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Cantor-Bendixson Spectra of ω-Stable Theories

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1 Introduction In the following, we shall mean by theory a first-order, countable, complete, quantifier-eliminable theory.

The idea of classifying ω-stable theories by the analysis of the Boolean algebras of the definable subsets of their countable models arises from [3] and is based on the remark that a theory T is ω-stable if and only if, for every countable model M of T, the Boolean algebra $B(M)$ of the definable subsets of M is superatomic. In fact, it is well-known that, for every Boolean algebra B, an ascending chain $\{I_\nu(B)\colon \nu \text{ ordinal}\}$ of ideals of B can be defined in this way:

1. $I_0(B) = \{0\}$
2. $I_1(B)$ is the ideal of finite elements of B
3. for every ordinal ν, $I_{\nu+1}(B)$ is the preimage in B of $I_1(B/_{I_\nu(B)})$ in the canonical homomorphism of B onto $B/_{I_\nu(B)}$
4. for every limit ordinal λ, $I_\lambda(B) = \bigcup_{\nu<\lambda} I_\nu(B)$.

In particular, when B is superatomic, there is an ordinal μ such that $I_\mu(B) = B$; let μ be the least ordinal with this property, then μ is a successor ordinal, and we may define:

α_B = predecessor of μ = least ordinal ν such that $I_\nu(B) \neq B$
d_B = number of atoms in $B/_{I_{\alpha_B}(B)}$.

We have the following:

(i) $\alpha_B < \omega_1$ if B is countable
(ii) $d_B < \omega$
(iii) for every ordered pair (α, d) with $1 \leq \alpha < \omega_1$, $1 \leq d < \omega$, there is a countable superatomic Boolean algebra B such that $(\alpha, d) = (\alpha_B, d_B)$

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(iv) for every countable superatomic Boolean algebras B_1, B_2, $B_1 \cong B_2$ if and only if $(\alpha_{B_1}, d_{B_1}) = (\alpha_{B_2}, d_{B_2})$.

(See [2].)

Then, for every ω-stable theory T and for every countable model M of T, we set $(\alpha_M, d_M) = (\alpha_{B(M)}, d_{B(M)})$, and we say that:

- α_M is the Cantor-Bendixson rank (CB-rank) of M
- d_M is the Cantor-Bendixson degree (CB-degree) of M
- (α_M, d_M) is the Cantor-Bendixson type (CB-type) of M.

We also define the Cantor-Bendixson spectrum of T (CB-Spec T) in the following way

$$\text{CB-Spec T} = \{(\alpha_M, d_M): M \models T, |M| = \aleph_0\}.$$

CB-Spec T can be ordered lexicographically, and has a minimal pair corresponding to the prime model M_0 of T, and a maximal pair corresponding to the countable saturated model $\overline{M}$ of T. Moreover, $\alpha_{\overline{M}}$ coincides with the Morley rank α_T of T, and $d_{\overline{M}}$ coincides with the Morley degree d_T of T (see [1]). The analysis of CB-Spec T gives a measure of complexity of T: for instance, |CB-Spec T| = 1 means T is pseudo-$\aleph_0$-categorical [4], CB-Spec $T = \{(1, 1)\}$ means T is strongly minimal.

Then we can classify ω-stable theories by the following equivalence relation $\sim$: if T_1, T_2 are ω-stable theories, we set $T_1 \sim T_2$ if and only if CB- Spec T_1 = CB-Spec T_2. Every $\sim$-class defines a subset of $(\omega_1 - \{0\}) \times (\omega - \{0\})$. Put for simplicity $\omega_1^* = \omega_1 - \{0\}$, $\omega^* = \omega - \{0\}$ and, for $X \subset \omega_1^* \times \omega^*$, define X a CB-set (Canton-Bendixson set) if there is an ω-stable theory T such that X = *CB-Spec* T. Our problem is to study the characterization of CB-sets. Of course, there are several subsets X of $\omega_1^* \times \omega^*$ which are not CB-sets; some restrictions are already provided by Lemmas 1.1 and 1.2 below, but stronger conditions must be satisfied by X when, for instance, $(1, 1) \in X$ (see [5]). This paper is a natural complement of [5]; in fact, our main goal is to provide a lot of general examples of CB-sets. More precisely, the program of this work is the following: Section 2 ("the bricks") is devoted to some basic examples of CB-sets; in Section 3 ("the project") we shall explain a simple project for combining these examples to construct more complicated CB-sets; this "construction" will be made in Section 4.

These results, together with those of [5], are a first step to a complete classification of the subsets $X \subset \omega_1^* \times \omega^*$ which are CB-sets.

Although this general problem seems to be very difficult, we may conjecture that, if the minimal rank of X is $\alpha \geq 3$, then some simple conditions should let X be a CB-set; but if the minimal rank is $\alpha < 3$, then deeper conditions must be satisfied by X. However, this will be the matter of some forthcoming notes.

Lemma 1.1 *Let $X \subset \omega_1^* \times \omega^*$ be a CB-set. Then X is countable and admits a maximal element.*

Proof: See the previous remarks.

Lemma 1.2 *If $(1, d) \in$ CB-Spec T for an ω-stable theory T, then $d_T \leq d$.*

Proof: Let $M \models T$, $|M| = \aleph_0$, $(\alpha_M, d_M) = (1, d)$. By the finite equivalence relation theorem [6], there is a 0-definable equivalence relation E on M admitting a finite number of equivalence classes, and, in particular, exactly d_T classes $E_1, \dots, E_{d_T}$ with Morley rank α_T. It follows that, for every $i = 1, \dots, d_T$, E_i is an infinite definable subset of M, so $B(M)/_{I_1(B(M))}$ has at least d_T atoms. Consequently, $d \geq d_T$.

Remark: It is not generally true that, if $(\alpha, d) \in$ CB-Spec T for $\alpha > 1$, then $d_T \leq d$. Counterexamples are implicit in the following.

A remark which will be useful is the following:

Lemma 1.3 *If* $(\alpha, d) \in \omega_1^* \times \omega^*$, $\{(\alpha, d)\}$ *is a CB-set.*

Proof: See [4]: it suffices to consider the pseudo-$\aleph_0$-categorical ω-stable theory T such that, for every countable $M \models T$, $(\alpha_M, d_M) = (\alpha, d)$.

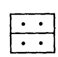
2 The bricks

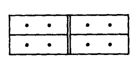
2.1 The structures M(α, i) For all $\alpha \in \omega_1^*$, $i \in \omega^*$, we construct a structure $M(\alpha, i)$, admitting an equivalence relation E_ν for every ν with $1 \leq \nu < \alpha$, in the following way:

- $M(1, i)$: i elements, no structure
- $M(\alpha + 1, i)$: i classes of the new equivalence relation E_α, every class isomorphic to $M(\alpha, i)$
- $M(\delta, i)$ for δ limit: fix a strictly increasing sequence $(\delta_n: n \in \omega)$ such that $\lim_n \delta_n = \delta$, set
  $M(\delta, i) = M(\delta_i, i)$
  $E_\nu = (M(\delta, i))^2$ for every ν with $\delta_i \leq \nu < \delta$.

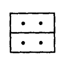
Some examples will explain the previous definitions.

