although $\phi \rightarrow_1 \alpha$ has an intuitively sound property regarding prohibition, it has a peculiar property regarding obligation and permissibility; e.g., $P(\phi \rightarrow_1 \alpha)$ implies that $\phi$ holds. This can be argued as follows: if $\neg\phi$ would hold the action $\phi \rightarrow_1 \alpha$ would fail, yielding no successor state. But then it cannot be permissible to do $\phi \rightarrow_1 \alpha$ since this guarantees that by doing $\phi \rightarrow_1 \alpha$ there *is* a successor state in which $\neg V$ holds, i.e., where there is no liability to punishment. So $\neg\phi$ cannot hold in case $P(\phi \rightarrow_1 \alpha)$ holds.

$\phi \rightarrow_2 \alpha$ has strange properties regarding all three: *F*, *O*, and *P*: in all three cases there is some requirement when $\neg\phi$ holds, e.g., $O(\phi \rightarrow_2 \alpha)$ implies that it is obligatory to do an idle step (to *do* nothing) if $\neg\phi$ holds.

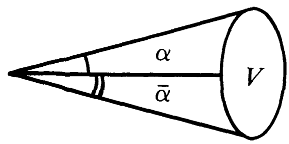
Personally, we find $\phi \rightarrow_3 \alpha$ the best choice. (59) and (56) are nice properties, e.g., (56) states that if $\neg\phi$ holds it is permitted to do *some* action. Only, (53) asserts that $F(\phi \rightarrow_3 \alpha)$ is stronger than just $\phi \supset F\alpha$, thus creating a difference between the prohibition of a conditional action and a conditional prohibition. In this interpretation, when it is forbidden to do $\alpha$ if $\phi$ holds (and we do not care otherwise), we have that it is also forbidden to do: "$\alpha$ if $\phi$ holds and something unspecified otherwise".

Anyway, this analysis shows that sometimes a too lighthearted view is taken of conditional actions and their relationship with conditional prohibitions, obligations, and permissions. It is worthwhile to ponder what it means to do a conditional action, especially when the condition involved is not fulfilled, before considering its prohibition, obligation, or permissibility.

***6.3 Additional axioms*** Thirdly, we can add directly axioms for assertions. For example we can augment the system by

(NCO) $\vdash \neg O(\alpha \,\&\, \bar{\alpha})$ ('no conflicting obligations')

which states that it is not obligatory to do some inconsistent (impossible) action. This is not necessarily true in the basic system, since it may be the case that for some $\alpha$, both $[\alpha]V$ and $[\bar{\alpha}]V$. In a picture:




However, this implies $O\bar{\alpha} \wedge O\alpha$ and hence also $O(\bar{\alpha} \,\&\, \alpha)$.

Using this extra axiom we are able to derive some extra theorems such as

(60) $\vdash \neg F(\bar{\alpha} \cup \alpha)$

stating that is is not forbidden to do something: either $\bar{\alpha}$ or $\alpha$.

*Proof:* $\neg O(\alpha \,\&\, \bar{\alpha}) \equiv \neg F(\overline{\alpha \,\&\, \bar{\alpha}}) = \neg F(\bar{\alpha} \cup \alpha)$.

(61) $\vdash O\alpha \supset P\alpha$.

When $\alpha$ is obligatory, it is also permissible.

*Proof:* $\neg O(\alpha \,\&\, \bar{\alpha}) \equiv \neg(O\alpha \wedge O\bar{\alpha}) \equiv \neg O\alpha \vee \neg O\bar{\alpha} \equiv O\alpha \supset \neg O\bar{\alpha} \equiv O\alpha \supset P\alpha$.

(62) $\vdash F\alpha \supset \neg O\alpha$.

When $\alpha$ is forbidden, it is not obligatory.

*Proof:* $O\alpha \supset P\alpha \equiv \neg P\alpha \supset \neg O\alpha \equiv F\alpha \supset \neg O\alpha$.

Another example of an additional axiom is the following:

(NP) $\vdash V \supset [\alpha]V$ ('no pardon')

stating that punishment cannot be remitted by any action.

***6.4 Further extensions*** Further extensions can be made by introducing a new constant $R$, meaning liability to some *reward*, and defining a new operator based upon this $R$: $\odot\alpha \equiv [\alpha]R$, expressing that doing $\alpha$ leads to some reward. For this $\odot$ it holds, e.g., that $\odot(\alpha_1 \cup \alpha_2) \equiv \odot\alpha_1 \wedge \odot\alpha_2$, suggesting a relationship with Kenney's Logic of Satisfactoriness (e.g., [13]).

By using Dynamic Logic it also becomes possible to express the executability of an action: $M\alpha \equiv \langle\alpha\rangle$ *true* now asserts that the action $\alpha$ is possible (executable). Obviously $P\alpha \supset M\alpha$, but not conversely. This means that permissibility implies possibility, which is not altogether counterintuitive.

The anonymous referee of this paper suggested one further extension, viz. that of the DO predicate for actions. DO($\alpha$) should have the meaning of stating that $\alpha$ is (will be) done. First, let me draw the reader's attention to the fact that every assertion in $PD_eL$ states something *hypothetical* about the effects of actions: $[\alpha]\phi$ asserts that if (or perhaps better: whenever) $\alpha$ is performed, $\phi$ will hold, regardless of the fact that $\alpha$ is or is not performed *actually*. So $PD_eL$ assertions involving actions are not merely 'material implications'. If we would have a DO predicate, it would certainly *not* hold that: $\neg DO(\alpha) \supset [\alpha]\phi$, for arbitrary $\phi$, since the truth of $[\alpha]\phi$ does not depend on the actual performance of $\alpha$. This is the reason that in the translation of the Chisholm paradox into $PD_eL$ we did not need a translation of $\neg p$ (meaning action $p$ is not done). The resulting $PD_eL$ formula holds whether $p$ (cq. $\alpha_1$) is done or not, and in case we know that $p$ (or $\alpha_1$) is not done, it says that one is already liable to punishment and still forbidden to do $q$ (cq. $\alpha_2$).

