Notre Dame Journal of Formal Logic
Volume 28, Number 2, April 1987

On Defining Sentential Connectives

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1 Introduction Since Lesniewski[1] first set the rules of definition for the introduction of new notation into a logical system, sentential connectives have been almost completely neglected. Traditional accounts of the rules of definition, like Carnap's ([2], pp. 66–73) and Suppes' ([9], pp. 154–162), treat only relation symbols, operation symbols, and individual constants. Even when not neglected, as by Tarski ([10], pp. 150–151), the rule for defining connectives covers their introduction in but a few expressively powerful languages. In this paper, then, we develop a sufficiently general, purely syntactical rule for defining sentential connectives. And because traditional accounts of the rules of definition usually call for us to frame definitions as material equivalences within the object language, these treatments cannot provide for the introduction of new notation into weaker languages which lack a suitable equivalence connective. So, we also extend our treatment of the definition of connectives to other parts of speech, thus allowing the introduction of new symbols into impoverished systems as well.

To begin, recall the rule which Patrick Suppes offers for defining relation symbols:

> An equivalence D introducing a new n-place relation symbol P is a proper definition in a theory if and only if D is of the form
>
> $$P(v_1,\ldots,v_n) \equiv S,$$
>
> and the following restrictions are satisfied: (i) $v_1,\ldots,v_n$ are distinct variables, (ii) S has no free variables other than $v_1,\ldots,v_n$, and (iii) S is a formula in which the only non-logical constants are primitive symbols and previously defined symbols of the theory. [9], p. 156

Because we find analogous concepts later, we call these three restrictions: (i) *n-arity* (since we require that P take n distinct variables), (ii) *parametric relevance*

*I am grateful to Gerald Massey of the University of Pittsburgh for his helpful comments and guidance.

Received June 4, 1985; revised August 27, 1985

(since we require that S contains only $v_1, \ldots, v_n$ as free variables), and (iii) *order* (since we require that the defined symbols of a theory be added in a fixed sequence).

Following Lesniewski, Suppes gives the definition of a new predicate P as a postulate of the theory. Flanking the biconditional sign are the *definiendum*, which contains the new symbol, and the *definiens*, which contains as its only nonlogical constants those that are primitive to the theory or have been previously defined. Further, Suppes demands, as we do, that definitions meet the traditional standards of *eliminability* so that we may remove the defined symbol from any context by replacing it with an expression containing as nonlogical constants only primitive and previously defined symbols, and *noncreativity* so that by the introduction of the new symbol we admit as a theorem no formula, lacking the new symbol, that was unprovable without the new definition.

But in general, definitions of given connectives using the biconditional, as an approach directly analogous to Suppes' would suggest, will not do. Too often, we sacrifice the requirements of eliminability and noncreativity. Instead, we offer a rule for defining connectives, and later, rules for relation, operation, and constant symbols, that require definitions to assume the form of *schematizable*, *two-way* inference rules. As a preliminary discussion, then, we devote the next section to inference rule schemes.

2 On inference rules A single *inference* takes the form $\Delta \vdash A$, where Δ is a set of wffs called *premises* and A is a wff called the *conclusion*. Thus, in a standard sentential language:

(1) $\{p \supset q, p\} \vdash q$,
(2) $\{(p \supset q) \supset q, p \supset q\} \vdash q$,
(3) $\{p \supset p, p\} \vdash p$

are all inferences. Similarly, in a standard first-order language:

(4) $\{\forall xFx \supset \forall xGx, \forall xFx\} \vdash \forall xGx$
(5) $\{\forall x\exists yHxy\} \vdash \forall x\exists zHxz$
(6) $\{\forall x\exists yHxy\} \vdash \forall z\exists yHzy$,

too, are inferences.

Rules, then, are sets of inferences; simply, the set of (1)–(6) is a rule. But if we look at the structure of the inferences above, we note that (1)–(4) have the same form, as do (5) and (6). And by expressing rules through *schemes* stated in the metalanguage, we can specify infinitely many inferences of the same form. So if we use 'A' and 'B' to stand for any wff, we find that inferences (1)–(4) are all instances of the rule scheme for modus ponens:

(7) $\{A \supset B, A\} \vdash B$.