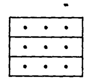
- $i = 1$ $M(\alpha, 1)$ 
- $i = 2$ $M(1, 2)$ 

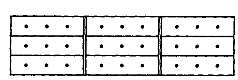
  $M(2, 2)$  E_1

  $M(3, 2)$  E_1

  E_2

  $M(\omega, 2)$ (put $\omega = \lim_n n$)  E_1 $(E_n = (M(\omega, 2))^2$ for $n \geq 2)$
- $i = 3$ $M(1, 3)$ 

  $M(2, 3)$ E_1 

  $M(3, 3)$ E_1 

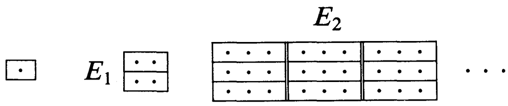
  E_2

  $M(\omega, 3) = M(3, 3)$ $(E_n = (M(\omega, 3))^2$ for $n \geq 3)$.

2.2 The structures $M_1(\delta)$ (δ limit) For all limit ordinals δ, we consider the following structure $M_1(\delta)$:

- domain: $\dot{\bigcup}_{n\in\omega^*} M(\delta, n)$
- an equivalence relation E_ν for every ν with $1 \leq \nu < \delta$.

Example: $M_1(\omega) = \dot{\bigcup}_{n\in\omega^*} M(\omega, n) = \dot{\bigcup}_{n\geq 1} M(n, n)$



Proposition *For every limit ordinal δ, $M_1(\delta)$ is ω-stable and CB-Spec $Th(M_1(\delta)) = \{(1, 1), (\delta, 1)\}$.*

Proof: It is obvious that $M_1(\delta)$ has CB-type (1, 1). Let M be a countable model of $Th(M_1(\delta))$, $M \supsetneqq M_1(\delta)$, $a \in M - M_1(\delta)$, then:

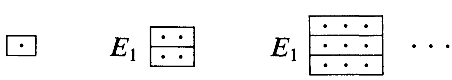
- $a|_{E_1}$ is infinite,
- for every μ, ν with $1 \leq \mu < \nu < \delta$, $a|_{E_\nu}$ contains infinitely many distinct E_μ-subclasses.

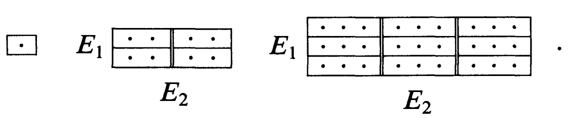
It follows that M has CB type $(\delta, 1)$.

2.3 The structures $M_1^*(\alpha)$ For all ordinals α, we consider the following structure $M_1^*(\alpha)$:

- domain $\dot{\bigcup}_{n\in\omega^*} M(\alpha + 1, n)$
- equivalence relations $E_\nu (1 \leq \nu \leq \alpha)$.

Examples:

$M_1^*(1) = \dot{\bigcup}_{n\in\omega^*} M(2, n)$ 

$M_1^*(2) = \dot{\bigcup}_{n\in\omega^*} M(3, n)$ 

Proposition *For every ordinal α, $M_1^*(\alpha)$ is ω-stable and CB-Spec $Th(M_1^*(\alpha)) = \{(1, 1)\} \cup \{(\alpha, n) : n \in \omega^*\} \cup \{(\alpha + 1, 1)\}$.*

Proof: Obviously $M_1^*(\alpha)$ has CB-type (1, 1). Let M be a countable model of $Th(M_1^*(\alpha))$, $M \supsetneqq M_1^*(\alpha)$, $a \in M - M_1^*(\alpha)$, then:

- $a|_{E_1}$ is infinite
- for every μ, ν such that $1 \leq \mu < \nu \leq \alpha$, $a|_{E_\nu}$ contains infinitely many disjoint E_μ-subclasses.

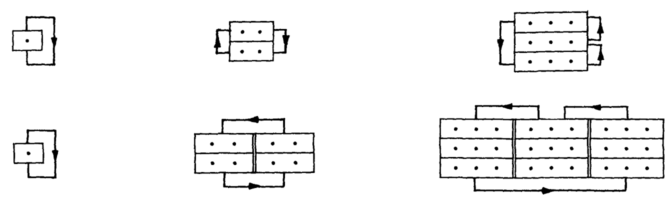
It follows that M has CB-type:

- (α, n) if M admits exactly n infinite E_α-classes
- $(\alpha + 1, 1)$ if M admits infinitely many infinite E_α-classes.

2.4 The structures $M_1(\alpha + 1)$ For all ordinals α, add to the previous structure $M_1^*(\alpha)$ an automorphism f, as the following examples describe:

$M_1(2)$

$M_1(3)$



$f = id$ $f^2 = id$ $f^3 = id$

We have:

Proposition *For every ordinal* α, $M_1(\alpha + 1)$ *is* ω*-stable and CB-Spec* $Th(M_1(\alpha + 1)) = \{(1, 1), (\alpha + 1, 1)\}$.

Proof: It follows from 2.3 and the definition of f.

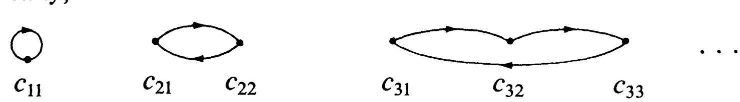
We are going now to construct more complicated examples.

2.5 The structure P_0 First, we consider a structure P_0 having domain $\{c_{i,j}: 1 \leq j \leq i < \omega\}$ and a unary function s such that

$$s(c_{i,j}) = \begin{cases} c_{i,j+1} & \text{if } j < i, \\ c_{i,1} & \text{if } j = i. \end{cases}$$

Graphically,

P_0



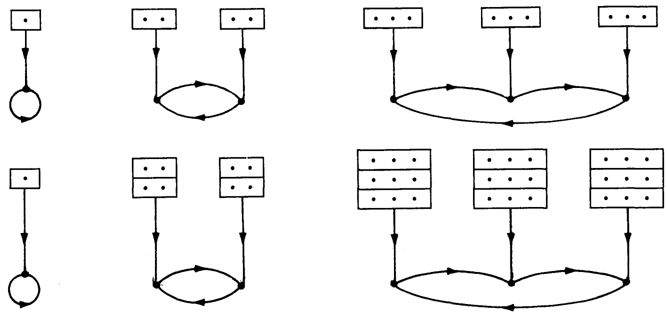
It is easy to see that $Th(P_0)$ is strongly minimal. Notice that P_0 is the prime model of $Th(P_0)$.

2.6 The structures $M_1'(\alpha + 1)(\alpha \geq 1)$ For every ordinal $\alpha \geq 1$, let $M_1'(\alpha + 1)$ be the disjoint union of $M_1^*(\alpha)$ and P_0, together with a function $\pi: M_1^*(\alpha) \to P_0$ such that, for every $1 \leq j \leq n < \omega$, $\pi^{-1}(c_{n,j})$ is an E_α-class of $M(\alpha + 1, n)$. The language of this structure will also have predicates Q, P for $M_1^*(\alpha)$, P_0, respectively, and constants for all elements. Note that $M_1'(\alpha + 1)$ is essentially just $M_1(\alpha + 1) \cup M_1(\alpha + 1)/E_\alpha$, together with a P_0-structure on the quotient set $M_1(\alpha + 1)/E_\alpha$.

Examples:

$M_1'(2)$

$M_1'(3)$



It is easy to see that $Th(M_1'(\alpha + 1))$ is ω-stable and that $M_1'(\alpha + 1)$ has CB-type (1, 2), while, if M is a countable model of $Th(M_1'(\alpha + 1))$ and $M \gneqq M_1'(\alpha + 1)$, then M has CB-type $(\alpha + 1, 1)$. Then we have:

Proposition *For every $\alpha \geq 1$, $M_1'(\alpha + 1)$ is ω-stable and CB-Spec* $Th(M_1'(\alpha + 1)) = \{(1, 2,), (\alpha + 1, 1)\}$.