Returning to our question about the suitability of a DO predicate, I think that, besides not being needed, it is not an interesting predicate either. For instance, what would one like to infer from $[\alpha]\phi \wedge \mathrm{DO}(\alpha)$? One cannot infer $\phi$, since we have not actually *done* $\alpha$ yet. However, this discussion suggests a more meaningful predicate: $\mathrm{DONE}(\alpha)$, meaning that action $\alpha$ *has been* performed. This predicate appears to have interesting properties such as, e.g., $[\alpha]\phi \vdash (\mathrm{DONE}(\alpha) \supset \phi)$, $[\alpha]\ \mathrm{DONE}(\alpha)$, $\mathrm{DONE}(\alpha_1 \cup \alpha_2) \equiv \mathrm{DONE}(\alpha_1) \vee \mathrm{DONE}(\alpha_2)$, $\mathrm{DONE}(\alpha_1\ \&\ \alpha_2) \equiv \mathrm{DONE}(\alpha_1) \wedge \mathrm{DONE}(\alpha_2)$ and $\mathrm{DONE}(\bar{\alpha}) \equiv \neg\mathrm{DONE}(\alpha)$. It will be interesting to research this in more depth.

Finally, of course, it is also possible to extend $PD_eL$ to first-order or higher-order logics, and it might as well be beneficial in some cases to introduce more or less the inverse $\phi[\alpha]$ of $[\alpha]\phi$, asserting the strongest postcondition of an action $\alpha$ given precondition $\phi$: $\phi[\alpha]$ holds after $\alpha$ has been done, if $\phi$ held before $\alpha$. These issues deserve further study.

*7 Conclusion* The system for (Propositional) Deontic Logic presented in this paper (together with its semantics) provides a very workable framework for reasoning with deontic concepts. It does not contain the very nasty paradoxes that often appear in other systems in the literature, especially where the connection between actions and assertions is concerned. Although based upon Anderson's idea for reduction (cf. [2], [15]), it lacks the undesirable consequences of Anderson's original reductions. Some of the troublesome theorems appearing there (such as $OOp \supset Op$) are not even well-formed expressions in our system. Instead, Anderson's idea is used in a modified version and leads to many desirable results.

An interesting open technical question is whether an expression can be found that is *equivalent* to $[\alpha_1\ \&\ \alpha_2]\phi$ instead of just an expression implying it. Also, more generally, the issue of completeness is still open. Preferably, this question should be settled upon two levels: upon the level of dynamic logic given some action calculus and, if possible, also upon the underlying level of the action calculus itself. However, for the latter, choices have to be made concerning the exact interpretation of the negated and joint actions.

We remark that by the reduction to dynamic logic we need not restrict ourselves to so-called 'deontically perfect worlds' to give a meaning to the deontic expressions in our system. These perfect worlds appear quite frequently in the literature (e.g., [6], [10]), and involve additional problems of a semantical nature. Since we do not need them, our system can deal with conflicting obligations without running into logical inconsistency. For example, $O(\alpha\ \&\ \bar{\alpha})(\equiv O\alpha \wedge O\bar{\alpha})$, more or less means that it is obligatory to do something impossible, or to put it differently (by $[\alpha]V \wedge [\bar{\alpha}]V$): whatever one will do or try, he will get into "trouble" ($V$) anyway. And this is a perfectly reasonable statement. We regard this ability to talk about such conflicting obligations (prohibitions) as a nice property of our logic, for once in a while obligations to impossibilities do occur in real life and are therefore not impossible themselves.

However, if one wants to exclude conflicting prohibitions and obligations, one can add another axiom to the basic system:

(NCO) $\vdash \neg O(\alpha \mathbin{\&} \bar{\alpha})$

which asserts that it cannot be the case that one is obliged to do conflicting actions. When one adopts this axiom of nonconflicting obligations, additional theorems can be obtained such as

(60) $\vdash \neg F(\bar{\alpha} \mathbin{\underline{\cup}} \alpha)$

asserting that is is not forbidden to do: either $\alpha$ or not-$\alpha$,

(61) $\vdash O\alpha \supset P\alpha$,

saying that if $\alpha$ is obligatory, it is also permissible, and

(62) $\vdash F\alpha \supset \neg O\alpha$,

"if some action is forbidden, it is not obligatory".

Finally, we have seen in Section 6 that the basic system can also be extended in other ways and provides a sound basis for the exploration of more specific deontic problems.

## Appendix: A Formal Semantics for Action Expressions

*A1 A semantical domain to deal with simultaneous actions* Since we have *simultaneous* or *concurrent* actions in our language, we must provide some way of dealing with these constructs semantically. To this end we first define the notion of a *synchronicity set*:

**Definition** A *synchronicity set* (s-set) is a (finite!) *nonempty* subset of $A$.

**Notation** We use $S, S_1, S_2, \ldots, S', \ldots$ for synchronicity sets, and in concrete cases we write such a set using square brackets. For instance, the s-set consisting of the atomic actions $a$ and $b$ is written as $\begin{bmatrix} a \\ b \end{bmatrix}$. The powerset of s-sets will be denoted by $\mathcal{P}^+(A)$.

A synchronicity set will be used to indicate which atomic actions are to be executed simultaneously. We must be able to compose s-sets sequentially and for this we introduce *synchronicity traces.*

**Definition** A *synchronicity trace* (s-trace) is a finite or infinite sequence $S_1, \ldots, S_n, \ldots$ *of s-sets* $S_i$. We refer to the number of s-sets in an s-trace $t$ as the *length* of $t$ (so possibly length $(t) = \infty$).