Using 'v' with subscripts to indicate any individual variable, we see that (5) and (6) are instances of the rule for the alphabetic change of a bound variable:

(8) $\{(\ldots A(v_1) \ldots)\} \vdash (\ldots A(v_2) \ldots)$.

In stating (8), however, we must impose the additional restrictions outside the metalanguage that: (i) v_1 is a variable that is not free in $A(v_1)$, (ii) v_2 is a variable that does not occur in $A(v_1)$, and (iii) $(\ldots A(v_2)\ldots)$ results from $(\ldots A(v_1)\ldots)$ by substituting $A(v_2)$ for $A(v_1)$ in $(\ldots A(v_1)\ldots)$, so to avoid incorrect inferences. Without these restrictions, we would have:

(9) $\{\forall xHxy\} \vdash \forall xHxx$

as a proper inference according to (8).

Although both are inference rules, (7) and (8) differ markedly: while (7) is *schematizable*,[2] (8) is not. Unlike the rule for the alphabetic change of a bound variable, the rule for modus ponens requires no restrictions; we can express it by referring to its structure alone. Indeed, the expressions in (7) are identical to formulas of the object language save that metavariables hold the place of wffs in the language. In (8), however, we must do more than refer to structure: we must comment upon it.

We formalize our notion of schematizable rules by first giving formation rules for *schematic expressions* which are analogous to the formation rules of the object language. For example, in a first-order system with identity, whose primitive connectives are negation, implication, and universal quantification,[3] we can exhaustively characterize all *schematic terms* as follows:

(i) If α is a metavariable standing for an individual variable, then α is a schematic term.
(ii) If γ is a metavariable standing for an individual constant, then γ is a schematic term.
(iii) If δ is a metavariable standing for a term (a variable, a constant, or an n-place operation symbol followed by n terms), then δ is a schematic term.
(iv) If η is a metavariable standing for an n-place operation symbol and $\delta_1, \ldots, \delta_n$ are schematic terms, then $\eta\delta_1, \ldots, \delta_n$ is a schematic term.

And all the *schematic wffs*:

(i) If ϕ is a metavariable standing for a wff, then ϕ is a schematic wff.
(ii) If π is a metavariable standing for an n-place relation symbol and $\delta_1, \ldots, \delta_n$ are schematic terms, then $\pi\delta_1, \ldots, \delta_n$ is a schematic wff.
(iii) If ϕ and ψ are schematic wffs, then so are $\ulcorner{\sim}\phi\urcorner$ and $\ulcorner\phi \supset \psi\urcorner$.
(iv) If ϕ is a schematic wff and α is a metavariable standing for an individual variable, then $\ulcorner\forall\alpha\phi\urcorner$ is a schematic wff.
(v) If δ_1 and δ_2 are schematic terms, then $\ulcorner\delta_1 = \delta_2\urcorner$ is a schematic wff.

We then call a rule *schematizable* if and only if we may state it using schematic expressions alone.

Much of our notion of schematizability rests upon our conventions about the metalanguage. We usually let the metavariables 'A' and 'B' stand for any wff, even identical wffs. But if one were to allow different metavariables to stand for only distinct wffs, (3) would not count as a proper inference according to (7). Modus ponens could then only be given by two rule schemes:

(10) $\{A \supset A, A\} \vdash A$

together with (7). Moreover, if one were to allow only bound variables in the metalanguage and by '$A(v_1)$' one were to mean that A contained v_1 as its only variable, then the rule for the alphabetic change of a bound variable would be schematizable, if somewhat limited. Here, however, we follow customary usage, allowing for none of these provisions. Later, we argue that unschematizable rules merely encode those that are schematizable.

Not only will our rule for defining connectives require that definitions be schematizable, but because we wish to interchange definiendum and definiens in any context, it also calls for definitions to be *two-way* inference rules. Put simply, in a two-way rule of exchange, a single premise and conclusion can shift roles. Consider the rules of double negation:

(11) $\{\sim\sim A\} \vdash A$
(12) $\{A\} \vdash \sim\sim A$.