2.7 The structures $M_1'(\delta)$ (δ limit) Let δ be a limit ordinal. Fix a strictly increasing sequence $(\delta_n: n \in \omega)$ such that $\delta = \lim_n \delta_n$, and let $M_1'(\delta)$ be the $\bigcup_{n\in\omega} M_1'(\delta_m + 1)$, where we identify the P_0's of each $M_1'(\delta_n + 1)$. The language of this structure will also have

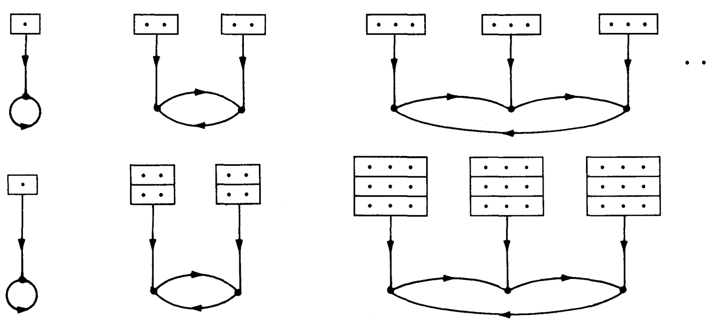
- a predicate P for P_0
- for every n, a predicate Q_n for $M_1^*(\delta_n)$ (the Q-part of $M_1'(\delta_n + 1)$)
- for every n, a function symbol π_n for the projection map of $M_1^*(\delta_n)$ onto P_0

(for each n, distinguish by symbols $E_{n,\nu}$, $1 \leq \nu \leq \delta_n$, the equivalence relations on $M_1{}^*(\delta_n)$).

Example: $M_1'(\omega)$, $\omega = \lim_n n$. We have

π_1 $M_1^*(1) \downarrow P_0$

π_2 $M_1^*(2) \downarrow P_0$



It is easy to see:

Proposition *For every limit ordinal δ, $M_1'(\delta)$ is ω-stable and CB-Spec* $Th(M_1'(\delta)) = \{(2, 1), (\delta, 1)\}$.

In fact, CB-type $M_1'(\delta) = (2, 1)$, while, if M is a countable model of $Th(M_1'(\delta))$ and $M \gneqq M_1'(\delta)$, then M has CB-type $(\delta, 1)$.

2.8 The structures $M(\alpha)$ ($\alpha \geq 1$) Finally, let $M(\alpha)$ be the structure whose domain is $P_{01} \dot{\cup} P_{02} \dot{\cup} Q(\alpha)$ where $P_{01}, P_{02} \simeq P_0$ and two projection maps π_1: $Q(\alpha) \to P_{01}$, π_2: $Q(\alpha) \to P_{02}$ are given such that, when $c^1_{i,j} \in P_{01}$ and $c^2_{h,k} \in P_{02}$, then

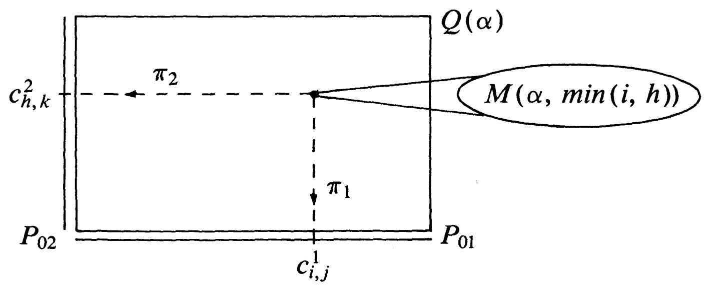
$$\pi_1^{-1}(c^1_{i,j}) \cap \pi_2^{-1}(c^2_{h,k}) \simeq M(\alpha, min(i, h)).$$

The language for this structure will have

- relation symbols P_1, P_2, Q for P_{01}, P_{02}, $Q(\alpha)$, respectively
- function symbols for π_1, π_2
- constants for all the elements of $M(\alpha)$,

together with the symbols for P_{0j} ($j = 1, 2$, see 2.5) and for the equivalence relations E_ν ($1 \leq \nu \leq \alpha$).

Graphically



Of course $M(\alpha)$ is the prime model of $Th(M(\alpha))$. Furthermore it is straightforward to see that, for any countable model M of $Th(M(\alpha))$,

- if $P_{01} = P_1^M$ or $P_{02} = P_2^M$ (in particular, if $M = M(\alpha)$), then M has CB-type (2, 1),
- if $P_{01} \neq P_1^M$ and $P_{02} \neq P_2^M$, then M has CB-type $(\alpha + 2, 1)$.

Then we have

Proposition *For every ordinal $\alpha \geq 1$, $M(\alpha)$ is ω-stable and CB-Spec $Th(M(\alpha)) = \{(2, 1), (\alpha + 2, 1)\}$.*

We conclude Section 2 recalling that the following are CB-sets (see 2.1–3):

- $\{(1, 1), (\alpha, 1)\}$ for every ordinal $\alpha \geq 1$
- $\{(1, 1)\} \cup \{(\alpha, n): n \in \omega^*\} \cup \{(\alpha + 1, 1)\}$ for every ordinal $\alpha \geq 1$.

3 The project We propose here a simple project to construct new CB-sets. Let $\{T_i: i \in I\}$ be a countable family of ω-stable theories. We define a theory $T = \dot{\bigcup}_{i \in I} T_i$ in this way:

- $L(T) = \dot{\bigcup}_{i \in I} L(T_i) \dot{\cup} \{U_i: i \in I\}$ where, for every $i \in I$, U_i is a 1-ary relation symbol
- $M \models T$ if and only if $M = \dot{\bigcup}_{i \in I} M_i \dot{\cup} M_\infty$ where $M_i = U_i^M$ is a model of T_i for every $i \in I$, while $M_\infty = \emptyset$ if I is finite, $M_\infty = M - \dot{\bigcup}_{i \in I} M_i$.

T is ω-stable. Furthermore, CB-Spec T can be easily determined because, for every countable model M of T, the algebra $B(M)$ is isomorphic to the weak direct product of the Boolean algebras $B(M_i)$ $(i \in I)$,

$$B(M) \cong w \underset{i \in I}{X} B(M_i)$$

(see [2]).

As an application, suppose I finite, $I = \{1, \ldots, d\}$, $T_1 = \ldots = T_d$. Consider the theory $T = \dot{\bigcup}_{1 \le i \le d} T_i$ defined as above, add to $L(T)$ new 1-ary functional symbols $h_{i,j}$ for $1 \le i, j \le d$ and in the enlarged language get a new theory T' adding to T axioms stating that M is a model of T' if and only if $M = \dot{\bigcup}_{1 \le i \le d} M_i$, where $M_1, \ldots, M_d \models T_1$ and, for every i, j with $1 \le i, j \le d$, $h_{i,j}$ is an isomorphism of M_i onto M_j, $h_{i,i} = id$, $h_{j,i} = h_{i,j}^{-1}$, $h_{i,j} \cdot h_{k,i} = h_{k,j}$. (We set for simplicity in this case $M = dM_1$.) T' is an ω-stable theory. Looking at the examples in Section 1, we put:

- $M_d(\alpha) = dM_1(\alpha)$: so CB-Spec $Th(M_d(\alpha)) = \{(1, d), (\alpha, d)\}$
- $M'_d(\alpha + 1) = dM'_1(\alpha + 1)$: CB-Spec $Th(M'_d(\alpha + 1)) = \{(1, 2d), (\alpha + 1, d)\}$
- $M'_d(\delta) = dM'_1(\delta)$ [δ a limit ordinal]: CB-Spec $Th(M'_d(\delta)) = \{(2, d), (\delta, d)\}$
- $M^*_d(\alpha) = dM^*_1(\alpha)$: so, CB-Spec $Th(M^*_d(\alpha)) = \{(1, d)\} \cup \{(\alpha, nd): n \in \omega^*\} \cup \{(\alpha + 1, d)\}$.

