Systems of Sentence Logic with Trans-Atomic Units

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Although there are only 16 possible full truth-functional connectives in the two-valued system of sentence logic (*SL*), there are an additional 65 partial connectives which can be introduced into such systems. Partial connectives are connectives which have determinate values for some value assignments to their components but otherwise receive arbitrary value assignments. By repeating each of the four truth-value assignments (TT, TF, FT, FF) a chart layout of the additional connectives can be presented to show a partial layout for partial connectives which are arbitrary in the first case (Chart 1) and in the first and second case (Chart 2).

Chart 1: Partial Connectives Arbitrary in the First (TT) Case Only

Connectives	1	2	3	4	5	6	7	8	
Case 1 TT	T	T	T	T	T	T	T	T	Connectives 1–8 may be either
TT	F	F	F	F	F	F	F	F	T or F in Case 1 (TT).
Case 2 TF	T	T	T	F	T	F	F	F	Determinate case. The connec-
TF	T	T	T	F	T	F	F	F	tives receive a unique value.
Case 3 FT	T	T	F	T	F	T	F	F	Determinate case.
FT	T	T	F	T	F	T	F	F	
Case 4 FF	T	F	T	T	F	F	T	F	Determinate case.
FF	T	F	T	T	F	F	T	F	

Comment: (Chart 1) There are eight possible connectives which have only one arbitrary case. Since there are four choices for the arbitrary case, there are 32 connectives with one arbitrary case.

Chart 2: Partial Connectives Arbitrary in the First Two Cases

Connectives	33	34	35	36	
Case 1 TT	T	T	T	T	arbitrary
TT	F	F	F	F	
Case 2 TF	T	T	T	T	arbitrary
TF	F	F	F	F	
Case 3 FT	T	T	F	F	determinate
FT	T	T	F	F	
Case 4 FF	T	F	T	F	determinate
FF	T	F	T	F	

Compounds formed by means of partial connectives will be termed "trans-atomic" units since they are partially atomic and partially compound. Such connectives are hybrids and may be expected to have unusual properties by comparison with standard truth-functional compounds. There are a number of junctures where either the atomic or compound nature of these transatomic (TA) units may be emphasized. In the following development it is the atomic aspect which is emphasized. In particular, as regards:

(1) Nesting: Let '*' be a partial connective. In the development here, nesting is ruled out. Thus while $(P * Q)$ is a TA unit, $(P * (Q * R))$ is not a TA unit. This is not to say that other developments are not possible.
(2) Sentence of a deductive system (set of sentences without the requirement of semantic closure): Given the deductive system $D = \{(P * Q), (R \& W \to L), \sim(R \& W)\}$, it does not follow that $(Q * P)$ is a sentence (as opposed to thesis (theorem)) of he system even though, obviously, $(R \lor W)$, for example, is a sentence though not a thesis of the system. $(Q * P)$ is syntactically different from $(P * Q)$ just as the atomic sentence letter P differs from Q. If P is a sentence of a deductive system, it does not follow thereby that Q is a sentence of the system. Again, this is not to rule out alternative approaches to TA units.

As a general remark, TA units, as developed here, serve to inject non-TA units into the deductive stream of a system under certain conditions rather than to become part of that deductive stream. The first system of logic considered which has a partial connective, '$-\#$' ($(P -\# Q)$ is read "P latch Q"), is the system $SL\#$:

The syntax of SL#

Vocabulary: The vocabulary of $SL\#$ is that of SL with the additional sign '$-\#$' for forming TA units.

Sentences:
1. If ϕ, ψ are sentences of *SL* then ϕ, ψ, $(\phi -\# \psi)$ are sentences of *SL#*.
2. If ϕ, ψ are sentences of *SL#* then $(\phi \& \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$, $\sim\phi$ are sentences of *SL#*. (This excludes nested *TA*'s.)

Rules of Inference: The inference rules of *SL#* are those of *SL* with the following additions:

$$1.\ Elim\ \frac{\phi -\# \psi}{\vdash \phi \to \psi}. \qquad\qquad 2.\ Intro\ \frac{\phi \& \psi}{\vdash \phi -\# \psi}.$$

Sample Theorems:
1. $P \to (P -\# P)$.
2. $(P \& \sim Q) \to \sim (P -\# Q)$.
3. $((P -\# R) \& \sim R) \to (\sim R -\# \sim P)$.