Dropping the set notation for convenience, we present (11) and (12) as the two-way rule:

(13) $\sim\sim A \dashv\vdash A$.

We further require that a definitional rule can operate on any part of a wff, so that we may exchange the new connective, together with its scope, in an arbitrary formula, for expressions containing only primitive and previously defined symbols as nonlogical constants. Unlike the rule for modus ponens, which would yield improper inferences if we did not restrict its application to entire formulas, a definitional rule may operate in any context. So, from the definition of an n-place connective $\otimes$ and $(\ldots\otimes(A_1,\ldots,A_n)\ldots)$, we can infer $(\ldots S\ldots)$, where S is an expression lacking the newly defined connective.

Finally, to ensure eliminability of the new connective, we assume that definitional rule schemes may operate on definienda or definientia which can contain free occurrences of variables. For if one were to do otherwise, one could not replace the wff '$Fx \star Gx$' which contains a newly defined, two-place connective $\star$ in a first-order language with a suitable expression lacking $\star$. Thus, in some systems where one can make inferences only when the premises and conclusion contain no free variables (e.g., see [7]), our proposed definitional rule schemes will not apply. With all this before us, then, we lay out our rule for defining connectives more formally in the next section.

3 *A rule for defining connectives*

With a clear understanding of the operation of inference rules, we put forth a rule for defining sentential connectives:

> A schematizable rule of interchange R introducing a new n-place connective $\otimes$ is a proper definition in a theory if and only if R is of the form
>
> $$\otimes(\phi_1,\ldots,\phi_n) \dashv\vdash \psi,$$
>
> and the following restrictions are satisfied: (i) *n-arity*: $\phi_1,\ldots,\phi_n$ are distinct metavariables standing for wffs; (ii) *parametric relevance*: ψ has no metavariables standing for wffs other than $\phi_1,\ldots,\phi_n$; and (iii) *order*: ψ is a schematic wff in which the only nonlogical constants are primitive symbols and previously defined symbols of the theory. (For a similar approach, see [3], p. 288.)

Although we cannot enter a definition given according to this rule into the course of a proof, as we can with the traditional postulate approach, nevertheless, we can introduce new notation within a deduction by applying a proper definitional inference rule. Underlying this shift from definition as postulate to definition as rule scheme is the equivalence of $\vdash (A \equiv B)$ and $A \dashv\vdash B$ in classical systems of logic with the material biconditional. This equivalence, however, does not hold for all languages. Consider the many-valued systems of Lukasiewicz,[4] where the first entails the second, but not the converse. Thus, by adopting the new rule for defining connectives, we offer a requirement for definability weaker than that of the standard treatment. Still, with this rule, we avoid many serious complications that would arise if we were to take Suppes' approach directly in defining connectives, as we shall see in the four examples below.

Strictly following Suppes, one might frame a rule for connectives so that their definitions take the form $\otimes(A_1, \ldots, A_n) \equiv S$, where $A_1, \ldots, A_n$ and S stand for wffs. But such a rule often runs afoul of the eliminability requirement.[5] Consider a propositional language L_1 with classical negation, material implication, and equivalence as primitive connectives. One might then offer a definitional postulate for disjunction:

(14) $p \vee q \equiv {\sim} p \supset q.$

Unless L_1 includes substitution as a rule of inference, 'v' would not be eliminable. We could not replace '$q \vee p$' with '${\sim} q \supset p$' given only (14) without a substitution rule. If, however, we were to adopt the definitional rule scheme:

(15) $A \vee B \dashv\vdash {\sim} A \supset B,$

then, following our rule for defining connectives, not only could we eliminate '$p \vee q$' in favor of '${\sim} p \supset q$', as (14) suggests, but also '$q \vee p$' in favor of '${\sim} q \supset p$', since A and B are metavariables standing for any sentential variable, indeed any wff, of L_1.