We will also denote by $M^1_d(\alpha)$ a countable ω-stable pseudo-$\aleph_0$-categorical structure whose theory has got CB spectrum $\{(\alpha, d)\}$ (see Lemma 1.3).

4 Some constructions Recall $\omega^* = \omega - \{0\}$, $\omega^*_1 = \omega_1 - \{0\}$. If $X \subset \omega^*_1 \times \omega^*$, we shall set:

- $X(\alpha) = \{d \in \omega^*: (\alpha, d) \in X\}$ for every $\alpha \in \omega^*_1$
- $X^* = \{\alpha \in \omega^*_1: X(\alpha) \neq \varnothing\}$.

We shall also use the following abbreviation: for α_i, α ordinals $\alpha_i \uparrow \alpha$ if and only if $\{\alpha_i: i \in N\}$ is a strictly increasing sequence such that $\lim_i \alpha_i = \alpha$.

Our main results are Theorems 4–6,7 and concern the sets X such that *min* $X^* \ge 3$.

We show now some results related to the case: $X(2) \neq \varnothing$.

Theorem 4.1 *Let $X \subset \omega^*_1 \times \omega^*$ be an infinite set such that:*
(a) *if $\alpha \in X^*$, $\alpha \ge 2$*
(b) *for every $\alpha \in \omega^*_1$, $|X(\alpha)| \le 1$*
(c) *if $\alpha_i \uparrow \alpha$ and $\alpha_i \in X^*$ for every $i \in N$, then $(\alpha, 1) \in X$.*
Then X is a CB-set.

Example 1: For every λ such that $\omega \le \lambda < \omega_1$, $\{(\alpha, 1): 2 \le \alpha \le \lambda\}$ is a CB-set.

Example 2: $\{(n, n): n \in \omega, n \ge 2\} \cup \{(\omega, 1)\}$ is a CB-set.

Proof: Letting (α_0, d_0) be the minimal element of X, we define a partition $X = X_0 \dot{\cup} X_1 \dot{\cup} X_2$ of X in the following way:

- $X_0 = \{(\alpha_0, d_0)\}$
- $X_1^* = \{\alpha \in X^*: \exists \beta \in X^*, \beta < \alpha,]\ \beta, \alpha[\cap X^* = \varnothing\}$, $X_1 = \{(\alpha, d) \in X: \alpha \in X_1^*\}$
- $X_2^* = \{\alpha \in X^*: \exists \alpha_i \in X_1^*, \alpha_i \uparrow \alpha\}$, $X_2 = \{(\alpha, 1): \alpha \in X_2^*\}$.

We set $M(X_1) = \dot{\bigcup}_{(\alpha,d)\in X_1} M_d(\alpha)$, and

$$M(X) = \begin{cases} M(X_1) & \text{if } (\alpha_0, d_0 = (2, 1) \\ M(X_1) \,\dot{\cup}\, M^1_{d_0-1}(2) & \text{if } \alpha_0 = 2, d_0 > 1 \\ M(X_1) \,\dot{\cup}\, M^1_{d_0}(\alpha_0) & \text{if } \alpha_0 > 2. \end{cases}$$

$M(X)$ is ω-stable. We claim that CB-Spec $Th(M(X)) = X$. For simplicity, we assume $(\alpha_0, d_0) = (2, 1)$. The proof can be easily modified to cover the remaining cases. Notice that the pseudo-$\aleph_0$-categorical ω-stable structure $M^1_{d_0-1}(2)$ $[M^1_{d_0}(\alpha_0)]$ lets (α_0, d_0) be the minimal pair of CB-Spec $Th(M(X))$. First we show that, for every $(\bar{\alpha}, \bar{d}) \in X$, there is $M \equiv M(X)$, $|M| = \aleph_0$ such that M has got CB-type $(\bar{\alpha}, \bar{d})$. Notice that we may suppose

$$M = \dot{\bigcup}_{(\alpha,d)\in X_1} M_{\alpha,d}(\dot{\cup} M_\infty)$$

where $M_{\alpha,d} \equiv M_d(\alpha)$, $|M_{\alpha,d}| = \aleph_0$ for every $(\alpha, d) \in X_1$. So we have:

- $(\bar{\alpha}, \bar{d}) = (2, 1)$: for every $(\alpha, d) \in X$, assume that the CB-type of $M_{\alpha,d}$ is $(1, d)$ so M has got CB-type $(2, 1)$
- $(\bar{\alpha}, \bar{d}) \in X_1$: take the following choice of $M_{\alpha,d}$: $M_{\alpha,d}$ has CB-type (α, d) when $(\alpha, d) = (\bar{\alpha}, \bar{d})$, $(1, d)$ otherwise; recall $\bar{\alpha} > \alpha_0 = 2$, so the CB-type of M is $(\bar{\alpha}, \bar{d})$
- $(\bar{\alpha}, \bar{d}) \in X_2$, so $\bar{d} = 1$: let $\{\alpha_i: i \in \mathbf{N}\}$ be a sequence of elements in X_1^* such that $\alpha_i \uparrow \alpha$, assume $M_{\alpha,d}$ has CB-type (α, d) when there is $i \in \mathbf{N}$ such that $\alpha = \alpha_i$, $(1, d)$ otherwise. In this case, the CB-type of M is $(\bar{\alpha}, 1)$.

Conversely, let $M \equiv M(X)$, $|M| = \aleph_0$, we will show that the CB-type of M belongs to X. Define $Y = \{\alpha \in X_1^*: M_{\alpha,d}$ has CB-rank $\alpha\}$. We can distinguish the following cases:

- $Y = \varnothing$; then M has CB-type $(2, 1)$
- $Y \neq \varnothing$, there is *max* $Y = \alpha$: let $X(\alpha) = d$, then M has CB-type $(\alpha, d) \in X_1$
- $Y \neq \varnothing$, but admits no maximal element: let $\alpha =$ *sup* Y, then $\alpha \in X_2^*$, $(\alpha, 1) \in X_2$, and we see $(\alpha, 1)$ is the CB-type of M.

The second step is to consider the finite disjoint unions of the theories given in Theorem 4.1. So we get:

Theorem 4.2 *Let $X \subset \omega_1^* \times \omega^*$ be an infinite set, (α_0, d_0) be the minimal element of X, and N be a positive integer. Suppose:*
(a) *if $\alpha \in X^*$, $\alpha \geq 2$*
(b) *$X(\alpha_0) = \{d_0\}$, where $d_0 \geq N$ if $\alpha_0 = 2$*

(c) *if* $\alpha \in X^* - \{\alpha_0\}$, *there are a positive integer* $N_\alpha \leq N$, *and* $d_1^\alpha, \ldots, d_{N_\alpha}^\alpha \in X(\alpha)$ *(not necessarily distinct) such that* $X(\alpha) = \left\{\sum_{i=1}^{N_\alpha} \epsilon_i d_i^\alpha : \epsilon_i = 0, 1, \sum_{i=1}^{N_\alpha} \epsilon_i \geq 1\right\}$
(d) *if* $\alpha_i \uparrow \alpha$ *and* $\alpha_i \in X^*$ *for every* $i \in \mathbf{N}$, *then* $N_\alpha = N$, $d_1^\alpha = \ldots = d_{N_\alpha}^\alpha = 1$.
Then, X is a CB-set.