We use $t, t_1, t_2, \ldots, t', \ldots$ for s-traces. Synchronicity traces may be concatenated in the following way. If $t = S_1 \ldots S_n$ and $t' = S_1' \ldots S_m'$ are s-traces, then the concatenation $t \circ t'$ is the s-trace $S_1 \ldots S_n\, S_1' \ldots S_m'$. If $t$ is infinite, $t \circ t' = t$.

To treat the nondeterministic operator $\underline{\cup}$ we need for our semantic domain (possibly infinite) *sets* of s-traces (or s-trace-sets). We use $T, T_1, T_2, \ldots, T', \ldots$

to denote s-trace-sets and the standard notation $T = \{t_1, t_2, \ldots\}$ for these sets. The concatenation $T \circ T'$ of s-trace-sets will stand for the set $\{t \circ t' \mid t \in T, t' \in T'\}$. Note that $T \circ \varnothing = \varnothing \circ T = \varnothing$.

Denotations of action expressions will be sets of infinite s-traces. Technically, this will facilitate the definitions of our semantical operators below. Intuitively, an action expression denotes a set of s-traces, which are specified up to a certain length and unspecified beyond this length. This unspecified part is always infinite corresponding with an unspecified future (cq. infinite amount of time). To indicate which part of an s-trace is specified and which is not, we, formally, use traces of pairs $\langle S, i \rangle$ where $S$ is an s-set and $i \in \{0,1\}$. $\langle S,1 \rangle$ denotes that the s-set $S$ is *specified* (or *relevant*), whereas $\langle S,0 \rangle$ denotes that $S$ is unspecified or irrelevant. We will view $\langle S, i \rangle$ as an s-set with additional information, and to stress this we use the more informal notation $S^{(i)}$ instead of $\langle S, i \rangle$. Concatenation of these 'annotated' s-sets is defined just as for s-sets. If $T$ is a collection of s-sets: $T = \{S_1, S_2, \ldots\}$, $T^{(i)} =_{df} \{S_1^{(i)}, S_2^{(i)}, \ldots\}$.

Our semantical domain will be the collection $\mathcal{C}$ of set of infinite s-traces that have a *finite* relevant (specified) part followed by an *infinite* irrelevant (unspecified) part. The empty set $\varnothing$ is included in $\mathcal{C}$.

For example, the following sets are in $\mathcal{C}$:

1. $[a]^{(1)} \circ [b]^{(1)} \circ [c]^{(0)} \circ [c]^{(0)} \circ \ldots$
2. $[a]^{(1)} \circ [b]^{(0)} \circ \mathcal{P}^+(A)^{(0)} \circ \mathcal{P}^+(A)^{(0)} \circ \ldots$
3. $[a]^{(1)} \circ [a]^{(1)} \circ \mathcal{P}^+(A)^{(1)} \circ \mathcal{P}^+(A)^{(0)} \circ \mathcal{P}^+(A)^{(0)} \circ \ldots$.

*Not* included in $\mathcal{C}$ are, e.g.,

4. $[b]^{(1)} \circ [a]^{(0)} \circ \mathcal{P}^+(A)^{(1)} \circ \mathcal{P}^+(A)^{(0)} \circ \ldots$
5. $[a]^{(0)} \circ [b]^{(1)} \circ [c]^{(0)} \circ [c]^{(0)} \circ \ldots$.

In order to compose elements in $\mathcal{C}$ in a meaningful way we need the following operator:

**Definition** Let $t \in T$ for some $T \in \mathcal{C}$, i.e., $t = S_1^{(1)} \ldots S_n^{(1)} S_{n+1}^{(0)} S_{n+2}^{(0)} \ldots$. Then $cut(t) =_{df} S_1^{(1)} \ldots S_n^{(1)}$ is called the *relevant* (or *specified*) part of $t$; $S_{n+1}^{(0)} S_{n+2}^{(0)} \ldots$ is called *irrelevant* or *unspecified*. For T $\in \mathcal{C}$, cut $(T) =_{df} \bigcup_{t \in T} cut(t)$.

**Definition** For $t \in T$ we refer to the length of $cut(t)$ as the *relevant length* of $t$.

*Example:* The relevant length of $t = [a]^{(1)} \circ [b]^{(1)} \circ [c]^{(0)} \circ \mathcal{P}^+(A)^{(0)} \circ \mathcal{P}^+(A)^{(0)} \circ \ldots$ is 2, since $cut(t) = [a]^{(1)} \circ [b]^{(1)}$, which has length 2.

Next we need operations $\Cap$ and $\sim$ on s-trace-sets in $\mathcal{C}$ that we shall use as semantical counterparts of the syntactical operators & and $\bar{}$ respectively.

**Definition** $\Cap$ on annotated s-sets $\langle S_1, i_1 \rangle$ and $\langle S_2, i_2 \rangle$ is given by:

$$\langle S_1, i_1 \rangle \Cap \langle S_2, i_2 \rangle = \begin{cases} \langle S, \max(i_1, i_2) \rangle & \text{if } S_1 = S_2 = S \\ \varnothing & \text{if } S_1 \neq S_2. \end{cases}$$

For $t = S_1^{(i_1)} S_2^{(i_2)} \ldots$ and $t' = S_1'^{(i_1)} S_2'^{(i_2)} \ldots$ we define:

$$t \,\dot{\cap}\, t' = (S_1^{(i_1)} \,\dot{\cap}\, S_1'^{(i_1')}) \circ (S_2^{(i_2)} \,\dot{\cap}\, S_2'^{(i_2')}) \circ \ldots .$$

For $T_1,\ T_2 \in \mathcal{C}$ we take:

$$T_1 \,\dot{\cap}\, T_2 = \bigcup_{\substack{t_1 \in T_1 \\ t_2 \in T_2}} t_1 \,\dot{\cap}\, t_2 .$$

Remark: $T_1 \,\dot{\cap}\, T_2$ is very similar to the normal intersection $T_1 \cap T_2$; the only difference is that we have to state what has to be done with the relevance markings. If we have, apart from annotation, the same s-trace in both $T_1$ and $T_2$, it is also in $T_1 \,\dot{\cap}\, T_2$ where the relevant length is taken to be maximal.