And still, if our language has substitution as a rule of inference, other difficulties set in. Consider a propositional language L_2 that has classical conjunction and material implication as primitive connectives. Following a rule strictly analogous to Suppes', one might offer:

(16) $p \equiv q . \equiv . (p \supset q) \cdot (q \supset p)$

as a definition of equivalence. But (16) is hopelessly circular. Not with the definitional scheme:

(17) $A \equiv B \dashv\vdash (A \supset B) \cdot (B \supset A),$

however. Indeed, by applying (17) twice to the theorem:

(18) $(p \supset q) \cdot (q \supset p) . \supset . (p \supset q) \cdot (q \supset p) : \cdot$
$: (p \supset q) \cdot (q \supset p) . \supset . (p \supset q) \cdot (q \supset p),$

we can establish (16) as a theorem of L_2. More generally, within a classical system, once we have (17), we may derive all instances of biconditional statements analogous to the definitional rule of an arbitrary n-place connective $\otimes$, namely $\otimes(\phi_1, \ldots, \phi_n) \dashv\vdash \psi$, replying only on the reflexivity of implication. Below, S is any instance of ψ.

Proof: $\vdash(S \supset S)\cdot(S \supset S)$
$\vdash(\otimes(A_1,\ldots,A_n) \supset S)\cdot(S \supset \otimes(A_1,\ldots,A_n))$
$\vdash\otimes(A_1,\ldots,A_n) \equiv S.$

Even worse, one can construct a language too impoverished to supply a suitable equivalence connective. Consider a first-order language L_3 with only conjunction and disjunction as its primitive connectives. There is no way to express the truth function of the material biconditional using just these two connectives. And without equivalence, a rule for defining connectives that is analogous to Suppes' for relations would fail. Moreover, in L_3, even Suppes' rule for relations fails since it is given for a standard first-order language that provides the material biconditional. But the rule given here for connectives and those offered later for other parts of speech enable us to introduce new notation into weaker languages.

Further, if one were to frame definitions of connectives as postulates using material equivalence in certain modal systems, the eliminability requirement would be violated. We show this indirectly. Consider a modal language L_4 which provides a rule of substitution and an equivalence connective but lacks a rule of necessitation. We propose the definition of a new n-place connective $\otimes$ as $\otimes(A_1,\ldots,A_n) \equiv S$. To demonstrate that $\otimes$ is eliminable, we let S_1 be a wff containing $\otimes$ in some context, $(\ldots\otimes(A_1,\ldots,A_n)\ldots)$. Given the order restriction, we should be able to produce a new wff S_2 where S_2 is $(\ldots S\ldots)$, which, though logically equivalent to S_1, does not contain the new symbol $\otimes$. But that S_1 is equivalent to S_2 depends solely on the substitutivity of the material biconditional. In L_4, however, this rule fails. Thus, we lose eliminability. For all these reasons, then, we reject a system of definitional postulates and, instead, opt to treat the definitions of new connectives as replacement rules of the theory.

We now show that a new connective defined according to our rule meets both the eliminability and noncreativity requirements. We assume: (i) that the definition of an arbitrary n-place connective $\otimes$ is a two-way, schematizable inference rule, $\otimes(\phi_1,\ldots,\phi_n) \dashv\vdash \psi$, which can operate on wffs containing free variables, and (ii) that the n-arity, parametric relevance, and order restrictions hold. To see that we can eliminate $\otimes$ from any context, we again consider a wff S_1, $(\ldots\otimes(A_1,\ldots,A_n)\ldots)$, which contains $\otimes$. Because our definitional scheme entitles us to write S, an appropriate instance of ψ, for each occurrence of $\otimes(A_1,\ldots,A_n)$ in S_1, we can produce a wff S_2, where S_2 is $(\ldots S\ldots)$, which contains only primitive and previously defined symbols as its nonlogical constants. Thus, $\otimes$ is eliminable, provided that its definitional scheme applies to wffs having free occurrences of variables, as noted in Section 2.