Nontheorems:
1. $P -\# P$
2. $\sim P \to (P -\# Q)$
3. $(P -\# R) \to (\sim R -\# \sim P)$

Comment: If deductive relationships are intended between *TA*s in a deductive system then such deductive relationships must be entered as postulates of the deductive system.

Deductive Systems: A deductive system is any set of sentences of *SL#*.
Sentence of a deductive system:
1. If ϕ is an *SL* sentence of *D* then ϕ is an (*SL#*) sentence of *D*.
2. If ϕ is an *SL#* sentence composed of the sentences and *TA* units of *D* then ϕ is a sentence of *D*.
Note: If $(P -\# Q)$ is a sentence of *D* it does not follow that $(\sim Q -\# \sim P)$ is a sentence of *D*.
Example of a complete deductive system:
$D = \{(H -\# L), \sim (H \& L), (L \vee H)\}$.
Every sentence of *D* is such that either it or its negation is a theorem of *D*.

The semantics of SL#

General Comment: *TA*s are partially atomic elements. In the case where the antecedent of a *TA* receives the value F, the *TA* may arbitrarily receive the value T or F. In the case where the antecedent receives the value T, a *TA* is similar to the material conditional.

Interpretation (M) of D: Any assignment of truth values to the sentence letters and *TA*s which are elements of *D*.

Legitimate Interpretation of D: Any interpretation *M* such that
1. If *M* assigns T to $\phi -\# \psi$ then *M* does not assign T to ϕ and F to ψ.
2. If *M* assigns T to ϕ, ψ, then *M* assigns T to $\phi -\# \psi$.

Validity: ϕ is valid iff every legitimate interpretation *M* of ϕ is such that $M \vDash \phi$.

Sample truth table analyses:

1. $\sim P \to (P -\# Q)$ $(= \phi)$ invalid

P	Q	$P -\# Q$	ϕ
T	T	T	T
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

2. $(P \ \& \sim Q) \to \sim(P -\# Q)$ $(= \phi)$ valid

P	Q	$P -\# Q$	ϕ
T	T	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

3. $((P -\# R) \ \& \sim R) \to (\sim R -\# \sim P)$ $(= \phi)$ valid

P	R	$P -\# R$	$\sim R -\# \sim P$	ϕ
T	T	T	T	T
T	T	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	T	F	T
F	T	F	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

Completeness of the system SL#: $D \vDash \phi \to D \vdash \phi$.   The completeness of the system follows as a corollary to a Gödel-Henkin *(GH)* type theorem to the effect that $(D)(D\text{-}cons \to (EM)(M\text{-}legit \ \& \ M \vDash D))$.

I. $(D)(D\text{-}cons \to (EM)(M\text{-}legit \ \& \ M \vDash D))$.
Assume D is consistent. By Lindenbaum's results, D may be extended to a maximal consistent set D^*. Hence if it can be proved that there exists a legitimate model M such that $M \vDash D^*$ then M would model D since D is a subset of D^*. M is constructed as follows. Since D^* is maximal, for every sentence letter S and *TA* unit U, $S \in D^*$ or $(\sim S) \in D^*$ and $U \in D^*$ or $(\sim U) \in D^*$. Let M assign T to S, U if $S, U \in D^*$ and F otherwise. To

prove $M \vDash D$ it suffices to prove that $\phi \in D^* \to M \vDash \phi$. This latter result follows from the stronger proposition (GH') $\phi \in D^* \leftrightarrow M \vDash \phi$ which will be proven by induction on the number of connectives in ϕ.

$|n = 0|$. Then ϕ is S or U and $\phi \in D^* \leftrightarrow M \vDash \phi$ follows by construction of M.