Examples:

1. $\left\{\begin{bmatrix} a \\ b \end{bmatrix}^{(0)}\right\} \dot{\cap} \left\{\begin{bmatrix} a \\ b \end{bmatrix}^{(1)}\right\} = \begin{bmatrix} a \\ b \end{bmatrix}^{(0)} \dot{\cap} \begin{bmatrix} a \\ b \end{bmatrix}^{(1)} = \left\{\begin{bmatrix} a \\ b \end{bmatrix}^{(1)}\right\}.$

2. $\left\{\begin{bmatrix} a \\ b \end{bmatrix}^{(1)} \circ [c]^{(0)} \circ \begin{bmatrix} d \\ e \end{bmatrix}^{(0)}\right\} \dot{\cap} \left\{\begin{bmatrix} a \\ b \end{bmatrix}^{(1)} \circ [c]^{(1)} \circ \begin{bmatrix} d \\ e \end{bmatrix}^{(0)}\right\} =$
$\left\{\begin{bmatrix} a \\ b \end{bmatrix}^{(1)} \circ [c]^{(1)} \circ \begin{bmatrix} d \\ e \end{bmatrix}^{(0)}\right\}.$

3. $\{[a]^{(1)}\} \,\dot{\cap}\, \left\{\begin{bmatrix} a \\ b \end{bmatrix}^{(1)}\right\} = [a]^{(1)} \,\dot{\cap}\, \begin{bmatrix} a \\ b \end{bmatrix}^{(1)} = \varnothing$, since $[a] \neq \begin{bmatrix} a \\ b \end{bmatrix}$.

4. $\left\{\begin{bmatrix} a \\ b \end{bmatrix}^{(1)} \circ [c]^{(0)}\right\} \dot{\cap} \left\{\begin{bmatrix} a \\ b \end{bmatrix}^{(1)} \circ \begin{bmatrix} c \\ d \end{bmatrix}^{(0)}\right\} = \varnothing$. Since $[c] \neq \begin{bmatrix} c \\ d \end{bmatrix}$.

5. $\{S \mid a \in S\}^{(0)} \,\dot{\cap}\, \left\{\begin{bmatrix} a \\ b \end{bmatrix}\right\}^{(1)} = \bigcup_{\substack{S \text{ such that} \\ a \in S}} \left(\{S^{(0)}\} \,\dot{\cap}\, \left\{\begin{bmatrix} a \\ b \end{bmatrix}^{(1)}\right\}\right) =$
$= \left(\begin{bmatrix} a \\ b \end{bmatrix}^{(0)} \dot{\cap} \begin{bmatrix} a \\ b \end{bmatrix}^{(1)}\right) \cup \varnothing = \left\{\begin{bmatrix} a \\ b \end{bmatrix}^{(1)}\right\}.$

**Proposition** $T_1,\ T_2 \in \mathcal{C}$ implies $T_1 \,\dot{\cap}\, T_2 \in \mathcal{C}$.

**Definition** For $S \in \mathcal{P}^+(A)$, $\langle S, i \rangle^{\sim} = \langle \mathcal{P}^+(A) \backslash S, i \rangle$; $\mathcal{P}^+(A) \backslash S$ is the complement of $S$ with respect to $\mathcal{P}^+(A)$, and is occasionally written as $S^c$. For $t = S_1^{(i_1)} \circ S_2^{(i_2)} \circ \ldots \circ S_n^{(i_n)} \circ \ldots$, $\tilde{t} = \bigcup_{n<\omega} S_1^{(i_1)} \circ S_2^{(i_2)} \circ \ldots \circ \tilde{S}_n^{(i_n)} \circ (\mathcal{P}^+(A)^{(0)})^{\omega}$.

For $T = \bigcup_{t \in T} t \in \mathcal{C}$: $\tilde{T} = \dot{\bigcap}_{t \in T} \tilde{t}$.

**Proposition** $T \in \mathcal{C}$ implies $\tilde{T} \in \mathcal{C}$.

**Proposition** If $T = T_1^{(i_1)} \circ T_2^{(i_2)} \circ \ldots \circ T_n^{(i_n)} \circ \ldots, T_i \subseteq \mathcal{P}^+(A)$,

$$\tilde{T} = \bigcup_{n<\omega} T_1 \circ T_2 \circ \ldots \circ \tilde{T}_n^{(i_n)} \circ (\mathcal{P}^+(A)^{(0)})^{\omega}.$$

Example:

$$
\begin{aligned}
([a]^{(1)} \circ (\wp^+(A)^{(0)})^\omega)^\sim
&= [a]^{(1)\sim} \circ (\wp^+(A)^{(0)})^\omega \cup [a]^{(1)} \circ \varnothing \circ (\wp^+(A)^{(0)})^\omega \\
&\quad \cup [a]^{(1)} \circ (\wp^+(A)^{(0)}) \circ \varnothing \circ (\wp^+(A)^{(0)})^\omega \\
&\quad \cup [a]^{(1)} \circ \wp^+(A)^{(0)} \circ \wp^+(A)^{(0)} \circ \varnothing \circ (\wp^+(A)^{(0)})^\omega \cup \ldots \\
&= [a]^{\sim(1)}(\wp^+(A)^{(0)})^\omega \cup \varnothing \cup \varnothing \cup \varnothing \cup \ldots \\
&= \{S^{(1)} \mid a \notin S\} \circ (\wp^+(A)^{(0)})^\omega.
\end{aligned}
$$

*A2 An s-trace semantics of actions* In this section we give a semantics for unconditional actions in *Act* based upon s-trace-sets. The semantical function $[\![ . ]\!]$ will yield all possible s-traces that can be executed by an action. We cannot yet consider conditional actions since this involves the evaluation of an assertion in a state (world) describing the values of all possible predicates. In this stage we do not yet have states; these will be introduced in Section A5.