Further, as long as the theory's axioms are given as schemes or the axioms contain only primitive symbols, the new connective's definition is noncreative. We assume there is a wff T, not containing $\otimes$, which is provable from the axioms and previous definitions of the theory along with the new definition of $\otimes$. Let $\mathcal{P}$ be a sequence of n wffs such that the nth wff is T and for each k, $1 \leq k \leq n$, the kth line of $\mathcal{P}$ is either an axiom or wff that follows from some previous lines by a rule of inference or definitional rule scheme. To construct $\mathcal{P}'$, for every k, if the kth line of $\mathcal{P}$ is a wff which does not contain $\otimes$, we enter that wff as the kth line of $\mathcal{P}'$. If the kth line of $\mathcal{P}$ contains $\otimes$, we eliminate it as its definition prescribes and enter the resulting wff as the kth line of $\mathcal{P}'$.

Now we see that $\mathcal{P}'$, too, is a proof of T. Suppose first that the kth line of $\mathcal{P}$ is an axiom. If the theory's axioms are given as schemes, the kth line of $\mathcal{P}'$ is an axiom, whether the kth line of $\mathcal{P}$ contains $\otimes$ or not. And, of course, in a theory whose axioms contain only primitive symbols, if the kth line of $\mathcal{P}$ is an axiom, then so is the kth line of $\mathcal{P}'$. But if the system's axioms contain defined symbols and are not given as schemes, and if the kth line in the proof $\mathcal{P}$ is an axiom with the newly defined symbol, then we could not claim that the wff resulting from the symbol's removal is also an axiom. The guarantee of noncreativity is lost. There is, however, good reason not to countenance such systems, for if we were to choose not to introduce the new symbol, since definitions are never mandatory, any axiom containing the new symbol would be meaningless.

Without loss of generality, suppose now that the kth line of $\mathcal{P}$ follows from its preceding line by an inference rule or previous definition. Then, the kth line of $\mathcal{P}'$, too, follows from its preceding line by the same inference rule or definition. If the kth line of $\mathcal{P}$ follows from its preceding line by the new definition, lines k and $k-1$ of $\mathcal{P}'$ will be identical. Thus, $\mathcal{P}'$ is a proof of T which does not appeal to the definition of $\otimes$.

We now examine the converse. Our three restrictions prevent innumerable catastrophes. If n-arity were not imposed, one could admit as a proper definition of a two-place connective:

(19) $A * A \vdash\!\dashv (A \cdot A) \vee A$

into a language with conjunction and disjunction previously defined. Yet with (19), it would be impossible to eliminate '$*$' from the expression '$p * q$'. Parametric relevance prevents the definition of a one-place connective like:

(20) $\odot A \vdash\!\dashv A \cdot B.$

If '$\odot p$' were in some context $(\ldots \odot p \ldots)$, it would follow that for any wff B, $(\ldots (p \cdot B) \ldots)$. Similarly, if for some wff B, $(\ldots (p \cdot B) \ldots)$, then $(\ldots \odot p \ldots)$would result. It is but one step to absurdity: if there is a B such that $(\ldots (p \cdot B) \ldots)$, then for all B, $(\ldots (p \cdot B) \ldots)$. (20) is creative. And finally, order prohibits circularity of definition, allowing for the eliminability of newly defined connectives.

4 On schematizability Impermissible according to our rule for defining connectives are those definitions that are unschematizable. Just as the definitional postulates of Suppes make no provision for restrictions outside the object language, so our definitional rule schemes prohibit restrictions outside the metalanguage. One might, for example, introduce the numerical existential quantifier by:

(21) $\exists_2 v A(v) \vdash\!\dashv \exists v_1 \exists v_2 (v_1 \neq v_2 \cdot A(v_1) \cdot A(v_2)).$

But (21) alone is not enough. As with the rule for the alphabetic change of a bound variable, one must impose the additional restrictions that: (i) v_1 and v_2 are distinct variables, (ii) v_1 and v_2 do not occur in A, and (iii) v_1 and v_2 are the first two allowable variables in the alphabetic sequence. For without condition (i), one could construe '$\exists_2 xFx$' as '$\exists x \exists x(x \neq x \cdot Fx \cdot Fx)$'. Without (ii), '$\forall y \exists_2 xGxy$' could be taken as '$\forall y \exists x \exists y(x \neq y \cdot Gxy \cdot Gyy)$'. And without (iii), a single instance of the definiendum would allow infinitely many interpretations of the definiens. Thus, the definition of the numerical extistential quantifier does not meet our demand of schematizability ([1] provides another, more complex, example). But each condition selectively eliminates instances of (21) from those inferences the rule should capture. (21) alone serves as a guide to the correct inferences which the added restrictions specify. And as with the rule for the alphabetic change of a bound variable, if one were to give a metalanguage which allowed only correct instances of (21), the definition would become purely schematic. Thus, (21) encodes a schematizable rule in a restricted metalanguage.