Example 3: $\{(2, n)\} \cup \{(\alpha, 1), \ldots, (\alpha, n) : 3 \leq \alpha \leq \lambda\}$ is a CB-set for every λ such that $\omega \leq \lambda < \omega_1$.

Proof: Notice that, for every $\alpha \in \omega_1^*$, $|X(\alpha)| < 2^N$. As above, we set

$$X_0 = \{(\alpha_0, d_0)\}$$
$$X_1^* = \{\alpha \in X^* : \exists \beta < \alpha, \beta \in X^*,]\beta, \alpha[\cap X^* = \emptyset\}$$
$$X_2^* = \{\alpha \in X^* : \exists \alpha_i \in X^*, \alpha_i \uparrow \alpha\}$$

so that $X^* = X_0^* \cup X_1^* \cup X_2^*$. We define now:

- $\overline{X}_0, \overline{X}_1, \ldots, \overline{X}_{N-1} \subset \omega_1^* \times \omega^*$
- for every $\alpha \in X^*$, an integer $P(\alpha)$ such that $0 \leq P(\alpha) < N$.

We proceed in the following way:

- $\alpha = \alpha_0 = 2$: $P(\alpha_0) = 0$; $(2, d_0 - N + 1) \in \overline{X}_0$, $(2, 1) \in \overline{X}_j$ if $j > 0$
- $\alpha = \alpha_0 > 2$: $P(\alpha_0) = 0$; $(\alpha_0, d_0) \in \overline{X}_0$, $(2, 1) \in \overline{X}_j$ if $j > 0$
- $\alpha \in X_1^*$: let $\beta \in X^*$, $\beta < \alpha,]\beta, \alpha[\cap X^* = \emptyset$, and set $P(\alpha) \equiv P(\beta) + 1 \ (mod\ N)$; $(\alpha, d_1^\alpha) \in \overline{X}_{P(\alpha)}$, $(\alpha, d_2^\alpha) \in \overline{X}_{P(\alpha)+1}, \ldots, (\alpha, d_{N_\alpha}^\alpha) \in \overline{X}_{P(\alpha)+N_\alpha-1}$ (we put $\overline{X}_r = \overline{X}_s$ when $r \equiv s \ (mod\ N)$)
- $\alpha \in X_2^*$: $P(\alpha) = 0$, $(\alpha, 1) \in \overline{X}_j$ for every $j = 0, 1, \ldots, N-1$
- let $\overline{X}_0, \ldots, \overline{X}_{N-1}$ contain no more elements.

Notice that, for every $j = 0, 1, \ldots, N-1$, $\overline{X}_j$ satisfies the hypotheses of Theorem 4.1, so there is an ω-stable structure M_j such that $\overline{X}_j =$ CB-Spec $Th(M_j)$. Let $M = M_0 \dot{\cup} \ldots \dot{\cup} M_{N-1}$, then M is ω-stable and we claim $X =$ CB-Spec $Th(M)$.

i. $X \subset$ CB Spec $Th(M)$.

It suffices to show that, for every $(\alpha, d) \in X$, there is $M' \equiv M$, $|M'| = \aleph_0$ such that M' has got CB-type (α, d). We notice that $M' = M_0' \dot{\cup} \ldots \dot{\cup} M_{N-1}'$ where $M_j' \equiv M_j$, $|M_j'| = \aleph_0$ (so that the CB-type of M_j' belongs to $\overline{X}_j$) for every $j = 0, 1, \ldots, N-1$.

- $(\alpha, d) = (\alpha_0, d_0)$: take M_0' having CB-type $(2, d_0 - N + 1)$ if $\alpha_0 = 2$, (α_0, d_0) if $\alpha_0 > 2$, M_j' having CB-type (2,1) if $j > 0$, so M' has CB-type (α_0, d_0)
- $\alpha \in X_1^*$, $d = \sum_{i=1}^{N_\alpha} \epsilon_i d_i^\alpha$: let M_j' have CB-type (α, d_i^α) if $j = P(\alpha) + i - 1$ and $\epsilon_i = 1$, M_j' have minimal CB-type otherwise, then M' has CB-type (α, d)
- $\alpha \in X_2^*$ (so that $P(\alpha) = 0$): assume M_j' has CB-type $(\alpha, 1)$ if $0 \leq j < d$, $(2, 1)$ if $d \leq j < N$, then M' has CB-type (α, d).

ii. Conversely, we show that, if $M' \equiv M$ and $|M'| = \aleph_0$, the CB-type of M' belongs to X. We have already seen that, if the CB-type of M'_j is the minimal one for every j, then (α_0, d_0) is the CB-type of M'. If this is not the case, let α be the maximal CB-rank of M'_j's $(0 \leq j < N)$, and put for $1 \leq i \leq N_\alpha$

$$\epsilon_i = \begin{cases} 1 & \text{if } (\alpha, d_i^\alpha) \text{ is the CB-type of } M'_{P(\alpha)+i-1}, \\ 0 & \text{otherwise;} \end{cases}$$

then M' has got CB-type $\left(\alpha, \sum_{i=1}^{N_\alpha} \epsilon_i d_i^\alpha\right) \in X$.

Remark: Of course, we can modify the previous proofs to get finite CB-sets. For instance, we have:

- if $n, d_0, d_1, \ldots, d_n \in \omega^*$, $\alpha_0, \alpha_1, \ldots, \alpha_n \in \omega_1^*$, $\left(1, \sum_{i=1}^{n} d_i\right) \leq (\alpha_0, d_0) < (\alpha_1, d_1)$, $1 < \alpha_1 < \ldots < \alpha_n$, then $\{(\alpha_j, d_j) : 0 \leq j \leq n\}$ is a CB-set
- let $n \in \omega^*$, $X \subset \omega_1^* \times \omega^*$, $X^* = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, where $1 \leq \alpha_0 \leq \alpha_1$, $1 < \alpha_1 < \ldots < \alpha_n$; if $\alpha_0 < \alpha_1$, suppose that, for every $i \geq 1$, there are $d_1^i, \ldots, d_{N_i}^i \in \omega^*$ – not necessarily distinct – such that $X(\alpha_i) = \left\{\sum_{j=1}^{N_i} \epsilon_j d_j^i : \epsilon_j = 0, 1, \sum_{j=1}^{N_i} \epsilon_j > 0\right\}$, while $X(\alpha_0) = \{(\alpha_0, d_0)\}$ where $d_0 \geq \sum_{i=1}^{N} \sum_{j=1}^{N_i} d_j^i$ if $\alpha_0 = 1$; if $\alpha_0 = \alpha_1$, suppose that, for every $i \geq 2$, there are $d_1^i, \ldots, d_{N_i}^i \in \omega^*$ such that $X(\alpha_i) = \left\{\sum_{j=1}^{N_i} \epsilon_j d_j^i : \epsilon_j = 0, 1, \sum_{j=1}^{N_i} \epsilon_j > 0\right\}$, while, looking at $X(\alpha_1)$, there are $d_0, d_1^1, \ldots, d_{N_1}^1 \in \omega^*$ such that $X(\alpha_1) = \left\{d_0 + \sum_{j=1}^{N_1} \epsilon_j d_j^1 : \epsilon_j = 0, 1\right\}$; then X is a CB-set.

(The previous results can be obtained as corollaries of the following Lemmas 4.3 and 4.4.)

Example 4: For every λ such that $2 \leq \lambda < \omega$, $\{(\alpha, 1) : 2 \leq \alpha \leq \lambda\}$ is a CB-set.