**Definition** $[\![ \cdot ]\!] \in Act_0 \to \mathcal{C}$ is given by:
$[\![ \underline{a} ]\!] = \{S \mid a \in S\}^{(1)} \circ (\wp^+(A)^{(0)})^\omega$;
$[\![ \alpha_1 ; \alpha_2 ]\!] = cut([\![ \alpha_1 ]\!]) \circ [\![ \alpha_2 ]\!]$;
$[\![ \alpha_1 \underline{\cup} \alpha_2 ]\!] = [\![ \alpha_1 ]\!] \cup [\![ \alpha_2 ]\!]$;
$[\![ \alpha_1 \,\&\, \alpha_2 ]\!] = [\![ \alpha_1 ]\!] \Cap [\![ \alpha_2 ]\!]$;
$[\![ \bar{\alpha}_1 ]\!] = [\![ \alpha_1 ]\!]^\sim$;
$[\![ \underline{\varnothing} ]\!] = \varnothing$; $[\![ \underline{U} ]\!] = \wp^+(A)^{(1)} \circ (\wp^+(A)^{(0)})^\omega$.

Remark 1: The action $\underline{a}$ is interpreted as the collection of s-trace-sets that contain *all* infinite s-traces that begin with elementary action $a$, possibly simultaneous with any other elementary action. For $\{S \mid a \in S\}$ we shall sometimes write the informal but suggestive notation $\begin{bmatrix} a \\ \cdot \end{bmatrix}$. This choice is motivated by the informal usage of actions in deontic logic: if an action $a$ is forbidden, then it is also forbidden to do *a together* with *any* other action, *followed by any* other action. In this case liability to punishment already arises after the doing of a set of actions in $\begin{bmatrix} a \\ \cdot \end{bmatrix}$, which includes the performance of $a$. So we denote in our semantics the action $\underline{a}$ as "$\begin{bmatrix} a \\ \cdot \end{bmatrix}$ followed by whatever else," where $\begin{bmatrix} a \\ \cdot \end{bmatrix}$ is the relevant or specified part and "whatever else" is the irrelevant part. This choice will render our formal system easier to use, since it keeps us closer to common parlance. However, it is, of course, possible to give a simpler semantics of $\underline{a}$ at the expense of having more cumbersome deontic assertions, stating explicitly which actions are to be executed jointly with $a$ and which actions are followed by $a$.

Remark 2: The semantics of the sequential composition of two actions is obtained by taking the *relevant* part of the interpretation of the first action (which is a set of *finite* s-traces) and concatenating this with the semantics of the second action. This can be illustrated by an example: For the meaning of $\underline{a}$ ; $\underline{b}$ we use the meanings of $\underline{a}$ and $\underline{b}$:

$$[\![\underline{a}]\!] = \begin{bmatrix} a \\ \cdot \end{bmatrix}^{(1)} \circ (\mathcal{P}^+(A)^{(0)})^\omega \text{ and } [\![\underline{b}]\!] = \begin{bmatrix} b \\ \cdot \end{bmatrix}^{(1)} \circ (\mathcal{P}^+(A)^{(0)})^\omega.$$

Now only the relevant part of $[\![\underline{a}]\!]$ is important for $[\![\underline{a} \,;\, \underline{b}]\!]$:

$$\begin{aligned} [\![\underline{a} \,;\, \underline{b}]\!] &= cut([\![\underline{a}]\!]) \circ [\![\underline{b}]\!] \\ &= cut\left(\begin{bmatrix} a \\ \cdot \end{bmatrix}^{(1)} \circ (\mathcal{P}^+(A)^{(0)})^\omega\right) \circ \begin{bmatrix} b \\ \cdot \end{bmatrix}^{(1)} \circ (\mathcal{P}^+(A)^{(0)})^\omega \\ &= \begin{bmatrix} a \\ \cdot \end{bmatrix}^{(1)} \circ \begin{bmatrix} b \\ \cdot \end{bmatrix}^{(1)} \circ (\mathcal{P}^+(A)^{(0)})^\omega, \end{aligned}$$

which was to be expected.

Remark 3: In this framework the syntactical operators $\underline{\cup}$, &, and $^{-}$ correspond simply with the semantical operations $\cup$, $\Cap$, and $\sim$. This renders the semantics *compositional* and similar to denotational semantics in the realm of programming (see, e.g., [5]).

Remark 4: $\underline{\varnothing}$ means 'failure': no s-set is executed anymore.

Remark 5: The whole set-up of the model may seem terribly complicated. The reason for this is that we want to maintain the axioms of $PD_eL$ in their precise form. For instance, the whole model would be simpler if instead of

(II) $\overline{\alpha_1 \,;\, \alpha_2} = \overline{\alpha_1} \cup (\alpha_1 \,;\, \overline{\alpha_2})$

we had the axiom

(II′) $\overline{\alpha_1 \,;\, \alpha_2} = (\overline{\alpha_2} \,;\, \underline{U}^m) \cup (\alpha_1 \,;\, \overline{\alpha_2})$,

where $m$ is the duration (or length) of $\alpha_2$ (or rather $[\![\alpha_2]\!]$). For instance, for elementary $a, b$, we would have by (II′):

$$\overline{\underline{a} \,;\, \underline{b}} = (\bar{\underline{a}} \,;\, \underline{U}) \,\underline{\cup}\, (\underline{a} \,;\, \bar{\underline{b}})$$

instead of by (II):

$$\overline{\underline{a} \,;\, \underline{b}} = \bar{\underline{a}} \,\underline{\cup}\, (\underline{a} \,;\, \bar{\underline{b}}).$$

However the latter formula is crucial when applied to deontic matters. For instance, $O(a \,;\, b) = Oa \wedge [a]Ob$ is perfectly intuitive and desirable, but it relies on the fact that

$$[\overline{\underline{a} \,;\, \underline{b}}]V \equiv [\bar{\underline{a}} \,\underline{\cup}\, \underline{a} \,;\, \bar{\underline{b}}]V \equiv [\bar{\underline{a}}]V \wedge [\underline{a} \,;\, \bar{\underline{b}}]V,$$

i.e., liability to punishment ($V$) should occur already *directly* after having done $\bar{a}$ and *not* only after $\bar{a}$ ; $U$ which we would get if we would use (II′): $[\overline{\underline{a} \,;\, \underline{b}}]\, V \equiv [\bar{\underline{a}} \,;\, \underline{U} \,\underline{\cup}\, \underline{a} \,;\, \bar{\underline{b}}]\, V \equiv [\bar{\underline{a}} \,;\, \underline{U}]\, V \wedge [\underline{a} \,;\, \bar{\underline{b}}]\, V$. This point is essential for a correct treatment of deontic assertions.