So why disallow definitions like (21)? Restricting our definitional rules to those that are schematizable enhances the formal character of the language. It is, after all, the formal structure of the language which we wish our definitions to mirror. By allowing unschematizable definitions as rules of the system, we sacrifice structure in order to expand the language. Insisting, then, that proper definitions be schematizable, we ensure that definitions given according to our rule will reflect only the structural aspects of the language, as definitions given according to the traditional rules do.

We have seen with the definition of the numerical existential quantifier how an unschematizable rule encodes a single schematizable rule in a restricted metalanguage. In turning to multigrade connectives, we shall see how a single unschematizable rule encodes denumerably many proper definitions. Our rule for defining connectives provides only for *unigrade* connectives which attach to a fixed number of arguments to form new wffs; we cannot account for *multigrade*[6] connectives like the word 'and' which may take any finite number of arguments. One might consider Wittgenstein's multigrade 'N' of [12] (Propositions 5.501–5.51) as defined by:

(22) $N(A_1, \ldots, A_n) \mathrel{\vert\!\!-\!\!\vert} (\ldots(\sim A_1 \cdot \sim A_2) \cdot \ldots \cdot \sim A_n)$.

But because the number of sentential arguments may vary with each instance, (22) is an unschematizable rule. Still, by entering the schemes which (22) encodes into a theory, we can find a suitable account of 'N'. Rather than treat (22) as n iterations of the connective '$\cdot$', as some propose, we view (22) as encoding proper definitions for infinitely many unigrade connectives[7] as follows:

(23) $N^1(A_1) \mathrel{\vert\!\!-\!\!\vert} \sim A_1$
$N^2(A_1, A_2) \mathrel{\vert\!\!-\!\!\vert} \sim A_1 \cdot \sim A_2$
$N^3(A_1, A_2, A_3) \mathrel{\vert\!\!-\!\!\vert} (\sim A_1 \cdot \sim A_2) \cdot \sim A_3$
.
.
.

The chain of definitions (23), where we indicate the arity of each connective by a superscript, expresses each instance of Wittgenstein's 'N' schematically. So, the recursive set U, whose members are the unigrade connectives 'N^1', 'N^2', 'N^3',... represents 'N', where for every positive integer n, $N(A_1, \ldots, A_n)$ is equivalent to $N^n(A_1, \ldots, A_n)$.

But suppose that one were to modify our rule for defining connectives to allow recursive definitions. The two rules of interchange:

(24) $N^1(A_1) \vdash\dashv {\sim}A_1$
(25) $N^n(A_1,\ldots,A_n) \vdash\dashv N^{n-1}(A_1,\ldots,A_{n-1}) \cdot {\sim}A_n$

would then completely characterize the connectives of U. Although (24) is a proper definition as prescribed by our rule, (25) will not do. Not only is it unschematizable but, without the resources of set theory, it does not meet the eliminability requirement. Thus, our treatment of multigrade connectives can provide only for the introduction of an infinite number of unigrade connectives into the language, nothing more. Though we lose rules given in their "fullest generality", we gain a schematizable account of 'N', for each instance of 'N' can be but one instance of a distinct connective of U.

*5 **Traditional rules revised*** We have already seen that for certain impoverished languages which cannot provide a suitable equivalence connective, the traditional rules of definition fail. So in revising these traditional rules, we extend our approach for defining connectives to treat relation symbols, operation symbols, and individual constants. Thus, the expressive power of the language will no longer limit the introduction of new notation into it.