Example 5: $\{(1, n(\alpha - 1))\} \cup \{(\nu, 1), \ldots, (\nu, n) : 2 \leq \nu \leq \alpha\}$ is a CB-set for every α such that $2 \leq \alpha < \omega$.

Lemma 4.3 *Let $X_1, \ldots, X_n$ be CB-sets, $(\alpha_j, d_j) = min\ X_j$, $X(\alpha_j) = \{d_j\}$, $X'_j = X_j - \{(\alpha_j, d_j)\}$ for $1 \leq j \leq n$. Put $\alpha_0 = max\{\alpha_1, \ldots, \alpha_n\}$, $d_0 = \sum_{\alpha_j = \alpha_0} d_j$ and suppose $\alpha_0 < (X'_1)^* < \ldots < (X'_n)^*$. Then $X'_1 \cup \ldots \cup X'_n \cup \{(\alpha_0, d_0)\}$ is a CB-set.*

Proof: Let M_j be a countable ω-stable structure such that X_j = CB-Spec $Th(M_j)$ $(1 \leq j \leq n)$; put $M = \dot{\bigcup}_{1 \leq j \leq n} M_j$, then it is easy to deduce that CB-Spec $Th(M) = X'_1 \cup \ldots \cup X'_n \cup \{(\alpha_0, d_0)\}$. In a similar way, we can deduce that, for every $(\alpha, d) \in \omega_1^* \times \omega^*$ such that $(\alpha_0, d_0) \leq (\alpha, d)$ and $\alpha < (X'_1)^*$, $X'_1 \cup \ldots \cup X'_n \cup \{(\alpha, d)\}$ is a CB-set.

Lemma 4.4 *Let $X_1, \ldots, X_n$; $\alpha_0, \alpha_1, \ldots, \alpha_n$; $d_0, d_1, \ldots, d_n$ be as above. Suppose now $\alpha_0 < (X_1')^* \leq \ldots \leq (X_n')^*$ and define, for $1 \leq i < j \leq n$,*

$$\delta(X_i, X_j) = \left\{\left(\alpha, \sum_{i \leq k \leq j} \epsilon_k d_k\right): \alpha \in \bigcap_{i \leq k \leq j} (X_k')^*, \right.$$

$$\left. d_k \in X_k(\alpha),\ \epsilon_k = 0, 1,\ \sum_{i \leq k \leq j} \epsilon_k \geq 2\right\}.$$

Then $X = X_1' \cup \ldots \cup X_n' \cup \left(\bigcup_{1 \leq i < j \leq n} \delta(X_i, X_j)\right) \cup \{(\alpha_0, d_0)\}$ is a CB-set.

Proof: Define $M_j (1 \leq j \leq n)$, M as above; let $M' \equiv M$, $|M'| = \aleph_0$, so that $M' = \dot{\bigcup}_{1 \leq j \leq n} M_j'$ where $M_j' \equiv M_j$, $|M_j'| = \aleph_0$. If M_j' has CB-type (α_j, d_j) for every j, M' has CB-type (α_0, d_0). Otherwise, let j be the maximal index such that M_j' has CB-type $(\alpha, d) \in X_j'$: if, for every $i < j$, the CB-rank of M_i' is less than α, M' has CB-type (α, d); if there is $i < j$ such that M_i' has CB-rank α (and i is the minimal index with this property), M' has CB-type $\left(\alpha, \sum_{i \leq k \leq j} \epsilon_k d_k\right)$, where

$$\epsilon_k = \begin{cases} 1 & \text{if the CB-rank of } M_k' \text{ is } \alpha \\ 0 & \text{otherwise} \end{cases}$$

so $\left(\alpha, \sum_{i \leq k \leq j} \epsilon_k d_k\right) \in \delta(X_i, X_j)$. Conversely, every element of X is the CB-type of a suitable structure $M' \equiv M$, $|M'| = \aleph_0$.

Looking now at the CB-sets X such that, for some $\alpha \in X^*$, $X(\alpha)$ is infinite, we prove the following.

Theorem 4.5 *Let $X \subset \omega_1^* \times \omega^*$, X^* infinite, and put $\alpha_0 = \min X^*$. Suppose:*
(a) *if $\alpha \in X^*$, $\alpha \geq 2$*
(b) *for every $\alpha \in X^*$, either $X(\alpha) = \omega^*$ or $X(\alpha) = \{1\}$*
(c) *when $X(\alpha) = \omega^*$, $X(\alpha + 1) \neq \varnothing$*
(d) *if $X(\alpha + 1) = \{1\}$ and $\alpha + 1 > \alpha_0$, $X(\alpha) = \omega^*$*
(e) *if $\alpha_i \in X^*$ for every $i \in N$ and $\alpha_i \uparrow \alpha$, $\alpha \in X^*$*
(f) *if $X(\alpha) = \{1\}$ and α is a limit ordinal and $\alpha > \alpha_0$, there is a sequence $\{\alpha_i \in X^*: i \in N\}$ such that $\alpha_i \uparrow \alpha$.*
Then X is a CB-set.

Example 6: $\{(\alpha, d): 2 \leq \alpha < \lambda, d \in \omega^*\} \cup \{\lambda, 1\}$ for $\omega \leq \lambda < \omega_1$.

Example 7: $\{(2n, d): n, d \in \omega^*\} \cup \{(2n + 1, 1): n \in \omega^*\} \cup \{(\omega, 1)\}$.

Example 8: $\{(2n, 1): n \in \omega^*\} \cup \{(2n + 1, d): n, d \in \omega^*\} \cup \{(\omega, 1)\}$.

Proof: We put $X_0^* = \{\alpha_0\}$, $\overline{X}^* = \{\alpha \in X^*: X(\alpha) = \omega^*\}$, $X_1^* = \overline{X}^* \cup (\overline{X}^* + 1)$. If $\alpha \in X^* - (X_0^* \cup X_1^*)$, then $X(\alpha) = \{1\}$, and α is a limit ordinal, so that there is a sequence $\{\alpha_i: i \in N\} \subset X^*$ such that $\alpha_i \uparrow \alpha$. We set $X_2^* = \{\alpha \in X^* - (X_0^* \cup X_1^*): \exists \alpha_0 < \alpha_1 < \alpha_2 < \ldots$ all in X_1^* such that $\alpha_i \uparrow \alpha\}$, notice that $X^* = X_0^* \cup X_1^* \cup X_2^*$, and X_1^*, X^* are infinite.

Case 1. $X(\alpha_0) = \{1\}$. We first suppose $\alpha_0 = 2$, and we define

$$M = \dot{\bigcup_{\alpha \in \overline{X}^*}} M_1^*(\alpha)$$

(notice that $\alpha_0 \notin \overline{X}^*$). Then, $Th(M)$ is ω-stable, and we claim $X =$ CB-Spec $Th(M)$. Recall that, for every countable model M' of $Th(M)$,

$$M' = \dot{\bigcup_{\alpha \in \overline{X}^*}} M_\alpha (\dot{\cup} M_\infty)$$

where $|M_\alpha| = \aleph_0$, $M_\alpha \equiv M_1^*(\alpha)$ for every α. We first show $X \subset$ CB-Spec $Th(M)$.