$|n = k + 1|$. *Case 1.* ϕ is $\sim\phi_1$. By hypothesis of induction (HI) $\phi_1 \in D^* \leftrightarrow M \vDash \phi$. Assume $\phi \in D^*$ (i.e., $(\sim\phi_1 \in D^*)$). Since D^* is consistent, $\sim(\phi_1 \in D^*)$. Using the HI, $\sim(M \vDash \phi_1)$. Then $M \vDash \sim\phi_1$. Hence $\phi \in D^* \to M \vDash \phi$. Assume $M \vDash \phi$ (i.e., $M \vDash \sim\phi_1$). Then $\sim(M \vDash \phi_1)$. Using the HI, $\sim(\phi_1 \in D^*)$. Since D^* is maximal $\sim\phi_1 \in D^*$. Hence $M \vDash \phi \to \phi \in D^*$.

   Case 2. ϕ is $\phi_1 \to \phi_2$. By HI, $\phi_1 \in D^* \leftrightarrow M \vDash \phi_1$ and $\phi_2 \in D^* \leftrightarrow M \vDash \phi_2$. Assume $\phi_1 \to \phi_2 \in D^*$. Since D^* is consistent, $\sim(\phi_1 \in D^*)$ or $\sim(\sim\phi_2 \in D^*)$. Since D^* is maximal, $(\sim\phi_1) \in D^*$ or $\phi_2 \in D^*$. Using HI, $M \vDash \sim\phi_1$ or $M \vDash \phi_2$. In either case $M \vDash \phi_1 \to \phi_2$. Proof that $M \vDash \phi_1 \to \phi_2$ only if $\phi_1 \to \phi_2 \in D^*$ is similar. Hence $\phi \in D^* \leftrightarrow M \vDash \phi$ follows by induction, and consequently $M \vDash D^*$ as well as D.

Legitimacy of M: It is to be shown

1. $M \vDash \phi$ $-\#$ ψ *only if* $\sim(M \vDash \phi, \sim\psi)$.
   Assume to the contrary $M \vDash \phi$ and $M \vDash \sim\psi$. By the result just proven, $\phi \in D^*$ and $\sim\psi \in D^*$. By hypothesis and GH', ϕ $-\#$ $\psi \in D^*$. Consequently, by *Elim*, $\phi \to \psi \in D^*$, contradicting the consistency of D^*.
2. $M \vDash \phi, \psi$ *only if* $M \vDash \phi$ $-\#$ ψ.
   By hypothesis, $M \vDash \phi, \psi$. By GH', $\phi, \psi \in D^*$. By *Intro*, ϕ $-\#$ $\psi \in D^*$. By GH', $M \vDash \phi$ $-\#$ ψ.

II. *Completeness $D \vDash \phi \to D \vdash \phi$.*
   Without loss of generality it may be assumed that D is consistent. By hypothesis, $D \vDash \phi$. Assume to the contrary $\sim(D \vdash \phi)$. Then D' $(=D \cup \{\sim\phi\})$ is consistent. By GH, $M_a \vDash D'$ and consequently $\sim\phi$. From the assumption $D \vDash \phi$ it follows $(M)(M \vDash D \to M \vDash \phi)$. Since D is a subset of D' and $M_a \vDash D'$, $M_a \vDash D$. Consequently $M_a \vDash \phi, \sim\phi$. This is not possible.

Systems SL#1–3: While it is not the opinion here, some may feel the logical independence of $(P$ $-\#$ $Q)$ and $(P \, \& \, P$ $-\#$ $Q)$ excessively restrictive. In the system *SL#* syntactically different *TA* units are logically independent. Systems with additional implicational relations between *TA* units can be constructed. These systems are constructed by adding additional inference rules and correspondingly restricting legitimate interpretations. The notion of a sentence of a deductive system must correspondingly be expanded since assigning truth values to the sentence letters and *TA* units in D does not mean that every *TA* unit built up from the sentence letters in D (i.e., in the sentences in D) receives a truth value.

   The system *SL#1* is a modest departure from *SL#* which makes *TA* units equivalent if their corresponding conditionals are logically equivalent. The basic changes needed are as follows:

Inference rules:

1. *Intro*
2. *Elim*
3. *LE* (Logical Equivalence) $\phi \; -\# \; \psi$

$$\frac{\vdash_{SL}(\phi \to \psi) \leftrightarrow (\phi_1 \to \psi_1)}{\vdash \phi_1 \; -\# \; \psi_1} \; .$$

Sentences of D:

1. If $\phi \; -\# \; \psi$ is an element (or a component of an element) of D and $\vdash_{SL}(\phi \to \psi) \leftrightarrow (\phi_1 \to \psi_1)$ then $\phi_1 \; -\# \; \psi_1$ is a sentence of D.
2. If ϕ is an *SL#* sentence composed of the *SL* sentences and *TA* sentences of D then ϕ is a sentence of D.