We offer the following rule for defining relation symbols:

> A schematizable rule of interchange R introducing a new n-place relation symbol P is a proper definition in a theory if and only if R is of the form
>
> $$P(\delta_1,\ldots,\delta_n) \vdash\dashv \psi,$$
>
> and the following restrictions are satisfied: (i) *n-arity*: $\delta_1,\ldots,\delta_n$ are distinct metavariables standing for terms, (ii) *parametric relevance*: ψ has no metavariables standing for terms other than $\delta_1,\ldots,\delta_n$ and no free metavariables standing for individual variables, and (iii) *order*: ψ is a schematic wff in which the only nonlogical constants are primitive symbols and previously defined symbols of the theory.

Proofs of the eliminability and noncreativity requirements follow those offered in Section 3.

Without stating the rule for defining operation symbols explicitly, we simply point out that their definitions take the form $f(\delta_1,\ldots,\delta_n) = \alpha \vdash\dashv \psi$, where f is the new symbol, $\delta_1,\ldots,\delta_n$ are metavariables standing for terms, α is a metavariable standing for an individual variable, and ψ is a schematic wff. Essentially, the n-arity, parametric relevance, and order restrictions remain unchanged. We do, however, provide that ψ can contain α as a free variable. And in order to meet the eliminability requirement, we also add a fourth restriction, the *justifying theorem*, which requires that the schematic expression $\exists!\alpha\psi$ be derivable from the axioms and preceding definitions of the theory. We can account for the definitions of individual constants by regarding then as 0-ary operation symbols. Thus, '0' is defined by the rule:

(26) $0 = v_1 \vdash\dashv \forall v_2(v_2 + v_1 = v_2)$,

provided that we can prove the theorem scheme:

(27) $\exists ! v_1 \forall v_2 (v_2 + v_1 = v_2)$.

We see the importance of the justifying theorem in the proof of the eliminability requirement below. Suppose we define the n-place operation symbol f by the rule $f(\delta_1, \ldots, \delta_n) = \alpha \vDash \psi$ and the n-arity, parametric relevance, order, and justifying theorem restrictions hold. Given a wff S_1, which contains f in some context $(\ldots f(t_1, \ldots, t_n) \ldots)$, we wish to produce a wff S_2 in which the new symbol does not appear. From an appropriate instance of the justifying theorem scheme, $\exists ! vS$, we can deduce S, where S is an instance of ψ. Applying the definitional rule for f to S, we have that $f(t_1, \ldots, t_n) = v$. And from existential generalization and conjunction introduction, it follows that $\exists v(S \cdot (\ldots v \ldots))$. Thus, we have S_2. Now that we have established a method to eliminate the new operation symbol from any context, the noncreativity requirement follows by a proof similar to the one in Section 3.

The restriction of the justifying theorem also prevents the introduction of a two-place operation symbol like:

(28) $t_1 \lhd t_2 = v \vDash (t_1 < v) \cdot (t_2 < v)$

into ordinary arithmetic. Since '1' is strictly less than both '2' and '3', (28) allows one to infer that '$1 \lhd 1 = 2$' and '$1 \lhd 1 = 3$'. The contradiction '$2 = 3$' follows immediately. But since we cannot prove that the operation is uniquely defined, (28) is not a proper definition according to our rule.

At last, our system for the rules of definition is complete. Thus, following the results of our earlier study of connectives, we can revise the usual rules of definition, thereby enabling us to introduce new notation into nearly any language, no matter how impoverished.

NOTES

1. Lesniewski [5] first formulated the rules of definition and the requirements of eliminability and noncreativity.
2. We owe the concept of schematizability to Professor Massey.
3. We consider quantifiers, when attached to variables, not as belonging to a distinct class of symbols, but rather as sentential connectives, for when a quantifier-variable complex prefixes a wff, a new wff results.
4. See Tarski [11], Paper IV, for the details of Lukasiewicz's systems.
5. This observation is due to Professor Massey.
6. Leonard and Goodman [4] first introduced the term 'multigrade', referring not to connectives, but to predicates. Hendry and Massey [3], and Massey [6] study the topic further.
7. McCawley [8] rightly argues that the first approach is misguided, but he also argues against the second, maintaining that rules must be given in their "fullest generality". See especially pp. 516–538.

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