So $[\![\alpha]\!]$ gives all possible executions of s-traces for $\alpha$. However, to define a semantics for conditional actions and also for a treatment of assertions of properties of the actions, we need resulting states given execution started in some initial state $\sigma_0$. The introduction of states is discussed in Section A4.

***A3 Algebraical properties of actions*** From the definition of $[\![\cdot]\!]$ we can easily derive the following equivalences of actions (with respect to $[\![\cdot]\!]$):

(i) $\alpha \cup \emptyset = \alpha$;
(ii) $\alpha \cup \alpha = \alpha$;
(iii) $\alpha \,\&\, \alpha = \alpha$;
(iv) $\alpha \,\&\, \bar{\alpha} = \emptyset$ and $(\alpha\,;\,\alpha_1) \,\&\, (\underline{\alpha}\,;\,\alpha_2) = \emptyset$;
(v) $\overline{\alpha_1\,;\,\alpha_2} = \bar{\alpha}_1 \cup (\alpha_1\,;\,\bar{\alpha}_2)$;
(vi) $\overline{\alpha_1 \cup \alpha_2} = \bar{\alpha}_1 \,\&\, \bar{\alpha}_2$;
(vii) $\overline{\alpha_1 \,\&\, \alpha_2} = \bar{\alpha}_1 \cup \alpha_2$;
(viii) $\bar{\bar{\alpha}} = \alpha$;
(ix) $\alpha_1 \,\&\, (\alpha_2 \cup \alpha_3) = (\alpha_1 \,\&\, \alpha_2) \cup (\alpha_1 \,\&\, \alpha_3)$;
(x) $\alpha_1 \cup (\alpha_2 \,\&\, \alpha_3) = (\alpha_1 \cup \alpha_2) \,\&\, (\alpha_1 \cup \alpha_3)$.

Furthermore if we introduce the notions *min dur* $(\alpha)$ and *max dur* $(\alpha)$ (if *min dur* $(\alpha)$ = *max dur* $(\alpha)$ we use the notation *dur* $(\alpha)$) by

$$\textit{min dur}\ (\alpha) = \min_i \{\text{length}(\textit{cut}(T_i))\}$$

and

$$\textit{max dur}\ (\alpha) = \max_i \{\text{length}(\textit{cut}(T_i))\}$$

where $[T_i\}_i$ is a partition of $[\![\alpha]\!]$ such that every s-trace in $T_i$ has the same relevant length and length $(\textit{cut}(T_i))$ is the length of the s-traces in $\textit{cut}(T_i)$, then we have the additional property

(xi) $\alpha_1 \,\&\, (\alpha_2\,;\,\alpha_3) = (\alpha_1 \,\&\, \alpha_2);\,\alpha_3$,
provided that *max dur* $(\alpha_1) \leq$ *min dur* $(\alpha_2)$.

This property states that if all relevant parts of $\alpha_1$ terminate before one of the relevant parts of $\alpha_2$ does, then if $\alpha_1$ is executed jointly with $\alpha_2\,;\,\alpha_3$, $\alpha_3$ cannot be started until $\alpha_1$ and $\alpha_2$ have been executed jointly, which is very clear, intuitively.

The soundness of all equivalences mentioned except (viii) follows directly from the definitions. Assuming these have been proved, it is easy to prove (viii) by induction on the complexity of action expression $\alpha$.

*Proof:* We distinguish the diverse cases of the form of $\alpha$:

(1) if $\alpha = \underline{a}$, $[\![\bar{\bar{\alpha}}]\!] = [\![\bar{\alpha}]\!]^{\sim} = ([\![\alpha]\!]^{\sim})^{\sim} =$

$$\left(\left(\begin{bmatrix} a \\ \cdot \end{bmatrix}^{(1)} \circ (\wp^{+}(A)^{(0)})^{\omega}\right)^{\sim}\right)^{\sim} =$$

$$\left(\begin{bmatrix} a \\ \cdot \end{bmatrix}^{c(1)} \circ (\wp^{+}(A)^{(0)})^{\omega}\right)^{\sim} = \left(\begin{bmatrix} a \\ \cdot \end{bmatrix}^{c}\right)^{c(1)} \circ (\wp^{+}(A)^{(0)})^{\omega} =$$

$$\begin{bmatrix} a \\ \cdot \end{bmatrix}^{(1)} \circ (\wp^{+}(A)^{(0)})^{\omega} = [\![\alpha]\!].$$