- (2, 1) is the CB-type of M' if M_α has CB-type (1, 1) for every $\alpha \in \overline{X}^*$
- if $\alpha \in \overline{X}^*$, (α, n) is the CB-type of M' when, for every $\beta \in \overline{X}^*$, M_β has CB-type (α, n) for $\beta = \alpha$, (1, 1) otherwise
- if $\alpha \in X_1^* - \overline{X}^*$, there exists $\nu \in \overline{X}^*$ such that $\alpha = \nu + 1$: suppose M_ν has CB-type $(\nu + 1, 1)$, while M_β has CB-type (1, 1) for every $\beta \in \overline{X}^* - \{\nu\}$: then M' has CB-type $(\alpha, 1)$
- if $\alpha \in X_2^*$, there is a sequence $\{\alpha_i \in X_1^*: i \in N\}$ such that $\alpha_i \uparrow \alpha$; we can suppose $\alpha_i \in X^*$ for every i; for every $\beta \in \overline{X}^*$, let M_β have CB-type $(\alpha_i, 1)$ if there is $i \in N$ such that $\beta = \alpha_i$, (1, 1) otherwise; we get a model M' of $Th(M)$, such that the CB-type of M' is $(\alpha, 1)$.

Conversely, CB-Spec $Th(M) \subset X$: let $M' \equiv M$, $|M'| = \aleph_0$, put $Y = \{\alpha \in \overline{X}^*$: the CB-rank of M_α is higher than $1\}$. We distinguish the following cases:

- $Y = \emptyset$: then M' has CB-type $(2, 1) \in X$
- $Y \neq \emptyset$, there exists $\alpha = max\ Y$: in this case, if M_α has CB-type (α, n) but $\alpha = \nu + 1$ for a suitable $\nu \in \overline{X}^*$ and M_ν has CB-type $(\nu + 1, 1)$, M' has CB-type $(\alpha, n + 1) \in X$; otherwise the CB-type of M' coincides with the CB-type of M_α, in particular belongs to X
- $Y \neq \emptyset$, Y admits no maximal element: let $\alpha = sup\ Y$, then M' has CB-type $(\alpha, 1)$. Furthermore there is a sequence $\{\alpha_i: i \in N\} \subset Y$ such that $\alpha_i \uparrow \alpha$, so $(\alpha, 1) \in X$.

When $\alpha_0 > 2$, we consider $M \dot{\cup} M_1^1(\alpha_0)$ instead of M.

Case 2. $X(\alpha_0) = \omega^*$. Follow the same procedure as above, recalling that $\alpha_0 \in \overline{X}^*$ in this case.

A more general version of Theorem 4.5 can be given starting from the structure $M_d^*(\alpha)$ $(d \in \omega^*)$ instead of $M_1^*(\alpha)$.

Remark: We have also that, if $X \subset \omega_1^* \times \omega^*$ and $X^* = \{1, \alpha_i, \alpha_i + 1: 1 \leq i \leq m\}$ where $1 < \alpha_1 < \ldots < \alpha_m$, $X(\alpha_i) = \{(\alpha_i, n): n \in \omega^*\}$ for $1 \leq i \leq m$, $X(\alpha_i + 1) = \{(\alpha_i + 1, 1)\}$ when $\alpha_i + 1 < \alpha_{i+1}$ or $i = m$, $X(1) = \{m\}$, then X is a CB-set. In fact, we can apply Lemma 4.4 when $(\alpha_0, d_0) = (1, m)$ and, for $1 \leq i \leq m$, $X_i = \{(1, 1)\} \cup \{(\alpha_i, n): n \in \omega^*\} \cup \{(\alpha_i + 1, 1)\}$, so that, for $1 \leq i < m$,

$$\delta(X_i, X_{i+1}) = \begin{cases} \emptyset & \text{if } \alpha_{i+1} > \alpha_i + 1 \\ \{(\alpha_{i+1}, n): n \geq 2\} \subset X_{i+1} & \text{otherwise,} \end{cases}$$

while, if $j > i + 1$,

$$\delta(X_i, X_j) = \emptyset.$$

Notice that we can assume the minimal element of X is (α_0, d_0) for every (α_0, d_0) such that $(1, m) \leq (\alpha_0, d_0) \leq (\alpha_1, 1)$.

Example 9: $\{(\alpha, d): 2 \leq \alpha < \lambda, d \in \omega^*\} \cup \{(\lambda, 1)\}$ (for $2 < \lambda < \omega$) is a CB-set.

We are going now to the main results of this paper: we follow a more complicated line of thought to construct more complicated CB-sets, assuming in particular Examples 2.5–8 as basic structures.

Theorem 4.6 *Let* $X \subset \omega_1^* \times \omega^*$ *with maximal element* $(\lambda + 3, 1)$ *and such that*:
(a) $X(\lambda + 2) = \omega^*$
(b) *if* $\alpha \in X^*$, $\alpha \geq 3$.
Then, X *is a CB-set*.

Proof: We may limit our examination to the case $\lambda > 1$. First we assume that $(3, 1)$ is the minimal element of X. We set $Y' = X - \{(\lambda + 3, 1)\}$ and we define $Y \subset \omega_1^* \times \omega^*$ in the following way: $Y^* - \{3\} = (Y')^* - \{3\}$, $Y(\nu) = Y'(\nu)$ if $\nu \in (Y')^* - \{3\}$, $Y(3) = \{d - 1: d \in Y'(3), d > 1\}$. Notice that (a) implies Y is infinite. We consider now the theory T whose models are the following structures:

$$M = \left(\dot{\bigcup_{(\alpha,d)\in Y}} M_{\alpha,d}\right) \dot{\cup} \left(\dot{\bigcup_{\substack{(\alpha,d),(\beta,e)\in Y\\(\alpha,d)<(\beta,e)}}} M_{\alpha,d;\beta,e}\right) (\dot{\cup} M_\infty)$$

where $M_{\alpha,d} \equiv M'_d(\alpha)$, $M_{\alpha,d;\beta,e} \equiv M(\lambda)$ and, for every (α, d), $(\beta, e) \in Y$ such that $(\alpha, d) < (\beta, e)$, two isomorphisms

$$h^{(1)}_{\alpha,d;\beta,e}: P^{M_{\alpha,d}} \to P_1^{M_{\alpha,d;\beta,e}}$$
$$h^{(2)}_{\alpha,d;\beta,e}: P^{M_{\beta,e}} \to P_2^{M_{\alpha,d;\beta,e}}$$

are given. (Remember that $M_{\alpha,d} = M_1 \dot{\cup} \ldots \dot{\cup} M_d$ where $M_1 \cong \ldots \cong M_d$, $M_1, \ldots, M_d \equiv M'_1(\alpha)$, so we mean by $P^{M_{\alpha,d}}$ P^{M_1}, for example; similarly for $P^{M_{\beta,e}}$). T is ω-stable; furthermore, if $M \models T$ and $|M| = \aleph_0$, then, for every (α, d), $(\beta, e) \in Y$, $(\alpha, d) < (\beta, e)$, the CB-type of $M_{\alpha,d;\beta,e}$ is

- (2,1) when either $M_{\alpha,d}$ or $M_{\beta,e}$ has minimal CB-type
- $(\lambda + 2, 1)$ otherwise.

We claim $X =$ CB-Spec T. First suppose $M \models T$, $|M| = \aleph_0$, we prove that the CB-type of M belongs to X. We can distinguish the following cases:

- for every $(\alpha, d) \in Y$, $M_{\alpha,d}$ admits minimal CB-type: M has CB-type (3,1)
- there is one and only one element $(\alpha, d) \in Y$ such that the CB-type of $M_{\alpha,d}$ is (α, d): if $\alpha = 3$, M has CB-type $(3, d + 1)$; if $\alpha > 3$, M has CB-type (α, d); in both cases, the CB-type of M belongs to X
- there is a finite number $n \geq 2$ of elements $(\alpha, d) \in Y$ such that $M_{\alpha,d}$ has CB-type (α, d): M has CB-rank $\lambda + 2$, and its CB-degree equals the

sum between $\sum_{1 \le i < n} i$ and, in case, all the CB-degrees d corresponding to the CB-rank $\alpha = \lambda + 2$

- there are infinitely many pairs $(\alpha, d) \in Y$ such that $M_{\alpha,d}$ has got CB-type (α, d): then M has CB-type $(\lambda + 3, 1)$.