Semantic Rules:

Legitimate Interpretation:
1. Rules for *SL#*
2. $M \vDash (\phi \; -\# \; \psi)$ and $\vDash_{SL}(\phi \to \psi) \leftrightarrow (\phi_1 \to \psi_1)$ only if $M \vDash (\phi_1 \; -\# \; \psi_1)$.

Completeness: A completeness proof for the system is similar to that given for *SL#* except that an additional section is required to show the legitimacy of M. By hypothesis $M \vDash (\phi \; -\# \; \psi)$ and $\vDash_{SL}(\phi \to \psi) \leftrightarrow (\phi_1 \to \psi_1)$. By the completeness of *SL*, $\vdash_{SL}(\phi \to \psi) \leftrightarrow (\phi_1 \to \psi_1)$. By *GH'*, $(\phi \; -\# \; \psi) \in D^*$. By *LE*, $D^* \vdash (\phi_1 \; -\# \; \psi_1)$. Consequently, $(c_1 \; -\# \; \psi_1) \in D^*$.

SL#2: The extension of *SL#1* to *SL#2* is accomplished, for example, by allowing the inference of $(P \; -\# \; Q)$ from $((P \vee R) \; -\# \; Q)$. Two additional inference rules are needed:

1. *LI* (Left implication) $\phi \; -\# \; \psi$

$$\frac{\vdash_{SL} \phi_1 \to \phi}{\vdash \phi_1 \; -\# \; \psi}$$

2. *RI* (Right implication) $\phi \; -\# \; \psi$

$$\frac{\vdash_{SL} \psi \to \psi_1}{\vdash \phi \; -\# \; \psi_1} \; .$$

SL#2, with the appropriate semantic correlates for *LI* and *RI*, can be proven complete. While $((R \; \& \; {\sim}R) \; -\# \; Q)$ can be proven from $(P \; -\# \; Q)$, $(R \to ({\sim}R \; -\# \; Q))$ does not follow from $((R \; \& \; {\sim}R) \; -\# \; Q)$, and hence *SL#2* skirts the Duns Scotus effect whereby $(P \to Q)$ follows from ${\sim}P$. The Duns Scotus effect is the primary difficulty with using the material conditional to define dispositional predicates such as 'is soluble'.

SL#3: The further extension of SL# systems which allows *LI* and *RI* to proceed from an assumption $(\phi_1 \to \phi)$ as opposed to a logical truth, i.e., from $(\phi_1 \to \phi)$, as opposed to $\vdash_{SL}(\phi_1 \to \phi)$, to infer the *TA* $(\phi_1 \; -\# \; \psi)$ from $(\phi, \; -\# \; \psi)$, is subject to a type of Duns Scotus effect. Where A is a given lump of sugar and B is a given diamond, let:

P be 'A is placed in water'
Q be 'A dissolves'
R be 'B is placed in water'
D be $\{(P -\# Q), \sim R\}$.

Then $D \vdash (R -\# Q)$. This would preclude using '$-\#$' as some sort of implicational relation as the inference that if a given diamond is placed in water then a given lump of sugar will dissolve hardly seems warranted.

In general it is not the claim here that any of the partial connectives considered are explicans for any natural language connectives. In fact the word 'then' in English may cover a multiple of sins other than the three or four for which it has been the preoccupation of a good deal of philosophical analysis to find explicans. It is the claim here, though it will not be argued here, that '$-\#$' is just what is needed to define a dispositional predicate such as 'is soluble' in the context of a deductive system. Other partial connectives are useful in formulating lawlike generalizations. Their utility is independent of the enterprise of explication.

It has been the purpose of this paper to demonstrate the logical viability of systems of logic with TA units or partial connectives. Whether such connectives prove to be useful is another question, a question which presupposes their logical viability.

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