(2) if $\alpha = \alpha_1 ; \alpha_2$, $\llbracket \bar{\bar{\alpha}} \rrbracket =$
$\llbracket \overline{\overline{\alpha_1 ; \alpha_2}} \rrbracket =$ (by property (v):)
$\llbracket \overline{\bar{\alpha}_1 \underline{\cup} (\alpha_1 ; \bar{\alpha}_2)} \rrbracket =$ (by (vi):)
$\llbracket \bar{\bar{\alpha}}_1 \mathbin{\&} (\overline{\alpha_1 ; \bar{\alpha}_2} \rrbracket =$ (by induction hypothesis and (v):)
$\llbracket \alpha_1 \mathbin{\&} (\bar{\alpha}_1 \underline{\cup} (\alpha_1 ; \bar{\bar{\alpha}}_2)) \rrbracket =$ (by induction again:)
$\llbracket \alpha_1 \mathbin{\&} (\bar{\alpha}_1 \underline{\cup} (\alpha_1 ; \alpha_2)) \rrbracket =$ (by (ix):)
$\llbracket (\alpha_1 \mathbin{\&} \bar{\alpha}_1) \underline{\cup} (\alpha_1 \mathbin{\&} (\alpha_1 ; \alpha_2)) \rrbracket =$ (by (iv):)
$\llbracket \underline{\varnothing} \underline{\cup} (\alpha_1 \mathbin{\&} (\alpha_1 ; \alpha_2)) \rrbracket =$ (by (i):)
$\llbracket \alpha_1 \mathbin{\&} (\alpha_1 ; \alpha_2) \rrbracket =$ (by (xi):)
$\llbracket (\alpha_1 \mathbin{\&} \alpha_1); \alpha_2 \rrbracket =$ (by (iii):)
$\llbracket \alpha_1 ; \alpha_2 \rrbracket = \llbracket \alpha \rrbracket$.

(3) if $\underline{\alpha = \alpha_1} \underline{\cup} \alpha_2$, $\llbracket \bar{\bar{\alpha}} \rrbracket =$
$\llbracket \overline{\overline{\alpha_1 \underline{\cup} \alpha_2}} \rrbracket =$ (by (vi):)
$\llbracket \overline{\overline{\alpha_1} \mathbin{\&} \overline{\alpha_2}} \rrbracket =$ (by (vii):)
$\llbracket \overline{\overline{\alpha_1}} \underline{\cup} \overline{\overline{\alpha_2}} \rrbracket =$ (by induction hypothesis:)
$\llbracket \alpha_1 \underline{\cup} \alpha_2 \rrbracket = \llbracket \alpha \rrbracket$.

(4) if $\underline{\alpha = \alpha_1} \mathbin{\&} \alpha_2$, $\llbracket \bar{\bar{\alpha}} \rrbracket =$
$\llbracket \overline{\overline{\alpha_1 \mathbin{\&} \alpha_2}} \rrbracket =$ (by (vii):)
$\llbracket \overline{\overline{\alpha_1} \underline{\cup} \overline{\alpha_2}} \rrbracket =$ (by (vi):)
$\llbracket \overline{\overline{\alpha_1}} \mathbin{\&} \overline{\overline{\alpha_2}} \rrbracket =$ (by induction hypothesis:)
$\llbracket \alpha_1 \mathbin{\&} \alpha_2 \rrbracket = \llbracket \alpha \rrbracket$.

(5) if $\alpha = \overline{\alpha_1}$, we remark that
$\llbracket (\overline{\alpha_1})^{=} \rrbracket = ((\llbracket \alpha_1 \rrbracket^{\sim})^{\sim\sim} =$
$(\llbracket \alpha_1 \rrbracket^{\sim\sim})^{\sim} = (\llbracket \overline{\overline{\alpha_1}} \rrbracket)^{\sim} =$ (by induction:)
$\llbracket \alpha_1 \rrbracket^{\sim} = \llbracket \overline{\alpha_1} \rrbracket$.

The properties (i) to (xi) will be of use when we consider assertions about the behaviour of the actions. However, we do not know whether this set of properties is complete in the sense that it specifies fully the semantical domain of actions that we have constructed. This issue of completeness remains a topic for further investigation.

***A4 A state transition semantics of actions*** In the last section we saw how we could associate s-trace-sets with (unconditional) actions. Now we want to speak about the states one can get into when pursuing these possible s-traces to the 'end', that is to say the end of the *relevant* part. (This consists of *finite* s-traces!)

To this end we assume that we have a given function $\imath : \mathcal{P}^{+}(A) \to (\Sigma \to \Sigma)$, where $\Sigma$ is the universe of states (worlds), assigning values to the propositional variables. This function gives for each s-set its behaviour in terms of state-transitions. So $\imath(S)(\sigma)$ yields the next state when one performs all elementary actions in $S$ jointly in state $\sigma$. Now we define for a finite s-trace $t = S_2 \circ \ldots \circ S_n$ the function $\mathcal{R}(t) \in \Sigma \to \Sigma$ inductively by

$$\mathcal{R}(S_1)(\sigma) = \imath(S_1)(\sigma)$$

and

$$\mathcal{R}(t_1 \circ t_2)(\sigma) = \mathcal{R}(t_2)(\mathcal{R}(t_1)(\sigma)).$$

We can extend $\mathcal{R}$ to s-trace-sets consisting of (only) *finite* s-traces as usual:

$$\mathcal{R}(T)(\sigma) = \{\sigma' \mid \sigma' = \mathcal{R}(t)(\sigma) \text{ for some } t \in T\}.$$

Using $\mathcal{R}$ we define the desired state transition semantics:

**Definition** The function $[\![\cdot]\!]_R: Act_0 \to (\Sigma \to \mathcal{P}(\Sigma))$ is defined by $[\![\alpha]\!]_R(\sigma) = \mathcal{R}(cut([\![\alpha]\!]))(\sigma)$. This function is extended to $Act_0 \to (\mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma))$ in the usual way.