Conversely, it is straightforward to see that, for every $(\alpha, d) \in X$, there is a countable model M of T such that the CB-type of M is (α, d).

Let now, more generally, (α_0, d_0) be the minimal element of X, $(\alpha_0, d_0) > (3, 1)$.

- $\alpha_0 > 3$: we define $X_0 \subset \omega_1^* \times \omega^*$ in the following way: $X_0^* - \{\alpha_0, 3\} = X^* - \{\alpha_0, 3\}$; $X_0(\alpha_0) = \{d - d_0: d \in X(\alpha_0), d > d_0\}$, $X_0(3) = \{1\}$, $X_0(\nu) = X(\nu)$ if $\nu \in X^* - \{\alpha_0\}$. Let T_0 be the ω-stable theory such that $X_0 = \text{CB-Spec } T_0$, choose a model M_0 of T_0 and consider the theory T of $M_0 \dot\cup M_{d_0}^1(\alpha_0)$: T is ω-stable, and CB-Spec $T = X$.
- $\alpha_0 = 3$, $d_0 > 1$: we define $X_0 \subset \omega_1^* \times \omega^*$ in the following way: $X_0^* = X^*$, $X_0(3) = \{d - d_0 + 1: d \in X(3)\}$, $X_0(\nu) = X(\nu)$ if $\nu > 3$; let T_0 be the ω-stable theory whose CB-spectrum is X_0, choose $M_0 \models T_0$, and consider the theory T of $M_0 \dot\cup M_{d_0 - 1}^1(3)$: T is ω-stable and CB-Spec $T = X$.

Remark: A similar proof shows the following is a CB-set:

$X \subset \omega_1^* \times \omega^*$, X admits a maximal element $(\lambda + 3, 1)$, and

a. $X(\lambda + 2) = \left\{ \sum_{1 \le k \le n} k: n \in \omega^* \right\}$

b. $\{(\alpha, d) \in X: \alpha < \lambda + 2\}$ is infinite

c. if $\alpha \in X^*$, $\alpha \ge 3$.

Similarly we have:

Theorem 4.7 *Let $X \subset \omega_1^* \times \omega^*$ admit as maximal element $(\lambda, 1)$ where λ is a limit ordinal and*:
(a) *$\{\alpha: X(\alpha + 2) \neq \emptyset\}$ is cofinal in λ*
(b) *for ever $\alpha \in X^*$, $\alpha \ge 3$.*
Then X is a CB-set.

Proof: First we assume that $(3, 1)$ is the minimal element of X. We set $Y' = X - \{(\lambda, 1)\}$ and we define $Y \subset \omega_1^* \times \omega^*$ as above: $Y^* - \{3\} = (Y')^* - \{3\}$, $Y(\nu) = Y'(\nu)$ if $\nu \in (Y')^* - \{3\}$, $Y(3) = \{d - 1: d \in Y'(3), d > 1\}$. Notice that (a) implies Y is infinite. However, $Z = \{((\alpha, d), (\beta, e)) \in Y^2: (\alpha, d) < (\beta, e)\}$ is a countable set, so we give some enumeration $\{((\alpha_n, d_n), (\beta_n, e_n)): n \in N\}$ to Z, and we take at the same time a sequence $\{\lambda_n: n \in N\}$ of ordinals such that $\lambda_n \uparrow \lambda$. We consider the following function ϕ having domain Z:

- $\phi((\alpha_0, d_0), (\beta_0, e_0)) = (\gamma_0, g_0)$ where $\gamma_0 = min\{\gamma: X(\gamma + 2) \neq \emptyset, \beta_0 < \gamma\}$ and $g_0 = min\ X(\gamma_0 + 2)$
- $\phi((\alpha_{n+1}, d_{n+1}), (\beta_{n+1}, e_{n+1})) = (\gamma_{n+1}, g_{n+1})$, where $\gamma_{n+1} = min\{\gamma: X(\gamma + 2) \neq \emptyset, \beta_{n+1}, \gamma_n, \lambda_n < \gamma\}$, and $g_{n+1} = min\ X(\gamma_{n+1} + 2)$.

Let T be the ω-stable theory whose models are the structures

$$M = \left(\dot{\bigcup}_{(\alpha,d)\in Y} M_{\alpha,d} \right) \dot{\bigcup} \left(\dot{\bigcup}_{((\alpha,d),(\beta,e))\in Z} M_{\alpha,d;\beta,e} \right) [\dot{\bigcup} M_{\infty}].$$

M is defined as above, in particular, for every $((\alpha, d), (\beta, e)) \in Z$, $M_{\alpha,d;\beta,e} \equiv gM(\gamma)$ where $(\gamma, g) = \phi((\alpha, d), (\beta, e))$, so that the CB-type of $M_{\alpha,d;\beta,e}$ is:

- $(2, g)$ if either $M_{\alpha,d}$ or $M_{\beta,e}$ has minimal CB-type
- $(\gamma + 2, g)$ otherwise.

We claim $X =$ CB-Spec T. First, let $M \models T$, $|M| = \aleph_0$. We shall prove that the CB-type of M belongs to X; we distinguish again four cases:

- for every $(\alpha, d) \in Y$, $M_{\alpha,d}$ admits minimal CB-type: M has CB-type $(3, 1)$
- there is one and only one $(\alpha, d) \in Y$ such that the CB-type of $M_{\alpha,d}$ is (α, d): if $\alpha = 3$, M has CB-type $(3, d + 1)$; if $\alpha > 3$, M has CB-type (α, d); in both cases, the CB-type of M belongs to X
- there is a finite number $n \geq 2$ of elements $(\alpha, d) \in Y$ such that $M_{\alpha,d}$ has CB-type (α, d): take the corresponding maximal pair $((\alpha, d), (\beta, e))$ in the enumeration of Z, let $(\gamma, g) = \phi((\alpha, d), (\beta, e))$, so M has CB-type $(\gamma + 2, g) \in X$
- there are infinitely many pairs $(\alpha, d) \in Y$ such that $M_{\alpha,d}$ has got CB-type (α, d): then M has CB-type $(\lambda, 1)$.

Conversely it is straightforward that, for every $(\alpha, d) \in X$, there is a model M of T such that $|M| = \aleph_0$ and the CB-type of M is (α, d). Finally, if the minimal pair of X is $(\alpha_0, d_0) > (3, 1)$, we can proceed as in Theorem 4.6.

Remarks: 1. Looking at Theorems 4.6 and 4.7, notice that similar results can be obtained about finite CB-sets.

2. Lemmas 4.3, 4.4, and disjoint unions with suitable pseudo-$\aleph_0$-categorical ω-stable structures can be used to construct new CB-sets, starting from the previous ones. In this way, we can partially cover the $\lambda + 1$, $\lambda + 2$ cases.

3. As a final remark, we note that there are $2^{\aleph_0}$ CB-sets, i.e., $2^{\aleph_0}$ classes of ω-stable theories of the equivalence relation: $T_1 \sim T_2$ if and only if T_1 and T_2 have got the same CB-spectrum.

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