It is now easy to extend $[\![\cdot]\!]_R$ to conditional actions. We extend $[\![\cdot]\!]$ to $Act \to (\Sigma \to \mathcal{C})$ and $[\![\cdot]\!]_R$ to $Act \to (\Sigma \to \Sigma)$ as follows:

$$[\![\underline{a}]\!](\sigma) = \{S \mid a \in S\}^{(1)} \circ (\mathcal{P}^+(A)^{(0)})^\omega;$$
$$[\![\alpha_1 ; \alpha_2]\!](\sigma) = cut([\![\alpha_1]\!](\sigma)) \circ [\![\alpha_2]\!]([\![\alpha_1]\!]_R(\sigma));$$
$$[\![\alpha_1 \cup\!\!\!\!\_\ \alpha_2]\!](\sigma) = [\![\alpha_1]\!](\sigma) \cup [\![\alpha_2]\!](\sigma);$$
$$[\![\alpha_1 \,\&\, \alpha_2]\!](\sigma) = [\![\alpha_1]\!](\sigma) \Cap [\![\alpha_2]\!](\sigma);$$
$$[\![\psi \to \alpha_1/\alpha_2]\!](\sigma) = \begin{cases} [\![\alpha_1]\!](\sigma) & \text{if } \sigma \models \psi; \\ [\![\alpha_2]\!](\sigma) & \text{if } \sigma \not\models \psi; \end{cases}$$
$$[\![\bar{\alpha}_1]\!](\sigma) = [\![\alpha_1]\!]^{\sim}(\sigma) \text{ (see Remark 1.)};$$
$$[\![\underline{\varnothing}]\!](\sigma) = \varnothing;$$
$$[\![\underline{U}]\!](\sigma) = \mathcal{P}^+(A)^{(1)} \circ (\mathcal{P}^+(A)^{(0)})^\omega;$$
$$[\![\alpha]\!]_R(\sigma) = \mathcal{R}(cut([\![\alpha]\!](\sigma)))(\sigma);$$

For $\tau \in \mathcal{P}(\Sigma)$ we define:

$$[\![\alpha]\!]_R(\tau) = \bigcup_{\sigma \in \tau} [\![\alpha]\!]_R(\sigma).$$

Remark 1: Note that to deal with $\overline{\psi \to \alpha_1/\alpha_2}$ we need to extend $\tilde{}$ to the conditional case:

$$(\text{if } \beta \text{ then } T_1 \text{ else } T_2)^{\sim} = \text{if } \beta \text{ then } \tilde{T}_1 \text{ else } \tilde{T}_2.$$

Remark 2: Up to now $\psi$ was assumed to be an ordinary propositional formula. In the next section we shall extend our language of assertions.

***A5 Semantics of assertions and soundness of PD$_e$L*** For the semantics of assertions we employ the semantics $[\![\cdot]\!]_R$ of actions and define for $\sigma \in \Sigma$, $\psi_1$, $\psi_2 \in Ass$ $\sigma \models \psi_1 \vee \psi_2, \ldots, \sigma \models \neg\psi_1$ as usual, and for $\alpha \in Act$, $\sigma \in \Sigma$, $\psi \in Ass$

$$\sigma \models [\alpha]\sigma \quad \text{iff } \forall \sigma' \in [\![\alpha]\!]_R(\sigma) \colon \sigma' \models \psi$$

and

$$\sigma \models \langle\alpha\rangle\sigma \quad \text{iff } \exists \sigma' \in [\![\alpha]\!]_R(\sigma) \colon \sigma' \models \psi.$$

Note that it now also becomes possible to allow assertions $\psi \in Ass$ in the conditional actions $\psi \to \alpha_1/\alpha_2$, using the definition of $\psi \to \alpha_1/\alpha_2$.

Finally we define

$$\models \psi \quad \text{iff } \forall \sigma \in \Sigma \colon \sigma \models \psi.$$

The soundness of (PC) and ($\underline{\varnothing}$) with respect to our formal model is obvious; that of ($\Box\supset$), (;), ($\cup\!\!\!\!\_$), ($\to$/), ($\Diamond$), and the rules (MP) and (N) is proved as in standard dynamic logic. The axioms involving $\bar{}$ follow from the algebraic properties of the actions in Section A3. The soundness of (&) is proved as follows. Take $\alpha_1$ and $\alpha_2$ such that *dur* $(\alpha_1)$ = *dur* $(\alpha_2)$. Suppose we have that $\sigma \models [\alpha_1]\psi$, i.e.,

$$\forall \sigma' \in [\![\alpha_1]\!]_R(\sigma): \sigma' \models \psi.$$

But $[\![\alpha_1 \mathbin{\&} \alpha_2]\!] = [\![\alpha_1]\!] \Cap [\![\alpha_2]\!]$, so $[\![\alpha_1 \mathbin{\&} \alpha_2]\!] \subseteq [\![\alpha_1]\!]$ and therefore also $[\![\alpha_1 \mathbin{\&} \alpha_2]\!]_R(\sigma) \subseteq [\![\alpha_1]\!]_R(\sigma)$. Hence $\forall \sigma' \in [\![\alpha_1 \mathbin{\&} \alpha_2]\!]_R(\sigma): \sigma' \models \psi$ as well, i.e.,

$$\sigma \models [\alpha_1 \mathbin{\&} \alpha_2]\psi.$$

Consequently $\sigma \models [\alpha_1]\psi \supset [\alpha_1 \mathbin{\&} \alpha_2]\psi$. Likewise $\sigma \models [\alpha_2]\psi \supset [\alpha_1 \mathbin{\&} \alpha_2]\psi$, and so $\sigma \models [\alpha_1]\psi \vee [\alpha_2]\psi \supset [\alpha_1 \mathbin{\&} \alpha_2]\psi$. This holds for all $\sigma$, therefore

$$\sigma \models [\alpha_1]\psi \vee [\alpha_2]\psi \supset [\alpha_1 \mathbin{\&} \alpha_2]\psi,$$

so (&) is sound.

The system PDL was shown to be logically complete by, e.g., [17]; i.e., it is possible to derive all valid formulas within the system. The natural question arises whether $PD_eL$ is complete with respect to the semantics we have defined. Settling this question seems to be more difficult than for PDL, since the semantics of actions is rather more complicated than that of 'programs' in the case of PDL (although we do not have the difficulty of dealing with the iterative operator *). This issue is still open.

## NOTE

1. The notion of duration is treated in Section 6 and the appendix.

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